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Operadic categories as a natural environment for Koszul duality

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This is the first paper of a series which aims to set up the cornerstones of Koszul duality for operads over operadic categories. To this end we single out additional properties of operadic categories under which the theory of quadratic operads and their Koszulity can be developed, parallel to the traditional one by Ginzburg–Kapranov. We then investigate how these extra properties interact with discrete operadic (op)fibrations, which we use as a powerful tool to construct new operadic categories from old ones. We pay particular attention to the operadic category of graphs, giving a full description of this category (and its variants) as an operadic category, and proving that it satisfies all the additional properties.

Our present work provides an answer to a question formulated in Loday's last talk, in 2012: "What encodes types of operads?". In the second and third papers of our series we continue Loday's program by answering his second question: "How to construct Koszul duals to these objects?", and proving Koszulity of some of the most relevant operads.

Contents

Introduction

Operads are a powerful foundation for handling composition and substitution of various structures. While at first the underlying combinatorics of operads concerned how trees are composed and contracted, mathematics and mathematical physics soon found the need for composing also more general graphs, leading to more complex notions.

The present work is the first one in a series of articles which lays down the basic stones of "operadic calculus" for our general theory of "operad-like" structures. By them we mean, besides the classical operads in the sense of Boardman–Vogt [13] and May [35], and their more recent variants such as cyclic, modular or wheeled operads [19, 20, 32], also diverse versions of PROPs such as properads [37], dioperads [16], $\frac{1}{2}$PROPs [34], and still more exotic stuff as permutads and

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pre-permutads [27] or protoperads [25]. Also Batanin's n-operads [3, 4] appear in our scope. One may vaguely characterize operad- and PROP-like structures as those generalizing compositions of multivalued functions.

History. To our knowledge, the first attempt to systematize this kind of objects was made by the second author in a 2008 paper [29]. He considered structures with operations modeled by contractions along edges of graphs (called "pasting schemes" in this context) of the type particular to a concrete situation. These schemes were required to satisfy an important property of *hereditarity*, which is a specific stability under contractions of subgraphs. This property was later redressed into categorical garment in the notion of a *Feynman category* [21]. Hereditarity however played an important rôle already in [14] and in unpublished work of Melliès–Tabareau [36]. Let us close this brief historical account by mentioning Getzler's work on regular patterns [18] predating Feynman categories, see also the follow-ups [10, 11]. Finally, in [9] an approach to general operad-like structures through the use of polynomial monads was developed. We are not commenting in this work on the connections between these approaches and ours, since this topic deserves a separate paper.

The setup. Our approach is based on the notion of an *operadic category*. The idea goes back to the first author's work on higher category theory using a higher generalization of non-symmetric operads [2]. In this formalism, a higher version of the Eckmann–Hilton argument was described by reformulating the classical notion of a (symmetric) operad and Batanin's notion of an n-operad in such a way that a comparison of the two notions became possible [3, 4]. The fruitfulness of this idea was then confirmed in [5].

In our work on the duoidal Deligne conjecture we came to understand that the same categorical scheme is very useful and, indeed, necessary for the study of many other standard and nonstandard operad-like structures. Thus the concept of operadic categories was introduced by the authors in [6].

Operadic categories are essentially the most distilled algebraic structures which contain all information determining operad-like structures of a given type along with their algebras. Morphisms in operadic categories possess *fibers* whose properties are modeled by the preimages of maps between finite sets. Unlike in Barwick's operator categories [1], the fibers need not be pullbacks. Each operadic category $\mathtt{O}$ has its *operads* and each $\mathtt{O}$-operad P has its category of P-*algebras*.

An archetypal operadic category is the skeletal category $\mathtt{Fin}$ of finite sets. Its operads are classical one-colored symmetric operads. As we demonstrate in this paper, various hereditary categories of graphs are operadic. Examples of different scent are Batanin's n-trees and n-ordinals, or the operadic category supporting permutads. For the reader's convenience we recall definitions of operadic categories and related notions in the opening Section 1. The background scheme of our approach is the triad

$$\boxed{\text{level 1: } \mathtt{O}^+\text{-operads}}$$
$$\Downarrow$$
$$\boxed{\text{level 0: } \mathtt{O}\text{-operads} = 1_{\mathtt{O}^+}\text{-algebras}}$$
$$\Downarrow$$
$$\boxed{\text{level -1: algebras of } \mathtt{O}\text{-operads}}$$

in which "$\Downarrow$" means "is governed by." At level 0 one sees operads over an operadic category $\mathtt{O}$. We consider algebras for these operads as objects at level -1. It turns out that $\mathtt{O}$-operads are algebras for the constant operad $1_{\mathtt{O}^+}$ over a certain operadic category $\mathtt{O}^+$ called the $+$-*construction* of $\mathtt{O}$, which we place at level 1. The triad can be continued upwards to infinity. The theory of $+$-constructions will be developed in a future paper.

An example is the *classical triad* in which $\mathtt{O}$ is the operadic category $\mathtt{Fin}$ of finite sets. $\mathtt{Fin}$-operads simultaneously appear as algebras of the constant operad $1_{\mathtt{RTr}}$ over the operadic category $\mathtt{RTr}$ of rooted trees, which is $\mathtt{Fin}^+$. At level -1 we find algebras for the classical operads.

Strong inspiration for our setup was the seminal paper by Getzler and Kapranov [20], who realized that modular operads are algebras over a certain (hyper)operad. They thus constructed levels 0 and -1 of the triad for the operadic category $\mathtt{ggGrc}$ of connected genus-graded ordered graphs, cf. Example 4.19. It turns out that $\mathtt{ggGrc}^+$ at level 1 is the category of graphs from $\mathtt{ggGrc}$ with a hierarchy of nested subgraphs. The resulting scheme is the *Getzler–Kapranov triad*.

The novelty of our approach is that we systematically put the structures we want to study at level -1 so that they appear as algebras over a certain operad. For instance, cyclic operads in our setup are algebras over the constant operad $\mathbf{1}_{\mathtt{Tr}}$ over the operadic category $\mathtt{Tr}$ of trees, though they themselves are *not* operads over any operadic category.

Aims of the present and future work. In this paper we focus on categorical and combinatorial foundations of Koszul duality for operads over operadic categories. In the follow-up [7] we introduce the notion of quadraticity for operads over operadic categories, and all other ingredients of the duality theory for operads including the Koszul property. We will then prove that operads describing the most common structures are Koszul. This provides an answer to the two questions in Loday's last talk [26] mentioned in the Abstract.

Our series of papers is continued by [8] in which we construct explicit minimal models for the (hyper)operads governing modular, cyclic and ordinary operads, and wheeled properads. The final paper of this series will be devoted to the +-construction in the context of operadic categories.

The plan. In Section 1 we recall operadic categories and related notions, using almost verbatim the material of [6]. In Section 2 we single out some finer additional properties of operadic categories which will ensure in our second paper [7] that free operads over these categories are of a particularly nice form. Section 3 is devoted to our construction of an important operadic category of graphs and we show that it satisfies all these extra requirements. We will also see that several subtle properties of graphs may be conveniently expressed in the language of our theory. In Section 4 we recall from [6] discrete (op)fibrations and the related Grothendieck construction, and use it as a tool for producing new operadic categories from old ones.

Free operads over operadic categories will play an important rôle both in the definition of quadraticity and of the dual dg operad needed for the formulation of the Koszul property in the follow-up [7]. As we noticed for classical operads in [28], the construction of free operads is more structured if one uses, instead of the standard definition, a modified one. Let us explain what we mean by this.

Traditional operads in the spirit of May [35] are collections $\{\mathcal{P}(n)\}_{n\geq 1}$ of Σ_n-modules with composition laws

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k),\ \ k, n_1, \ldots, n_k \geq 1, \tag{1a}$$

satisfying appropriate associativity and equivariance axioms; notice that we implicitly assume that $\mathcal{P}(0)$ is empty. However, in [28, Definition 1.1] we suggested a definition based on binary composition laws

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m + n - 1),\ \ n \geq 1,\ \ 1 \leq i \leq m. \tag{1b}$$

It turned out that under some quite standard assumptions, for instance in the presence of units, augmentations or connectivity, the two notions are equivalent, see e.g. [28, Observation 1.2] or [29, Proposition 13], though there are structures possessing composition laws (1a) only [29, Example 19]. Operad-like structures based on "partial compositions" in (1b) were later called Markl operads.

Operads over general operadic categories also exist in two disguises which are, under favorable conditions, equivalent – in the form where the compositions in all inputs are made simultaneously; this is how they were introduced in [6] – and in Markl form whose composition laws are binary. The crucial advantage of the latter is, as in the classical case, that free Markl operads are naturally graded by the length of the chain of compositions. The theory of Markl operads and its relation to the original formulation of operad theory over operadic categories given in [6, Section 1] together with the necessary background material occupies Sections 5 and 6. To help the reader navigate the paper, we include an index of terminology and notation.

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1 Operadic categories and their operads

In this preliminary section we recall, for the convenience of the reader, some basic definitions from [6]. The reader may also wish to look at Lack's paper [23] for a characterization of operadic categories in the context of skew monoidal categories, or at [17] by Garner, Kock and Weber for yet another point of view. For brevity we use the terms *operadic category* and *operadic functor* for what have been defined as a *strict operadic category* and a *strict operadic functor* in [6]. More general equivalence-invariant operadic categories will be the subject of upcoming work of Batanin, Kock and Weber.

Let $\mathtt{Fin}$ be the skeletal category of finite sets (denoted in [6] by $\mathtt{sFSet}$). The objects of this category are the linearly ordered sets $\bar{n} = \{1, \ldots, n\}, n \in \mathbb{N}$. Morphisms are arbitrary maps between these sets. We define the ith fiber $f^{-1}(i)$ of a morphism $f : T \to S$, $i \in S$, as the pullback of f along the map $\bar{1} \to S$ which picks up the element i, so this is the object $f^{-1}(i) = \bar{n}_i \in \mathtt{Fin}$ which is isomorphic as a linearly ordered set to the preimage $\{j \in T \mid f(j) = i\}$. Any commutative triangle in $\mathtt{Fin}$

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "h"'] & & S \arrow[dl, "g"] \\ & R & \end{tikzcd}$$

then induces a map $f_i : h^{-1}(i) \to g^{-1}(i)$ for any $i \in R$. This assignment is a functor $Fib_i : \mathtt{Fin}/R \to \mathtt{Fin}$. Moreover, for any $j \in S$ we have the equality $f^{-1}(j) = f^{-1}_{g(j)}(j)$. The above structure on the category $\mathtt{Fin}$ motivates the following abstract definition.

Recall that an object t in a category $\mathtt{O}$ is a *local terminal object* if it is a terminal object in its connected component. An *operadic category* is a category $\mathtt{O}$ equipped with a *cardinality functor* $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$ having the following properties. We require that each connected component of $\mathtt{O}$ has a chosen local terminal object U_c, $c \in \pi_0(\mathtt{O})$. We also assume that for every $f : T \to S$ in $\mathtt{O}$ and every element $i \in |S|$, there is given an object $f^{-1}(i) \in \mathtt{O}$, which we will call *the i-th fiber* of f, such that $|f^{-1}(i)| = |f|^{-1}(i)$. We also require that

(i) For any $c \in \pi_0(\mathtt{O})$, $|U_c| = 1$.

(ii) For any $T \in \mathtt{O}$ and each $i \in |T|$, the fiber $1\!\!1_T^{-1}(i)$ of the identity $1\!\!1_T : T \to T$ is a chosen local terminal object.

(iii) For any commutative diagram in $\mathtt{O}$

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "h"'] & & S \arrow[dl, "g"] \\ & R & \end{tikzcd} \tag{2}$$

and every $i \in |R|$, one is given a map

$$f_i : h^{-1}(i) \to g^{-1}(i)$$

such that $|f_i| : |h^{-1}(i)| \to |g^{-1}(i)|$ is the map $|h|^{-1}(i) \to |g|^{-1}(i)$ of sets induced by

$$\begin{tikzcd} {|T|} \arrow[rr, "|f|"] \arrow[dr, "|h|"'] & & {|S|} \arrow[dl, "|g|"] \\ & {|R|} . & \end{tikzcd}$$

We moreover require that this assignment forms a functor $Fib_i : \mathtt{O}/R \to \mathtt{O}$ called *fiber functor*. Moreover, if $R = U_c$, the functor Fib_1 must be the domain functor $\mathtt{O}/R \to \mathtt{O}$. The last condition says that the unique fiber of the canonical morphism $!_T : T \to U_c$ is T.

(iv) In the situation of Axiom (iii), for any $j \in |S|$, one has the equality

$$f^{-1}(j) = f^{-1}_{|g|(j)}(j). \tag{3}$$

(v) Let

$$\begin{tikzcd} & S \arrow[dr, "a"] \arrow[dd, "g"] & \\ T \arrow[ur, "f"] \arrow[rr, "b"] \arrow[dr, "h"'] & & Q \arrow[dl, "c"] \\ & R & \end{tikzcd}$$

be a commutative diagram in $\mathtt{O}$ and let $j \in |Q|, i = |c|(j)$. Then by Axiom (iii) the diagram

$$\begin{tikzcd} h^{-1}(i) \arrow[rr, "f_i"] \arrow[dr, "b_i"'] & & g^{-1}(i) \arrow[dl, "a_i"] \\ & c^{-1}(i) & \end{tikzcd}$$

commutes, so it induces a morphism $(f_i)_j : b_i^{-1}(j) \to a_i^{-1}(j)$. By Axiom (iv) we have

$$a^{-1}(j) = a_i^{-1}(j) \text{ and } b^{-1}(j) = b_i^{-1}(j).$$

We then require the equality $f_j = (f_i)_j$.

We will also assume that the set $\pi_0(\mathtt{O})$ of connected components is *small* with respect to a sufficiently big ambient universe.

An *operadic functor* between operadic categories is a functor $F : \mathtt{O} \to \mathtt{P}$ over $\mathtt{Fin}$ which preserves fibers in the sense that $F\big(f^{-1}(i)\big) = F(f)^{-1}(i)$, for any $f : T \to S \in \mathtt{O}$ and $i \in |S| = |F(S)|$. We also require that F preserves the chosen local terminal objects, and that $F(f_i) = F(f)_i$ for f as in (2). This gives the category $\mathtt{OpCat}$ of operadic categories and their operadic functors.

Let $(\mathtt{V}, \otimes, \Bbbk)$ be a (closed) symmetric monoidal category. Thanks to MacLane's coherence theorem we will also assume that associativity and unit constrains in $\mathtt{V}$ are identities. For a family $E = \{E(T)\}_{T \in \mathtt{O}}$ of objects of $\mathtt{V}$ and a morphism $f : T \to S$ let

$$E(f) = \bigotimes_{i \in |S|} E(T_i),\ T_i := f^{-1}(i).$$

In the following definition we tacitly use equalities (3).

Definition 1.1. An *operad over* $\mathtt{O}$ (or simply an $\mathtt{O}$-*operad*) in $\mathtt{V}$ is a family $\mathcal{P} = \{\mathcal{P}(T)\}_{T \in \mathtt{O}}$ of objects of $\mathtt{V}$ together with units

$$I \to \mathcal{P}(U_c),\ c \in \pi_0(\mathtt{O}),$$

and composition laws

$$\gamma_f : \mathcal{P}(f) \otimes \mathcal{P}(S) \to \mathcal{P}(T),\ f : T \to S,$$

satisfying the following axioms.

(i) Let $T \xrightarrow{f} S \xrightarrow{g} R$ be morphisms in $\mathtt{O}$ and $h := gf : T \to R$ as in (2). Then the following diagram of composition laws of $\mathcal{P}$ combined with the canonical isomorphisms of products in $\mathtt{V}$ commutes:

$$\begin{tikzcd} \bigotimes_{i \in |R|} \mathcal{P}(f_i) \otimes \mathcal{P}(g) \otimes \mathcal{P}(R) \arrow[drr, bend left, "\bigotimes_i \gamma_{f_i} \otimes \mathbb{1}"] \arrow[dd, "\mathbb{1} \otimes \gamma_g"'] & & \\ & & \mathcal{P}(h) \otimes \mathcal{P}(R) \arrow[dl, bend left, "\gamma_h"] \\ \bigotimes_{i \in |R|} \mathcal{P}(f_i) \otimes \mathcal{P}(S) \cong \mathcal{P}(f) \otimes \mathcal{P}(S) \arrow[r, "\gamma_f"] & \mathcal{P}(T) & \end{tikzcd}$$

(ii) The composite

$$\mathcal{P}(T) \longrightarrow \bigotimes_{i\in|T|} \Bbbk \otimes \mathcal{P}(T) \longrightarrow \bigotimes_{i\in|T|} \mathcal{P}(U_{c_i}) \otimes \mathcal{P}(T) \xrightarrow{=} \mathcal{P}(\mathbb{1}_T) \otimes \mathcal{P}(T) \xrightarrow{\gamma_{\mathbb{1}}} \mathcal{P}(T)$$

is the identity for each $T \in \mathtt{O}$.

(iii) The composite

$$\mathcal{P}(T) \otimes \Bbbk \longrightarrow \mathcal{P}(T) \otimes \mathcal{P}(U_c) \xrightarrow{=} \mathcal{P}(!_T) \otimes \mathcal{P}(U_c) \xrightarrow{\gamma_{!_T}} \mathcal{P}(T)$$

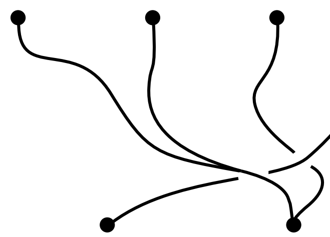
is the identity for each $T \in \mathtt{O}$, where $!_T : T \to U_c$ is the unique morphism.

A *morphism* $\mathcal{P}' \to \mathcal{P}''$ of $\mathtt{O}$-operads in $\mathtt{V}$ is a collection of $\mathtt{V}$-morphisms $\mathcal{P}'(T) \to \mathcal{P}''(T)$, $T \in \mathtt{O}$, commuting with the composition laws and units. We denote by $\mathtt{O}\text{-}\mathtt{Oper}(\mathtt{V})$ (or simply by $\mathtt{O}\text{-}\mathtt{Oper}$ if $\mathtt{V}$ is understood) the category of $\mathtt{O}$-operads in $\mathtt{V}$. Each operadic functor $F : \mathtt{O} \to \mathtt{P}$ induces the restriction $F^* : \mathtt{P}\text{-}\mathtt{Oper} \to \mathtt{O}\text{-}\mathtt{Oper}$.

Example 1.2. A primary example of an operadic category is the category $\mathtt{Fin}$, while the cardinality functor $|-| : \mathtt{O} \to \mathtt{Fin}$ is an example of an operadic functor. Thus $\mathtt{Fin}$ is the terminal object in the category of operadic categories and operadic functors. The category of $\mathtt{Fin}$-operads is isomorphic to the category of classical one-colored (symmetric) operads.

Example 1.3. The subcategory $\mathtt{Fin}_{\mathrm{semi}} \subset \mathtt{Fin}$ of nonempty finite sets and surjections is also an operadic category. Operads over $\mathtt{Fin}_{\mathrm{semi}}$ are classical one-colored symmetric operads without nullary operations. We will call such operads *constant-free*.

Example 1.4. The category of vines $\mathtt{Vines}$ [24, 38] is another example of an operadic category. It has the same objects as $\mathtt{Fin}$ but a morphism $\bar{n} \to \bar{m}$ is an isotopy class of merging descending strings in $\mathbb{R}^3$ (called *vines*) like in the following picture:



There is a canonical identity-on-object functor $|-| : \mathtt{Vines} \to \mathtt{Fin}$ which sends a vine to the function assigning to the top endpoint of a string its bottom endpoint. A fiber of a vine $v : \bar{n} \to \bar{m}$ is equal to the fiber of $|v| : \bar{n} \to \bar{m}$. The rest of the operadic category structure on $\mathtt{Vines}$ is quite obvious. The category of $\mathtt{Vines}$-operads is isomorphic to the category of braided operads [15]. This fact can be easily proved using the equivalent definition of braided operad given in [5].

In fact, using Weber's theory [38] one can associate an operadic category $\mathtt{O}(G)$ to each group operad G (see [39] for the definition) such that $\mathtt{O}(G)$-operads are exactly G-operads. The operadic categories $\mathtt{Fin}$ and $\mathtt{Vines}$ are special cases $\mathtt{O}(\Sigma)$ and $\mathtt{O}(\mathit{Braid})$ of this construction for the symmetric group and braid group operads, respectively. We will provide the details elsewhere.

Example 1.5. Let $\mathfrak{C}$ be a set. Recall from [6, Example 1.7] (see also [23, Example 10.2]) that a $\mathfrak{C}$-*bouquet* is a map $b : X + 1 \to \mathfrak{C}$, where $X \in \mathtt{Fin}$. In other words, a $\mathfrak{C}$-bouquet is an ordered $(k+1)$-tuple $(i_1, \ldots, i_k; i)$, $X = \bar{k}$, of elements of $\mathfrak{C}$. It can also be thought of as a planar corolla all of whose edges (including the root) are colored. The extra color $b(1) \in \mathfrak{C}$ is called the *root color*. The finite set X is the *underlying set* of the bouquet b.

A map of $\mathfrak{C}$-bouquets $b \to c$ whose root colors coincide is an arbitrary map $f : X \to Y$ of their underlying sets. There are no maps between $\mathfrak{C}$-bouquets with different root colors. We denote the resulting category of $\mathfrak{C}$-bouquets by $\mathtt{Bq}(\mathfrak{C})$.

The cardinality functor $|\text{-}| : \mathtt{Bq}(\mathfrak{C}) \to \mathtt{Fin}$ assigns to a bouquet $b : X + 1 \to \mathfrak{C}$ its underlying set X. The fiber of a map $b \to c$ given by $f : X \to Y$ over an element $y \in Y$ is the $\mathfrak{C}$-bouquet whose underlying set is $f^{-1}(y)$, the root color coincides with the color of y and the colors of the elements are inherited from the colors of the elements of X.

Operads over the category $\mathtt{Bq}(\mathfrak{C})$ of $\mathfrak{C}$-bouquets are ordinary $\mathfrak{C}$-colored operads. Therefore, for each $\mathfrak{C}$-colored collection $E = \{E_c\}_{c \in \mathfrak{C}}$ of objects of $\mathtt{V}$ one has the *endomorphism* $\mathtt{Bq}(\mathfrak{C})$*-operad* $\mathcal{E}nd_E^{\mathtt{Bq}(\mathfrak{C})}$, namely the ordinary colored endomorphism operad [12, §1.2].

Example 1.6. The category Δ_{alg} of finite ordinals (including the empty one) has an obvious structure of an operadic category. Operads over Δ_{alg} are ordinary nonsymmetric operads [4, Prop. 3.1].

Example 1.7. The cartesian product in the category of operadic categories exists and is given by a pullback over $\mathtt{Fin}$ using cardinality functors. In particular, for any operadic category $\mathtt{O}$ and any set of coulours $\mathfrak{C}$, the $\mathtt{Bq}(\mathfrak{C}) \times \mathtt{O}$-operads are $\mathfrak{C}$-colored $\mathtt{O}$-operads [6, page 1637]. Likewise, the product $\mathtt{Vines} \times \mathtt{O}$ with the operadic category of vines of Example 1.4 describes braided versions of $\mathtt{O}$-operads. The product $\Delta_{alg} \times \mathtt{O}$ is isomorphic to the subcategory $\mathtt{O}_{\mathtt{ord}} \subset \mathtt{O}$ of morphisms for which $|f|$ is order preserving.

Example 1.8. Another important example is the operadic category $\mathtt{Ord}_n$ of n-ordinals, $n \in \mathbb{N}$, see [3, Sec. II]. $\mathtt{Ord}_n$-operads are Batanin's pruned n-operads which are allowed to take values not only in ordinary symmetric monoidal categories, but in more general *globular* monoidal n-categories. Although $\mathtt{Ord}_n$ does not fulfill the additional properties required for some constructions in this work, it was a crucial motivating example for our definition of operadic categories.

For each operadic category $\mathtt{O}$ with $\pi_0(\mathtt{O}) = \mathfrak{C}$, there is a canonical operadic "arity" functor

$$Ar : \mathtt{O} \to \mathtt{Bq}(\mathfrak{C}) \tag{4}$$

giving rise to the factorization

$$\begin{tikzcd} & \mathtt{O} \arrow[dl, bend right, "Ar"'] \arrow[dr, bend left, "|\text{-}|"] & \\ \mathtt{Bq}(\mathfrak{C}) \arrow[rr, "|\text{-}|"] & & \mathtt{Fin} \end{tikzcd} \tag{5}$$

of the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$. It is constructed as follows.

Recall that the *ith source* $s_i(T)$ of an object $T \in \mathtt{O}$ is the ith fiber of the identity automorphism of T, i.e. $s_i(T) := \mathbb{1}_T^{-1}(i)$ for $i \in |T|$. We denote by $s(T)$ the set of all sources of T. For an object $T \in \mathtt{O}$ we denote by $\pi_0(T) \in \pi_0(\mathtt{O})$ the connected component to which T belongs. Similarly, for a subset X of objects of $\mathtt{O}$,

$$\pi_0(X) := \{\pi_0(T) \mid T \in X\} \subset \pi_0(\mathtt{O}).$$

The bouquet $Ar(T) \in \mathtt{Bq}(\mathfrak{C})$ is defined as $b : s(T) + 1 \to \mathfrak{C}$, where b associates to each fiber $U \in s(T)$ the corresponding connected component $\pi_0(U) \in \mathfrak{C}$, and $b(1) := \pi_0(T)$. The assignment $T \mapsto Ar(T)$ extends to an operadic functor.

Example 1.9. For a $\mathfrak{C}$-colored collection $E = \{E_c\}_{c \in \mathfrak{C}}$ in $\mathtt{V}$ and an operadic category $\mathtt{O}$ with $\pi_0(\mathtt{O}) = \mathfrak{C}$, one defines the *endomorphism* $\mathtt{O}$*-operad* $\mathcal{E}nd_E^{\mathtt{O}}$ as the restriction

$$\mathcal{E}nd_E^{\mathtt{O}} := Ar^*\big(\mathcal{E}nd_E^{\mathtt{Bq}(\mathfrak{C})}\big)$$

of the $\mathtt{Bq}(\mathfrak{C})$-endomorphism operad of Example 1.5 along the arity functor Ar of (4).

The following definition was given in [6, Definition 1.20].

Definition 1.10. An *algebra* over an $\mathtt{O}$-operad $\mathcal{P}$ in $\mathtt{V}$ is a collection $A = \{A_c\}_{c \in \pi_0(\mathtt{O})}$, $A_c \in \mathtt{V}$, equipped with an $\mathtt{O}$-operad map $\alpha : \mathcal{P} \to \mathcal{E}nd_A^{\mathtt{O}}$.

An algebra structure is thus provided by suitable structure maps

$$\alpha_T : \mathcal{P}(T) \otimes \bigotimes_{c \in \pi_0(s(T))} A_c \to A_{\pi_0(T)}, \ T \in \mathtt{O},$$

We denote by $\mathcal{P}\text{-}\mathtt{Alg}(\mathtt{V})$ (or simply by $\mathcal{P}\text{-}\mathtt{Alg}$ when $\mathtt{V}$ is clear from the context) the category of $\mathcal{P}$-algebras and their morphisms.

2 Sundry facts about operadic categories

The aim of this section is to study some finer properties of operadic categories and formulate some additional axioms and their consequences required for our future constructions concerning the category of graphs and Koszul duality theory.

Conventions. Chosen local terminal objects of an operadic category $\mathtt{O}$ will be denoted by U with various decorations such as U', U'', etc. The notation U_c will mean the chosen local terminal object of a component $c \in \pi_0(\mathtt{O})$. We will sometimes call these chosen local terminal objects the *trivial* ones. Quasibijections will be indicated by $\sim$, isomorphisms by $\cong$; a preferred notation for both of them will be something resembling permutations, like σ, ω, π, etc.

A *quasibijection* is a morphism $f : T \to S$ in $\mathtt{O}$ such that, for each $i \in |S|$, we have $f^{-1}(i) = U_{d_i}$ for some $d_i \in \pi_0(\mathtt{O})$. To avoid any possible confusion, we assert that this definition implies that a map between objects with the empty set of fibers is also a quasibijection; in [6] such morphisms were called *trivial*. Note that an isomorphism is not necessary a quasibijection and that a quasibijection is not necessary an isomorphism.

We will denote by $\mathtt{O}_{\mathtt{qb}} \subset \mathtt{O}$ the subcategory of quasibijections, and by $\mathtt{O}_{\mathtt{ord}} \subset \mathtt{O}$ the subcategory of morphisms for which $|f|$ is order preserving. Notice that $\mathtt{O}_{\mathtt{ord}}$ unlike $\mathtt{O}_{\mathtt{qb}}$ has a natural structure of an operadic category, cf. Example 1.7.

The following Lemma 2.1(iii) and Corollary 2.3(i) were also proved by Lack as [23, Lemma 8.2].

Lemma 2.1. *Consider the commutative diagram in an operadic category*

$$\begin{tikzcd} S \arrow[d, "f'"'] \arrow[dr, "f''"] & \\ T' \arrow[r, "\sigma"] & T''. \end{tikzcd}$$

Let $j \in |T''|$ and $|\sigma|^{-1}(j) = \{i\}$ for some $i \in |T'|$. Then

(i) The unique fiber of the induced map $f'_j : f''^{-1}(j) \to \sigma^{-1}(j)$ equals $f'^{-1}(i)$.

(ii) If $\sigma^{-1}(j)$ is trivial, in particular if σ is a quasibijection, then

$$f'^{-1}(i) = f''^{-1}(j).$$

(iii) If both σ and f'' are quasibijections then f' is a quasibijection.

Proof. By Axiom (iv) of an operadic category, $f'^{-1}(i) = f_j'^{-1}(i)$ which readily gives the first part of the lemma. If $\sigma^{-1}(j)$ is trivial, then the fiber of f'_j equals $f''^{-1}(j)$ by Axiom (iii). This proves the second and third part of the lemma. □

Lemma 2.2. *Consider the commutative diagram in an operadic category*

$$\begin{tikzcd} S' \arrow[rr, "\pi", "\sim"'] \arrow[dr, "f'"'] & & S'' \arrow[dl, "f''"] \\ & T & \end{tikzcd}$$

where π is a quasibijection. Then all $\pi_i : f'^{-1}(i) \to f''^{-1}(i)$, $i \in |T|$, are quasibijections too.

Proof. Immediate from Axiom (iv). □

Corollary 2.3. *In any operadic category $\mathtt{O}$,*

(i) quasibijections are closed under composition.

(ii) If a quasibijection is an isomorphism, then its inverse is also a quasibijection.

Proof. The first statement follows from Lemma 2.2 when f'' is a quasibijection. Indeed, in this case we have a quasibijection $! = \pi_i : f'^{-1}(i) \to U_{c_i}$, for each $i \in |T|$, but the fiber of such a morphism must be equal to $f'^{-1}(i)$.

The second statement follows readily from part (iii) of Lemma 2.1. $\square$

Lemma 2.4. *Consider the commutative diagram in an operadic category*

$$\begin{tikzcd} S' \arrow[r, "\pi"] \arrow[d, "f'"'] \arrow[dr, "f"] & S'' \arrow[d, "f''"] \\ T' \arrow[r, "\sigma"] & T''. \end{tikzcd} \tag{6}$$

Let $j \in |T''|$ and $|\sigma|^{-1}(j) = \{i\}$ for some $i \in |T'|$. Diagram (6) determines:

(i) the map $f'_j : f^{-1}(j) \to \sigma^{-1}(j)$ whose unique fiber equals $f'^{-1}(i)$, and

(ii) the induced map $\pi_j : f^{-1}(j) \to f''^{-1}(j)$.

If $\sigma^{-1}(j)$ is trivial, in particular if σ is a quasibijection, then π induces a map

$$\pi_{(i,j)} : f'^{-1}(i) \to f''^{-1}(j) \tag{7}$$

which is a quasibijection if π is.

Proof. The first part immediately follows from Lemma 2.1 and Axiom (iii). Under the assumption of the second part, one applies Lemma 2.1(ii) to get an equality $f'^{-1}(i) = f^{-1}(j)$. Then $\pi_{(i,j)}$ is defined as the composite

$$\pi_{(i,j)} : f'^{-1}(i) = f^{-1}(j) \xrightarrow{\pi_j} f''^{-1}(j).$$

The rest follows from Lemma 2.2. $\square$

Thus, in the situation of Lemma 2.4 with σ a quasibijection, one has the *derived sequence*

$$\left\{\pi_{(i,j)} : f'^{-1}(i) \to f''^{-1}(j), j = |\sigma|(i)\right\}_{i\in|T'|} \tag{8}$$

consisting of quasibijections if π is a quasibijection.

Central constructions of this work will require the following:

Blow-up axiom. *Let $\mathtt{O}$ be an operadic category. Consider the corner*

$$\begin{tikzcd} S' \arrow[d, "f'"'] & \\ T' \arrow[r, "\sigma", "\sim"'] & T'' \end{tikzcd} \tag{9}$$

in which σ is a quasibijection and $f' \in \mathtt{O}_{\mathtt{ord}}$. Assume we are given objects F''_j, $j \in |T''|$ together with a collection of maps

$$\{\pi_{(i,j)} : f'^{-1}(i) \to F''_j,\ j = |\sigma|(i)\}_{i\in|T'|}. \tag{10}$$

Then the corner (9) can be completed uniquely into the commutative square

$$\begin{tikzcd} S' \arrow[r, "\pi"] \arrow[d, "f'"'] & S'' \arrow[d, "f''"] \\ T' \arrow[r, "\sigma", "\sim"'] & T'' \end{tikzcd} \tag{11}$$

in which $f'' \in \mathtt{O}_{\mathtt{ord}}$, $f''^{-1}(j) = F''_j$ for $j \in |T''|$, and such that the derived sequence (8) induced by π coincides with (10).

The requirement that $f', f'' \in \mathtt{O_{ord}}$ is crucial, otherwise the factorization would not be unique even in "simple" operadic categories such as $\mathtt{Fin}$. It will sometimes suffice to assume the blow-up for $\sigma = 1\!\!1$ only, i.e. to assume

Weak blow-up axiom. *For any $f' : S' \to T$ in $\mathtt{O_{ord}}$ and morphisms $\pi_i : f'^{-1}(i) \to F''_i$ in $\mathtt{O}$, $i \in |T|$, there exists a unique factorization of f'*

$$\begin{tikzcd} S' \arrow[rr, "\omega"] \arrow[dr, "f'"'] & & S'' \arrow[dl, "f''"] \\ & T & \end{tikzcd}$$

such that $f'' \in \mathtt{O_{ord}}$ and $\omega_i = \pi_i$ for all $i \in |T|$.

Notice that $\omega \in \mathtt{O_{ord}}$ (resp. $\omega \in \mathtt{O_{qb}}$) if and only if $\pi_i \in \mathtt{O_{ord}}$ (resp. $\pi_i \in \mathtt{O_{qb}}$) for all $i \in |T|$.

Remark 2.5. If we require the weak blow-up axiom only for order-preserving π_i then it simply means that the fiber functor

$$\mathtt{O_{ord}}/T \to \mathtt{O_{ord}}^{|T|}$$

is a discrete opfibration. Such a condition for an operadic category $\mathtt{O}$ (not only for its subcategory $\mathtt{O_{ord}}$) is closely related to Lack's condition [23, Proposition 9.8] which ensures that the natural tensor product of $\mathtt{O}$-collections, which is only skew associative in general, is genuinely associative. In fact, as we will show elsewhere, under some restrictions natural in our context, the weak blow-up axiom implies Lack's condition (see also Remarks 13–14 of [17] for other important connections).

Corollary 2.6. *If the weak blow-up axiom is satisfied in $\mathtt{O}$, then*

$$\mathtt{O_{qb}} \cap \mathtt{O_{ord}} = \mathtt{O_{disc}},$$

the discrete category with the same objects as $\mathtt{O}$. In particular, the only quasibijections in $\mathtt{O_{ord}}$ are the identities.

Proof. It is clear that each identity belongs to $\mathtt{O_{qb}} \cap \mathtt{O_{ord}}$. On the other hand, assume that $\phi : S \to T \in \mathtt{O_{qb}} \cap \mathtt{O_{ord}}$. Since it is a quasibijection, all its fibers are trivial, $\phi^{-1}(i) = U_i$ for $i \in |T|$. Consider now the two factorizations of ϕ,

$$\begin{tikzcd} & S \arrow[d, "\phi"] \\ S \arrow[ur, equal, "1\!\!1_S"] \arrow[r, "\phi"'] & T \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\phi"'] & T \arrow[dl, equal, "1\!\!1_T"] \\ T\,. & \end{tikzcd} \tag{12}$$

In the left triangle we have $(1\!\!1_T)_i : U_i = \phi^{-1}(i) \to \phi^{-1}(i) = U_i$, for $i \in |T|$, and therefore $(1\!\!1_T)_i = 1\!\!1_{U_i}$ by the terminality of U_i. Let us turn our attention to the right triangle.

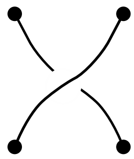
By Axiom (ii) of an operadic category, all fibers of an identity are trivial, thus

$$(\phi)_i : U_i = \phi^{-1}(i) \to {1\!\!1_T}^{-1}(i) = U_c$$

for some chosen local terminal U_c. Since any morphism between trivial objects is an identity we see that both factorizations in (12) are determined by the collection $1\!\!1_{U_i} : \phi^{-1}(i) \to U_i$, $i \in |T|$, so by the uniqueness in the blow-up axiom, they are the same. □

Example 2.7. Corollary 2.6 shows the power of the blow-up axiom and illustrates how it determines the nature of an operadic category. While it is satisfied in operadic categories underlying "classical" examples of operads, it is violated e.g. in the category of vines recalled in Example 1.4, whose operads are braided operads, or in Batanin's category of n-trees, whose operads are $(n-1)$-terminal (but not pruned) globular n-operads [4, Section 4].

Let us look at vines first. For the automorphism $s \in \mathtt{Vines}(\bar{2}, \bar{2})$ represented by

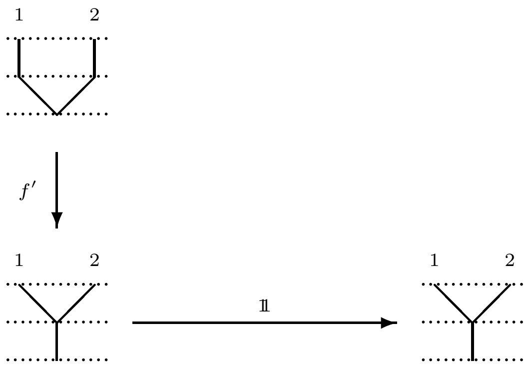


and an integer k, one has the commutative diagram

$$
\begin{tikzcd}
\bar{2} \arrow[r, "s^{2k}"] \arrow[d, "\mathbb{1}"'] & \bar{2} \arrow[d, "s^{-2k}"] \\
\bar{2} \arrow[r, "\mathbb{1}"] & \bar{2}\,.
\end{tikzcd}
\tag{13}
$$

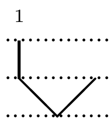
The sequences of the fibers of both vertical maps are the same for each k, namely $\{\bar{1}, \bar{1}\}$, and the derived sequence (8) consists of the identities, $\pi_{(1,1)} = \pi_{(2,2)} = \mathbb{1}$. Hence the upper horizontal map in (13) is *not* uniquely determined by its associated derived sequence, which violates the requirement that the map π in diagram (11) is unique. The map s^{2k} is a nontrivial quasibijection in $\mathtt{Vines}_{\mathtt{Ord}}$.

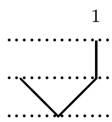
In some categories the extension of the corner (9) into (11) may not exist. We illustrate it on the category of Batanin's n-trees, cf. [4, Section 4] for necessary definitions and notation. The fibers of the map f' of 2-trees in



(14)

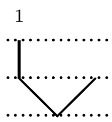
are

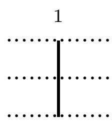
$f'^{-1}(1) =$ 

and $f'^{-1}(2) =$ 

.

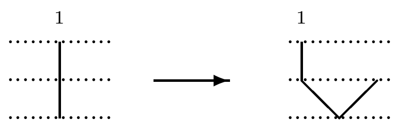
Take

$F_1'' :=$ 

and $F_2'' :=$ 

.

Defining the maps in (10) as $\pi_{(1,1)} = \mathbb{1}$ and taking $\pi_{(2,2)}$ to be the obvious unique morphism, it is easy to check that the corner in (14) cannot be completed to (11). The unique map



provides an example of a quasibijection in $2\text{-}\mathtt{Trees}_{\mathtt{Ord}}$ which is not the identity, not even an isomorphism.

Definition 2.8. An operadic category $\mathtt{O}$ is *factorizable* if each morphism $f \in \mathtt{O}$ decomposes, not necessarily uniquely, as $\phi\sigma$ for some $\phi \in \mathtt{O}_{\mathtt{ord}}$ and $\sigma \in \mathtt{O}_{\mathtt{qb}}$ or, symbolically, $\mathtt{O} = \mathtt{O}_{\mathtt{ord}}\mathtt{O}_{\mathtt{qb}}$.

Definition 2.9. An operadic category $\mathtt{O}$ is *strongly factorizable* if each morphism $f : T \to S$ decomposes *uniquely* as $\phi\sigma$ for some $\phi \in \mathtt{O}_{\mathtt{ord}}$ and $\sigma \in \mathtt{O}_{\mathtt{qb}}$ such that the induced map between the fibers

$$\sigma_i : f^{-1}(i) \to \phi^{-1}(i)$$

is an order-preserving quasibijection for each $i \in |S|$.

The first part of the following lemma has the same conclusion as Corollary 2.6 but the assumptions are different.

Lemma 2.10. *In a strongly factorizable operadic category, any order-preserving quasibijection is an identity. In particular, the morphisms on fibers induced by the quasibijection σ in the unique factorization $f = \phi\sigma$ are always the identities.*

Proof. Let $\sigma : T \to S$ be an order-preserving quasibijection. Then there are two factorizations of the unique morphism $! : T \to U_c$. Namely $T \xrightarrow{\mathbb{1}} T \to U_c$ and $T \xrightarrow{\sigma} S \to U_c$. Since such a factorization must be unique we have $\sigma = \mathbb{1}$. ☐

Lemma 2.11. *Assume that in $\mathtt{O}$ all quasibijections are invertible, $\mathtt{O}$ is factorizable and satisfies the weak blow-up axiom. Then $\mathtt{O}$ is strongly factorizable and satisfies the blow-up axiom. Schematically*

$$\mathsf{QBI} \ \& \ \mathsf{Fac} \ \& \ \mathsf{WBU} \Longrightarrow \mathsf{BU} \ \& \ \mathsf{SFac},$$

with the obvious meaning of the abbreviations.

Proof. Let $f : T \to S$ be a morphism in $\mathtt{O}$. We factorize it into a quasibijection ω followed by an order-preserving $\eta : T' \to S$ as in the left upper triangle of

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\omega", "\sim"'] & S \\ T' \arrow[ur, "\eta"] \arrow[r, "\pi", "\sim"'] & Q . \arrow[u, "\phi"'] \end{tikzcd} \tag{15}$$

By virtue of Lemma 2.2 for $i \in |S|$, let $\pi_i : \eta^{-1}(i) \to f^{-1}(i)$ be the quasibijection inverse to $\omega_i : f^{-1}(i) \to \eta^{-1}(i)$. Using the weak blow-up axiom we uniquely factorize η into $\phi\pi$ such that π on fibers induces the morphisms π_i, $i \in |S|$, see the lower right triangle of (15). Notice that π is a quasibijection as well. We thus have a factorization of f into a quasibijection $\sigma := \pi\omega$ followed by $\phi \in \mathtt{O}_{\mathtt{ord}}$. By functoriality of the fiber functor, σ induces the identities $\sigma_i = \pi_i\omega_i = \mathbb{1}$ of the fibers.

Suppose we have two such factorizations of f, namely

$$\begin{tikzcd} & Q' \arrow[dr, "\phi'"] \arrow[dd, dashed, "p", "\sim"'] & \\ T \arrow[ur, "\sigma'", "\sim"'] \arrow[dr, "\sigma''", "\sim"'] & & S . \\ & Q'' \arrow[ur, "\phi''"'] & \end{tikzcd}$$

By our assumptions quasibijections are invertible, hence we have a unique quasibijection $p : Q' \to Q''$ which induces identities of the fibers over S by Corollary 2.6. It follows from the uniqueness part of the weak blow-up axiom that $p = \mathbb{1}$. So the decomposition $f = \phi\sigma$ is unique, thus $\mathtt{O}$ is strongly factorizable.

It remains to prove the general version of the blow-up axiom. Let

$$\begin{tikzcd} S' \arrow[d, "f'"'] & \\ T' \arrow[r, "\sigma", "\sim"'] & T'' \end{tikzcd} \tag{16}$$

be the corner for the blow-up axiom as in (9) and $\pi_{(i,j)} : f'^{-1}(i) \to F''_j$, $j = |\sigma|(i)$ for $i \in |T'|$, a collection of maps. By the weak blow-up axiom we have a unique factorization $S' \xrightarrow{\gamma} S'' \xrightarrow{g} T'$ of f' as in

$$\begin{tikzcd} S' \arrow[r, "\gamma"] \arrow[d, "f'"'] & S'' \arrow[r, "\varpi", "\sim"'] \arrow[dl, "g"] & Q \arrow[dl, "\eta"] \\ T' \arrow[r, "\sigma", "\sim"'] & T'' & \end{tikzcd}$$

such that $\gamma_i = \pi_{(i,j)}$ for $i \in |T'|$. We then apply the strong factorization axiom to σg and get a factorization $S'' \xrightarrow{\varpi} Q \xrightarrow{\eta} T''$ where $\varpi \in \mathtt{O}_{\mathtt{qb}}$ and $\eta \in \mathtt{O}_{\mathtt{ord}}$.

Since ϖ is a quasibijection, the derived sequence $\varpi_{(i,j)}$ consists of order-preserving quasibijections. We already established that $\mathtt{O}$ is strongly factorizable, thus each $\varpi_{(i,j)}$ is the identity by Lemma 2.10, therefore

$$(\varpi\gamma)_{(i,j)} = \varpi_{(i,j)}\gamma_i = \gamma_i = \pi_{(i,j)}, \text{ for each } i \in |T'|.$$

We conclude that

$$\begin{tikzcd} S' \arrow[r, "\pi"] \arrow[d, "f'"'] & Q \arrow[d, "\eta"] \\ T' \arrow[r, "\sigma", "\sim"'] & T'' \end{tikzcd} \tag{17}$$

with $\pi := \varpi\gamma$ is a completion of the corner (16) required by the blow-up axiom.

To prove that the completion (17) is unique, assume that $S' \xrightarrow{\pi'} Q' \xrightarrow{\eta'} T''$ is another completion of (16), and let

$$\begin{tikzcd} & P \arrow[dr, "\lambda"] & \\ S' \arrow[ur, "\rho"] \arrow[rr, "\pi"] & & Q \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} & P' \arrow[dr, "\lambda'"] & \\ S' \arrow[ur, "\rho'"] \arrow[rr, "\pi'"] & & Q' \end{tikzcd}$$

be the unique factorizations of π resp. π' to a quasibijection inducing identities on fibers followed by an order-preserving map. The existence of factorizations of this type is guaranteed by the already proven SFac combined with QBI assumed in Corollary 2.6. Consider the commutative diagrams

$$\begin{tikzcd} & P \arrow[dr, "\lambda"] \arrow[dd, "\eta\lambda"] & \\ S' \arrow[ur, "\rho"] \arrow[rr, "\pi"] \arrow[dr, "a"'] & & Q \arrow[dl, "\eta"] \\ & T'' & \end{tikzcd}, \quad \begin{tikzcd} & P' \arrow[dr, "\lambda'"] \arrow[dd, "\eta'\lambda'"] & \\ S' \arrow[ur, "\rho'"] \arrow[rr, "\pi'"] \arrow[dr, "a"'] & & Q' \arrow[dl, "\eta'"] \\ & T'' & \end{tikzcd}, \tag{18}$$

where $a := \eta\pi = \eta'\pi'$, and take $j \in |Q| = |Q'|$, $i := |\eta|(j) = |\eta'|(j)$. By Axiom (iii) of an operadic category the diagram

$$\begin{tikzcd} a^{-1}(i) \arrow[rr, "\rho_i"] \arrow[dr, "\pi_i"'] & & (\eta\lambda)^{-1}(i) \arrow[dl, "\lambda_i"] \\ & \eta^{-1}(i) & \end{tikzcd} \tag{19a}$$

related to the left diagram in (18) commutes and induces a morphism $(\rho_i)_j : \pi_i^{-1}(j) \to \lambda_i^{-1}(j)$ between fibers. By Axiom (iv) we have

$$\lambda^{-1}(j) = \lambda_i^{-1}(j) \text{ and } \pi^{-1}(j) = \pi_i^{-1}(j).$$

Axiom (v) thus gives $\rho_j = (\rho_i)_j$, but $\rho_j = \mathbb{1}$ by construction, hence $(\rho_i)_j = \mathbb{1}$.

Since λ and η are order preserving, the morphism λ_i in (19a) is also order preserving. We see that (19a) represents the unique factorization of the morphism π_i to a quasibijection inducing identities on fibers followed by an order-preserving map. In exactly the same manner we obtain a unique factorization

$$\begin{tikzcd} a^{-1}(i) \arrow[rr, "\rho'_i"] \arrow[dr, "\pi'_i"'] & & (\eta'\lambda')^{-1}(i) \arrow[dl, "\lambda'_i"] \\ & \eta'^{-1}(i) & \end{tikzcd} \tag{19b}$$

related to the right diagram in (18).

Notice that, by assumption, the induced morphism $\pi_i : a^{-1}(i) \to \eta^{-1}(i)$ in (19a) is equal to $\pi'_i : a^{-1}(i) \to \eta'^{-1}(i)$ in (19b). Hence $\rho_i = \rho'_i$, and thus the quasibijection $q := \rho\rho'^{-1}$ in the diagram

$$\begin{tikzcd} & P' \arrow[dd, dashed, "q", "\sim"'] \arrow[dr, "\eta'\lambda'"] & \\ S' \arrow[ur, "\rho'", "\sim"'] \arrow[dr, "\rho"', "\sim"] & & T'' \\ & P \arrow[ur, "\eta\lambda"'] & \end{tikzcd}$$

induces the identity maps between the fibers of $\eta\lambda$ and $\eta'\lambda'$. The uniqueness part of the weak blow-up axiom tells us that $q = \mathbb{1}$, $P = P'$ and $\eta\lambda = \eta'\lambda'$. Using the above facts, we can modify the right diagram in (18) so that we will now be comparing the diagrams

$$\begin{tikzcd} & P \arrow[dr, "\lambda"] \arrow[dd, "\eta\lambda"] & \\ S' \arrow[ur, "\rho"] \arrow[rr, "\pi"] \arrow[dr, "a"'] & & Q \arrow[dl, "\eta"] \\ & T'' & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & P \arrow[dr, "\lambda'"] \arrow[dd, "\eta\lambda"] & \\ S' \arrow[ur, "\rho"] \arrow[rr, "\pi'"] \arrow[dr, "a"'] & & Q' \arrow[dl, "\eta'"] \\ & T'' & \end{tikzcd}.$$

By assumption, the morphisms between fibers of a and η, resp. a and η', induced by π, resp. π', coincide. Since ρ is invertible by QBI, the morphisms induced by $\lambda = \pi\rho^{-1}$, resp. $\lambda' = \pi'\rho^{-1}$, between the fibers of $\eta\lambda$ and η, resp. $\eta\lambda$ and η', coincide as well, thanks to the functoriality of the fiber functors. Using the uniqueness in WBU we conclude that $\lambda = \lambda'$ and $\eta = \eta'$, thus finally $\pi = \pi'$ and $\eta = \eta'$ as required. Notice that Axiom (v) of an operadic category played a crucial role in the second part of the proof. □

Lemma 2.12. *Any isomorphism in an operadic category has local terminal objects as its fibers. Conversely, in a factorizable operadic category in which all quasibijections are isomorphisms and the weak blow-up axiom is fulfilled, a morphism whose fibers are local terminals is an isomorphism.*

Proof. Let $\phi : S \to T$ be an isomorphism with inverse ψ. Consider the commutative diagram over T:

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[dr, "\phi"'] & T \arrow[r, "\psi"] \arrow[d, "\mathbb{1}"] & S \arrow[dl, "\phi"] \\ & T. & \end{tikzcd}$$

By functoriality of the fiber functor this diagram induces isomorphisms from the fibers of ϕ to the fibers of the identity morphism of T. Therefore the fibers of ϕ are isomorphic to trivial objects, so they are all local terminal.

Conversely, suppose an operadic category $\mathtt{O}$ is factorizable with all quasibijections isomorphisms, and suppose that all fibers of $\phi : A \to T$ are local terminals. By assumption, one can factorize ϕ as a quasibijection σ followed by $\xi \in \mathtt{O}_{\mathtt{ord}}$. The quasibijection σ induces quasibijections, hence isomorphisms, between the fibers of ϕ and ξ. So it will be enough to show that any $\xi : R \to S$ in $\mathtt{O}_{\mathtt{ord}}$ whose fibers are local terminals is an isomorphism.

Let F_i denote the fiber of ξ over $i \in |S|$. Since each F_i is local terminal, we have by assumption the unique isomorphism $\xi_i : F_i \to U_{c_i}$ for each i and some $c_i \in \pi_0(\mathtt{O})$, and its inverse $\eta_i : U_{c_i} \to F_i$. By the weak blow-up axiom there exists a unique factorization of $\mathbb{1} : S \to S$ as $S \xrightarrow{a} Q \xrightarrow{b} S$ such that a induces the morphisms η_i on the fibers. The following diagram

$$\begin{tikzcd} R \arrow[r, "\xi"] \arrow[dr, "\xi"'] & S \arrow[r, "a"] \arrow[d, "\mathbb{1}"] & Q \arrow[dl, "b"] \\ & S & \end{tikzcd}$$

in $\mathtt{O}_{\mathtt{ord}}$ commutes and by functoriality it induces the identity morphisms between the fibers of ξ and b. By the uniqueness part of the weak blow-up axiom $Q = R$, we have $b = \xi$ and $\xi a = \mathbb{1}_R$. Repeating the same argument we find also that $a\xi = \mathbb{1}_S$, hence ξ is an isomorphism. □

Lemma 2.13. *In a factorizable operadic category* $\mathtt{O}$ *in which all quasibijections are isomorphisms and the weak blow-up axiom is fulfilled, each* $f \in \mathtt{O}$ *decomposes as* $\psi\omega$*, where* ω *is an isomorphism,* ψ *is order preserving and all local terminal fibers of* ψ *are trivial.*

Proof. Decompose f into $A \xrightarrow{\sigma} X \xrightarrow{\phi} B$ with σ a quasibijection and $\phi \in \mathtt{O}_{\mathtt{ord}}$ using the factorizability in $\mathtt{O}$. By the weak blow-up axiom, one has the diagram

$$\begin{tikzcd} X \arrow[rr, "\tilde{\sigma}"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & B & \end{tikzcd} \tag{20}$$

in which ψ has the same non-terminal fibers as ϕ and all its terminal fibers are trivial, and $\tilde{\sigma}$ induces the identity maps between non-terminal fibers. By Axiom (iv) of operadic categories a fiber of $\tilde{\sigma}$ is either equal to a fiber of the identity or to a fiber of a morphism $! : t \to U_c$ for some $c \in \pi_0(\mathtt{O})$ where t is a local terminal. Thus all fibers of $\tilde{\sigma}$ are local terminals, so it is an isomorphism by Lemma 2.12. The desired factorization of f is then given by ψ in (20) and $\omega := \tilde{\sigma}\sigma$. ∎

In the rest of the paper, the notation $F \blacktriangleright T \xrightarrow{\phi} t$ or $F \blacktriangleright T \to t$ when ϕ is understood will express that $\phi : T \to t$ is the unique map to a local terminal object t and that F is the fiber of ϕ. It follows from Axiom (i) of an operadic category that each local terminal object has cardinality 1, so F is unique and is uniquely determined by t.

Definition 2.14. The *unique fiber condition* for an operadic category $\mathtt{O}$ requires that, if the fiber of the unique morphism $\phi : T \to t$ to a local terminal object t is T, then t is a chosen local terminal object U_c for some $c \in \pi_0(\mathtt{O})$. In other words, the only situation when $T \blacktriangleright T \to t$ is when t is trivial.

Lemma 2.15. *Let* $F \blacktriangleright T \xrightarrow{\phi'} t'$ *and* $F \blacktriangleright T \xrightarrow{\phi''} t''$ *be morphisms to local terminal objects with the same fiber* F*. If the weak blow-up and unique fiber conditions are satisfied, then* $\phi' = \phi''$*.*

Proof. Consider the commutative triangle

$$\begin{tikzcd} T \arrow[rr, "\phi''"] \arrow[dr, "\phi'"'] & & t'' \arrow[dl, "\xi"] \\ & t' & \end{tikzcd}$$

in which ξ is the unique map between the local terminal objects. We have the induced morphism of fibers $\phi''_1 : F \to \xi^{-1}(1)$. By Axiom (iv) of the operadic categories the fiber of this morphism is F. As $\xi^{-1}(1)$ is local terminal by Lemma 2.12, the unique fiber condition implies that $\phi''_1 = ! : F \to U_c$ is the unique map to a chosen local terminal object. This means that the fiber $\xi^{-1}(1)$ is U_c, so ξ is a quasibijection. By Corollary 2.6, ξ must be the identity. ∎

Definition 2.16. An operadic category $\mathtt{O}$ is *rigid* if, given $\phi \in \mathtt{O}_{\mathtt{ord}}$, the only isomorphism σ that makes

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d, "\phi"'] & S \arrow[d, "\phi"] \\ T \arrow[r, "\sigma", "\cong"'] & T \end{tikzcd} \tag{21}$$

commutative is the identity $1\!\!1 : T = T$.

Example 2.17. The category $\mathtt{Fin}$ is not rigid, but its subcategory $\mathtt{Fin}_{\mathrm{semi}}$ of nonempty finite sets and their surjections is.

Definition 2.18. An operadic category $\mathtt{O}$ is *constant-free* if $|f|$ is surjective for each $f \in \mathtt{O}$. Equivalently, $\mathtt{O}$ is constant-free if the cardinality functor $\mathtt{O} \to \mathtt{Fin}$ factorizes through the operadic category $\mathtt{Fin}_{\mathrm{semi}}$.

Lemma 2.19. *If a constant-free operadic category satisfies the weak blow-up and the unique fiber conditions, then it is rigid. Schematically*

$$\mathsf{UFib}\ \&\ \mathsf{WBU} \Longrightarrow \mathsf{Rig}.$$

Proof. Since the category of finite sets and surjections is obviously rigid, one has $|\sigma| = \mathbb{1}$ for σ in (21). For each $i \in |T|$ we have the induced morphism of the fibers

$$\phi_i : \phi^{-1}(i) \to \sigma^{-1}(i)$$

whose unique fiber is $\phi^{-1}(i)$ by Axiom (iv) of an operadic category. The fiber $\sigma^{-1}(i)$ is local terminal by Lemma 2.12, thus, by the unique fiber condition, $\sigma^{-1}(i)$ is trivial, so σ is a quasi-bijection. Hence, it must be the identity by Corollary 2.6. $\square$

3 The operadic category of graphs

In this section we introduce the operadic category $\mathtt{Gr}$ of ordered graphs. The adjective "ordered" indicates that the sets of flags, vertices and legs of graphs in $\mathtt{Gr}$ have prescribed total orders. The category $\mathtt{Gr}$ and its modifications will play the fundamental rôle in the second paper of the series [7]. We will prove that it is a constant-free strongly factorizable operadic category satisfying the blow-up axiom in which all quasibijections are invertible. Moreover, $\mathtt{Gr}$ is strictly graded (see Definitions 3.22 and 3.24) by the number of edges. We also show that $\mathtt{Gr}$ satisfies the unique fiber condition and is rigid. We start by a more structured version of the standard concept of graphs as recalled e.g. in [33, Definition II.5.23].

Definition 3.1. A *preordered graph* Γ is a pair (g, σ) consisting of an order-preserving map

$$g : F \to V,\ V \neq \emptyset$$

in the category $\mathtt{Fin}$ together with an involution σ on F.

Notice that we do not require the geometric realization [33, Section II.5.3] of preordered graphs to be connected. Elements of $\mathrm{Flg}(\Gamma) := F$ are the *flags* (also called *half-edges*) of Γ and elements of $\mathrm{Ver}(\Gamma) := V$ are its *vertices*. The elements of the set $\mathrm{Leg}(\Gamma)$ of fixed points of σ are called the *legs* of Γ while nontrivial orbits of σ are its *edges*. The *endpoints* of an edge $e = \{h_1, h_2\}$ are $g(h_1)$ and $g(h_2)$.

For any $v \in V$, the set $g^{-1}(v)$ of flags adjacent to v inherits a linear order from F which we call the *local order* at v. We may thus equivalently define a preordered graph as a map $g : F \to V$ from a finite set F into a linearly ordered set V with the additional data consisting of linear orders of each $g^{-1}(v)$, $v \in V$. The lexicographic order combining the order of V with the local orders makes F a finite ordinal, and the two definitions coincide.

We will often use a short notation (F, V) or (F, g, V) for a preordered graph (g, σ) if we want to specify its set of vertices and flags only. We hope it will not lead to any confusion.

A *morphism* of preordered graphs $\Phi : \Gamma \to \Gamma'$ is a pair (ψ, ϕ) of morphisms of finite sets such that the diagram

$$\begin{tikzcd} F \arrow[d, "g"'] & F' \arrow[l, hook', "\psi"'] \arrow[d, "g'"] \\ V \arrow[r, two heads, "\phi"] & V' \end{tikzcd} \tag{22}$$

commutes. We moreover require ϕ to be a surjection and require ψ to be equivariant with respect to the involutions and induce a bijection on fixed points. Thus ψ injectively maps flags to flags and bijectively legs to legs. The pair (ψ, ϕ) must satisfy the following condition: If $\phi(i) \neq \phi(j)$ and e is an edge with endpoints i and j then there exists an edge e' in Γ' with endpoints $\phi(i)$ and $\phi(j)$ such that $e = \psi(e')$. Notice that we denote by the same symbol both the map of flags and the obvious induced map of edges. Preordered graphs and their morphisms form a category *of preordered graphs* $\mathtt{prGr}$.

Remark 3.2. Our definition of morphism of graphs goes back to Getzler–Kapranov [20] and Borisov–Manin [14]. It is simultaneously more structured than the former one since we want to take orders of flags, legs and vertices into account, and less complicated (but still more structured) that the latter. A detailed discussion of Borisov–Manin's definition can be found in [22].

The *fiber* $\Phi^{-1}(i)$ of a map $\Phi = (\psi, \phi) : \Gamma \to \Gamma'$ in (22) over $i \in V'$ is a preordered graph whose set of vertices is $\phi^{-1}(i)$ and whose set of flags is $(\phi g)^{-1}(i)$. The involution τ of $\Phi^{-1}(i)$ is defined as

$$\tau(h) := \begin{cases} h & \text{if } h \in Im(\psi) \\ \sigma(h) & \text{if } h \notin Im(\psi), \end{cases}$$

where σ is the involution of Γ. Observe that $h \notin Im(\psi)$ if and only if $\sigma(h) \notin Im(\psi)$.

Definition 3.3. Let $(\psi, \phi) : \Gamma \to \Gamma'$ be a map of preordered graphs.

(i) The map (ψ, ϕ) is called a *local reordering* if $\phi = \mathbb{1}$ and ψ is an isomorphism.

(ii) The map (ψ, ϕ) is called a *local isomorphism* if ϕ is a bijection and ψ restricts to an order preserving isomorphism ${g'}^{-1}(j) \cong g^{-1}(i)$ for each $i \in V'$, $j = \phi(i)$.

(iii) The map (ψ, ϕ) is called a *contraction* if ϕ is order preserving. A contraction will be also called an *order preserving* morphism.

(iv) The map (ψ, ϕ) is called a *pure contraction* if both ψ and ϕ are order preserving.

Lemma 3.4. *Let $\Phi_0 = (\psi_0, \phi_0) : (F, g, V) = \Gamma \to \Gamma_0 = (F_0, g_0, V_0)$ and $\Phi_1 = (\psi_1, \phi_1) : \Gamma \to \Gamma_1 = (F_1, g_1, V_1)$ be two pure contractions such that*

$$\phi := \phi_0 = \phi_1 : V \to V_0 = V_1,$$

that is Φ_0 and Φ_1 are equal on vertices, and

$$\Phi_0^{-1}(i) = \Phi_1^{-1}(i)$$

for $i \in V_0$, that is, Φ_0 and Φ_1 have equal fibers. Then $\Gamma_0 = \Gamma_1$ and $\Phi_0 = \Phi_1$.

Proof. We have to prove that $F_0 = F_1$. For this let $h_0 \in F_0$, $i = g_0(h_0)$ and consider $h = \psi_0(h_0) \in (\phi g)^{-1}(i)$. Since $h \in Im(\psi_0)$ the flag h is a fixed point of the involution τ_0 of the fiber $\Phi_0^{-1}(i)$. Then it is also a fixpoint of the involution τ_1 of the fiber $\Phi_1^{-1}(i)$. Hence, there is a unique $h_1 \in F_1$ such that $h = \psi_1(h_1)$. We then have a map $p : F_0 \to F_1$ which is obviously an order preserving bijection which commutes with g_0 and g_1. So $F_0 = F_1$ and, moreover, $g_0 = g_1$. □

Definition 3.5. Let $\Gamma = (F, g, V)$ be a preordered graph.

(i) *Contraction data* for Γ consists of an order preserving surjection $\phi : V \twoheadrightarrow V'$ and for each $i \in V'$ a σ-free and σ-closed subset $E_i \subset (\phi g)^{-1}(i)$.

(ii) For $i \in V'$, the ith *fiber associated to the contraction data* is the graph Γ_i given as the restriction $F_i := (\phi g)^{-1}(i) \to \phi^{-1}(i) =: V_i$ of g, along the involution which agrees with σ for the flags in E_i and is otherwise trivial.

The following Lemma shows that the contraction data are in one-to-one correspondence with pure contraction morphisms with domain Γ in which E_i, $i \in V'$ play the roles of sets of edges which we contract to a vertex i.

Lemma 3.6. *Given a contraction data (ϕ, E_i) for $\Gamma = (F, g, V)$ there is a preordered graph Γ' together with a contraction $(\psi, \phi) : \Gamma \to \Gamma'$ whose vertex map ϕ is that from the contraction data and whose fibers are the fibers associated to the contraction data. Moreover, there is a unique such graph for which (ψ, ϕ) is a pure contraction.*

Proof. We construct Γ' as the graph whose set of vertices is V' and whose set of flags is $F' := F \setminus \bigcup_{i\in V'} E_i$. The map $g' : F' \to V'$ is the restriction of the composite ϕg, as shown in

$$\begin{tikzcd} F \arrow[d, "g"'] & F' := F \setminus \bigcup_{i\in V'} E_i \arrow[l, hook', "\psi"'] \arrow[d, "g'"] \\ V \arrow[r, two heads, "\phi"] & V' . \end{tikzcd}$$

It is easy to see that (ψ, ϕ) is a pure contraction. Uniqueness follows from Lemma 3.4. □

Despite the fact that preordered graphs do not form an operadic category, the following version of the weak blow-up condition for pure contractions makes sense.

Lemma 3.7. *Let $\Phi = (\psi, \phi) : \Gamma = (F, g, V) \to \Gamma' = (F', g', V')$ be a pure contraction with fibers $\Gamma_i = (F_i, V_i)$, $i \in V'$. Given pure contractions $\Xi_i : \Gamma_i \to \Lambda_i$ for each $i \in V'$, there exists a unique factorization of Φ as a composite of pure contractions*

$$\begin{tikzcd} \Gamma \arrow[rr, "a"] \arrow[dr, "\Phi"'] & & \Lambda \arrow[dl, "b"] \\ & \Gamma' & \end{tikzcd} \tag{23}$$

such that the induced map a_i of the fibers equals Ξ_i, $i \in V'$.

Proof. Assume that the pure contraction Φ is given, as in Lemma 3.6, by an order-preserving map $\phi : V \to V'$ and subsets E_i, $i \in V'$, of edges. Suppose also that the pure contractions Ξ_i are given by order-preserving maps $\phi_i : \phi^{-1}(i) \twoheadrightarrow C_i$, $i \in V'$, and subsets $E_{ij} \subset E_i$ of edges of $\Xi_i^{-1}(j)$ for each $j \in C_i$. We then use Lemma 3.6 again to build Λ with the set of vertices C, and a pure contraction a as follows. As C we take the ordinal sum $\bigcup_{i\in V'_i} C_i$ and

$$\phi_a := \bigcup_{i\in V'_i} \phi_i : V = \bigcup_{i\in V'_i} \phi^{-1}(i) \longrightarrow C. \tag{24}$$

The pure contraction a is then determined by $\phi_a : V \to C$ and the subsets of edges E_{ij}, $j \in C_i$, $i \in V'$. It is easy to check that Γ' is a result of a further pure contraction b associated to the subsets of edges $E_i \setminus \bigcup_j E_{ij}$. The uniqueness of the construction is clear. □

Another version of the weak blow-up condition is described in

Lemma 3.8. *Let $\Phi : \Gamma = (F, g, V) \to \Gamma' = (F', g', V')$ be a pure contraction with fibers Γ_i, $i \in V'$. Given local isomorphisms $\Xi_i : \Gamma_i \to \Lambda_i$ for each $i \in V'$, there exists a unique factorization of Φ as in (23) in which a is a local isomorphism inducing the prescribed maps Ξ_i on the fibers, and b a pure contraction.*

Proof. Let $\Lambda_i = (G_i, C_i)$. We construct Λ as the graph whose set of vertices C equals the ordinal sum $\bigcup_{i\in V'_i} C_i$, and the set F'' of flags the ordinal sum $\bigcup_{i\in V'_i} G_i$. There is an obvious isomorphism ψ_a between the set F'' of flags of Λ and the set F of flags of Γ induced by the local isomorphism between the fibers. We transport the involution of Γ to the flags of Λ along this isomorphism. Then $a := (\psi_a, \phi_a)$ with ϕ_a as in (24) is the requisite local isomorphism. It is easy to check that Γ' is a result of a pure contraction of Λ for which the contraction data consist of the order preserving map $\phi_b : \bigcup_{i\in V'_i} C_i \to V'$, $\phi_b(c) = i$ if $c \in C_i$ together with the collection of subsets $\{\psi_a^{-1}(E_i), i \in V'\}$ where $\{E_i, i \in V'\}$ is the collection of the contraction subsets for Φ. The uniqueness of factorization is clear again. □

The last version of the weak blow-up axiom which we will need is

Lemma 3.9. *Let $\Phi : \Gamma = (F, g, V) \to \Gamma' = (F', g', V')$ be a morphism between preordered graphs with fibers Γ_i, $i \in V'$. Given local reorderings $\Xi_i : \Gamma_i \to \Lambda_i$ for each $i \in V'$, there exists a unique factorization of Φ as in (23) in which a is a local reordering that induces the prescribed maps on the fibers.*

Notice that we did not require Φ to be a pure contraction. When Φ is a pure contraction, $b : \Lambda \to \Gamma'$ need not be pure, but it is a contraction.

Proof of Lemma 3.9. Each vertex j of Γ belongs to a unique fiber of Φ. So the prescribed reorderings of the fibers determine a reordering at each vertex of Γ. We thus construct Λ as the graph with the same vertices as Γ but with the local orders modified according to the above reorderings. The map $a : \Gamma \to \Lambda$ is then the related local reordering map. Since it is an isomorphism, it determines the map $b : \Lambda \to \Gamma'$ uniquely. $\square$

Proposition 3.10. *Any morphism Φ of preordered graphs*

$$\begin{tikzcd} F \arrow[d, "g"'] & F' \arrow[l, hook', "\psi"'] \arrow[d, "g'"] \\ V \arrow[r, two heads, "\phi"] & V' \end{tikzcd}$$

can be factorized as a local isomorphism followed by a pure contraction followed by a local reordering. Symbolically

$$\Phi = Reo\,Cont\,Li. \tag{25}$$

Proof. We first factorize ϕ as a bijection $\pi : V \to V''$ followed by an order-preserving map $\xi : V'' \to V'$ such that π restricts to an order-preserving isomorphism $\phi^{-1}(i) \cong \xi^{-1}(i)$ for each $i \in V'$, cf. the bottom row of

$$\begin{tikzcd} F \arrow[r, "\eta", "\cong"'] \arrow[d, "g"'] & F'' \arrow[d, "g''"'] & F' \arrow[l, hook', "\eta\psi"'] \arrow[d, "g'"'] \arrow[ll, bend right, "\psi"'] \\ V \arrow[r, "\pi", "\cong"'] \arrow[rr, bend right, two heads, "\phi"'] & V'' \arrow[r, two heads, "\xi"] & V' . \end{tikzcd} \tag{26}$$

We then factorize πg into the composite $F \xrightarrow{\eta} F'' \xrightarrow{g''} V''$ where η induces an order-preserving isomorphism $(\pi g)^{-1}(j) \cong (g'')^{-1}(j)$ for each $j \in V''$, cf. the left square in (26). We induce an involution on F'' from F via the isomorphism η. The pair (η^{-1}, π) is the required local isomorphism Li in (25).

The pair $(\eta\psi, \xi)$ in the right square of (26) is a morphism of graphs as well. We factorize $\eta\psi$ as a bijection $\mu : F' \to F'''$ followed by an order-preserving monomorphism $\lambda : F''' \to F''$ as in

$$\begin{tikzcd} F'' \arrow[d, "g''"'] & F''' \arrow[l, hook', "\lambda"'] \arrow[dr, "g'''"] & F' \arrow[l, "\mu"', "\cong"] \arrow[d, "g'"] \arrow[ll, bend right, "\eta\psi"'] \\ V'' \arrow[rr, two heads, "\xi"] & & V' . \end{tikzcd}$$

We finally define $g''' : F''' \to V'$ as $\xi g'' \lambda$. Since $\xi g'' \lambda \mu = g'$, the diagram

$$\begin{tikzcd} F''' \arrow[d, "g'''"'] & F' \arrow[l, "\mu"', "\cong"] \arrow[d, "g'"] \\ V' \arrow[r, "\mathbb{1}"] & V' \end{tikzcd}$$

commutes. It is a reordering morphism playing the rôle of Reo in (25). The pair (λ, ξ), which is clearly a pure contraction, is $Cont$ in (25). $\square$

Corollary 3.11. *Any isomorphism of preordered graphs can be factorized into a local isomorphism followed by a local reordering, symbolically $Iso = Reo\,Li$.*

Proof. The statement follows from Proposition 3.10 combined with the obvious fact that the only pure contractions that are isomorphisms are the identity maps. □

Corollary 3.12. *Any morphism $\Phi = (\psi, \phi)$ such that $\phi : V' \to V''$ is order preserving is a composite of a pure contraction followed by a local reordering.*

Proof. Another consequence of Proposition 3.10. Notice that the decomposition of ϕ in the bottom row of (26) was specified so that if ϕ is order preserving, π must be the identity, thus $\eta = \mathbb{1}$ as well, so Li in (25) is the identity morphism. □

For each natural number $n \geq 0$, let 1_n (the *corolla*) be the graph $\bar{n} \to \bar{1}$ with the trivial involution. The corollas are not local terminal objects in $\mathtt{prGr}$ since there are exactly $n!$ morphisms from any graph Γ with n legs to 1_n. Any such a morphism is completely determined by a linear order of the legs of Γ.

Definition 3.13. The category of *ordered graphs* $\mathtt{Gr}$ is the coproduct of the categories $\mathtt{prGr}/1_n$ for $n \geq 0$.

Lemma 3.14. *The category of ordered* $\mathtt{Gr}$ *is equipped with an operadic category structure.*

Proof. We describe the main ingredients of the operadic category $\mathtt{Gr}$ and leave the verification of the axioms as an exercise. Since objects and morphisms of $\mathtt{Gr}$ are defined using finite sets and their morphism as building blocks, such a verification reduces to the properties of the operadic category $\mathtt{Fin}$.

The cardinality functor assigns to a graph the (linearly ordered) set of its vertices. Because of this we will often identify a vertex i of a graph Γ with its image in the ordinal $|\Gamma|$.

A morphism $\Phi : \Gamma \to \Gamma'$ of ordered graphs, i.e. a diagram

$$\begin{tikzcd}
F \arrow[d, "g"'] & & F' \arrow[ll, hook', "\psi"'] \arrow[d, "g'"] \\
& \bar{n} \arrow[ul, hook'] \arrow[ur, hook] \arrow[d] & \\
V \arrow[rr, "\phi"] \arrow[r, two heads] & \bar{1} & V' \arrow[l, two heads]
\end{tikzcd} \tag{27}$$

induces for each $i \in V'$ a commutative diagram

$$\begin{tikzcd}
(\phi g)^{-1}(i) \arrow[d, "g"'] & (g')^{-1}(i) \arrow[l, hook', "\psi"'] \arrow[d, "g'"] \\
\phi^{-1}(i) \arrow[r, "\phi"] & \bar{1}
\end{tikzcd} \tag{28}$$

in $\mathtt{Fin}$ in which the morphisms g, ϕ, g' and ψ are the restrictions of the corresponding morphisms from (27). We interpret the right vertical morphism as a corolla by imposing the trivial involution on $(g')^{-1}(i)$. Due to the definition of fibers of maps of preordered graphs, the diagram above represents a map of the fiber of Φ over i to a corolla, which makes it an ordered graph. We take it as the definition of the *fiber in* $\mathtt{Gr}$. In other words, the fiber gets a linear order on its legs from the ordinal $(g')^{-1}(i)$. Finally, the chosen local terminal objects in $\mathtt{Gr}$ are $c_n = \mathbb{1} : 1_n \to 1_n$, that is corollas whose global order of legs coincides with the local order at this unique vertex. □

It follows from the commutativity of the upper triangle in (27) that the map ψ preserves the global orders of legs, therefore morphisms of ordered graphs induce order-preserving bijections of the legs of graphs. Thus the category $\mathtt{Gr}_{\mathtt{ord}}$ then consists of morphisms (27) in which, moreover, ϕ is order preserving, that is, the order of vertices is preserved.

A quasibijection $\Phi : \Gamma \to \Gamma'$ in $\mathtt{Gr}$ is a morphism (27) each of whose fibers is the chosen local terminal object c_n for some $n \geq 0$. It is clear that in this case both ϕ and ψ must be bijections and, moreover, the local orders on $(g')^{-1}(i)$ and $(\phi g)^{-1}(i)$ coincide for each $i \in V'$. In other words, quasibijections are local isomorphisms over 1_n. So Γ' as an ordered graph is obtained from Γ by reordering its vertices. We thus have:

Lemma 3.15. *All quasibijections in* $\mathtt{Gr}$ *are invertible.*

Lemma 3.16. *The operadic category* $\mathtt{Gr}$ *is factorizable.*

Proof. Given a morphism $\Gamma \to \Gamma'$ over 1_n we use Proposition 3.10 to factorize it as a local isomorphism $\Gamma \to \Gamma''$ followed by a composite of a pure contraction and a local reordering. This last composite is an order-preserving morphism $\Gamma'' \to \Gamma'$. We have a commutative diagram

$$\begin{tikzcd}
F'' \arrow[d, "\cong"'] & F' \arrow[l, hook'] \\
F & \bar{n} \arrow[l, hook'] \arrow[u, hook] \arrow[ul, dashed, hook']
\end{tikzcd}$$

of flags of the corresponding graphs. All maps in this diagram induce isomorphisms of the sets of legs. Thus there is a unique monomorphism $\bar{n} \hookrightarrow F''$ which makes the diagram commutative and which, moreover, induces an isomorphism of the sets of legs. Therefore the factorization described above is the factorization over 1_n as required. ◻

Lemma 3.17 below involves a local reordering morphism $\Upsilon = (\sigma, \mathbb{1}) : \Gamma'' \to \Gamma'''$ of ordered graphs. Recall that such an Υ induces the identity between the vertices of the graphs Γ'' and Γ''', i.e. their vertices are "the same." The σ part of this morphism amounts to a permutation of flags adjacent to a vertex i of Γ'. This observation is important for the formulation of:

Lemma 3.17. *Consider a commutative diagram*

$$\begin{tikzcd}
\Gamma \arrow[rr, "f"] \arrow[dr, "\Phi=(\psi,\phi)"'] & & \Gamma' \arrow[dl] \arrow[ddl] \\
& \Gamma'' \arrow[d, "\Upsilon=(\sigma,\mathbb{1})"'] & \\
& \Gamma''' &
\end{tikzcd}$$

of ordered graphs in which Υ *is a local reordering. Then the fiber* $\Phi^{-1}(i), i \in |\Gamma''|$, *is obtained from the fiber* $(\Upsilon\Phi)^{-1}(\mathbb{1}(i))$ *by changing the global order of its legs according to the permutation of flags induced by* σ *at the vertex* i.

Analogously the map between the fibers induced by f *over* $i \in |\Gamma''|$ *can be obtained from the map induced by* f *over* $\mathbb{1}(i) \in |\Gamma'''|$ *by a permutation of orders of legs according to the permutation* σ.

Proof. Direct verification. ◻

Lemma 3.18. *The operadic category* $\mathtt{Gr}$ *satisfies the weak blow-up axiom.*

Proof. Let $\Phi : \Gamma \to \Gamma'$ be an order-preserving map with fibers Γ_i, $i \in |\Gamma'|$. Assume we are given a morphism $\Xi_i : \Gamma_i \to \Lambda_i$ for each i.

Let us first ignore the global orders of graphs involved, i.e. work in the category $\mathtt{prGr}$ of preordered graphs. Using Proposition 3.10, we first factorize Φ into a pure contraction c followed by a local reordering ρ as in the bottom of

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[dr, "u"] & B \arrow[r, "\gamma"] \arrow[d, "v"] & C \arrow[d, "b"] \arrow[dl, "w"'] \\
\Gamma \arrow[u, "\alpha"] \arrow[r, "c"] \arrow[rr, bend right, "\Phi"'] & \Gamma'' \arrow[r, "\rho"] & \Gamma'.
\end{tikzcd} \tag{29}$$

Let $\widehat{\Gamma}_i$ be the graph obtained from Γ_i by modifying its global order according to the action of the local reordering ρ as in Lemma 3.17. Notice that $\widehat{\Gamma}_i$ is the fiber of c over $i \in |\Gamma''|$. Let $\widehat{\Lambda}_i$ be the graph Λ_i with the global order modified in the same manner, and $\widehat{\Xi}_i : \widehat{\Gamma}_i \to \widehat{\Lambda}_i$ the induced map. We factorize $\widehat{\Xi}_i$ as a quasibijection followed by a pure contraction and a local reordering, as in

$$\widehat{\Xi}_i : \widehat{\Gamma}_i \xrightarrow{\alpha_i} A_i \xrightarrow{\beta_i} B_i \xrightarrow{\gamma_i} \widehat{\Lambda}_i, \ i \in |\Gamma''|.$$

We then realize these families of maps as the induced maps between fibers step by step using Lemmas 3.7, 3.8 and 3.9 giving rise to preordered graphs A, B, C together with the morphisms α, β and γ in (29). It is clear that diagram

$$\begin{tikzcd} \Gamma \arrow[rr, "a"] \arrow[dr, "\Phi"'] & & C \arrow[dl, "b"] \\ & \Gamma' & \end{tikzcd} \tag{30}$$

with $a := \gamma\beta\alpha$ and $b := \rho w$ commutes. By Lemma 3.17, a induces the requisite maps between the fibers in the category of preordered graphs. Since the forgetful functor $\mathtt{Gr} \to \mathtt{prGr}$ is faithful, the same is true also in the category of ordered graphs.

We must prove that the graph C in (30) thus constructed carries a compatible global order. Since morphisms in $\mathtt{prGr}$ map legs to legs bijectively, the unique dashed arrow in

$$\begin{tikzcd} \mathrm{Leg}(\Gamma) & & \mathrm{Leg}(C) \arrow[ll, "\cong"'] \\ & \bar{n} \arrow[ul, "\cong"] \arrow[ur, dashed] \arrow[d, "\cong"'] & \\ & \mathrm{Leg}(\Gamma') \arrow[uul, "\cong"] \arrow[uur, "\cong"'] & \end{tikzcd}$$

provides the requisite global order of C.

We need to prove that the factorization (30) is unique. Let $\Gamma = (F, V)$, $\Gamma' = (F', V')$, $\Lambda_i = (F_i, V_i)$ for $i \in V'$ and $C = (G, W)$. Since the map $b : C \to \Gamma'$ is order preserving, the set W of vertices of C must be the ordinal sum $\bigcup_{i \in V'} V_i$ of the sets of vertices of the fibers. Likewise, the set of flags G of C equals the ordinal sum $\bigcup_{i \in V'} F_i$. It is not difficult to show that also the involution on G is determined by the involutions on F' and F_i, $i \in V'$. Thus the graph C is uniquely determined by the input data, namely by Γ' and the fibers Λ_i, $i \in V'$.

Let us discuss the uniqueness of the maps in (30). As each vertex of Γ belongs to a unique fiber of Φ, the horizontal arrow in the diagram

$$\begin{tikzcd} V \arrow[rr, two heads] \arrow[dr, two heads] & & W \arrow[dl, two heads] \\ & V' & \end{tikzcd}$$

of the induced maps of vertices is uniquely determined by the maps $\mathrm{Ver}(\Gamma_i) \to \mathrm{Ver}(\Lambda_i)$, $i \in V'$, induced by the prescribed maps Ξ_i of the fibers. Since both down-going maps are order preserving by assumption, the right down-going map is uniquely determined by the remaining two. By a similar argument, the horizontal inclusion in the diagram

$$\begin{tikzcd} F & & G \arrow[ll, hook'] \\ & F' \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

of the induced maps of flags is uniquely determined by the maps $\mathrm{Flg}(\Lambda_i) \to \mathrm{Flg}(\Gamma_i)$, $i \in V'$, induced by the prescribed maps of the fibers, so the right up-going inclusion is unique as well. This finishes the proof. $\square$

Corollary 3.19. *The operadic category* $\mathtt{Gr}$ *satisfies the blow-up axiom.*

Proof. The proof follows from Lemma 2.11 whose assumptions for the operadic category $\mathtt{Gr}$ were verified in Lemmas 3.15, 3.16 and 3.18. $\square$

Lemma 3.20. *The category* $\mathtt{Gr}$ *satisfies the unique fiber condition.*

Proof. Assume that the ordered graph Γ is given by the left diagram below

$$T = \begin{tikzcd} F \arrow[d, "g"'] & \bar{n} \arrow[l, hook', "u"'] \arrow[d] \\ V \arrow[r, two heads] & \bar{1} \end{tikzcd}, \qquad t = \begin{tikzcd} \bar{n} \arrow[d] & \bar{n} \arrow[l, hook', "\alpha"'] \arrow[d] \\ \bar{1} \arrow[r, two heads] & \bar{1} \end{tikzcd}$$

and the local terminal object by the right one. A morphism $\Phi : \Gamma \to t$ in $\mathtt{Gr}$ is characterized by a monomorphism $\psi : \bar{n} \to F$ in the diagram

$$\begin{tikzcd} F \arrow[dd, "g"'] & & \bar{n} \arrow[ll, hook', "\psi"'] \arrow[dd] \\ & \bar{n} \arrow[ul, hook', "u"] \arrow[ur, hook, "\alpha"'] \arrow[d] & \\ & \bar{1} & \\ V \arrow[ur, two heads] \arrow[rr, two heads] & & \bar{1} \arrow[ul, two heads] \end{tikzcd} \tag{31}$$

and by (28) its fiber $\Phi^{-1}(1)$ equals

$$\begin{tikzcd} F \arrow[d, "g"'] & \bar{n} \arrow[l, hook', "\psi"'] \arrow[d] \\ V \arrow[r, two heads] & \bar{1}. \end{tikzcd}$$

Thus $\Phi^{-1}(1) = \Gamma$ if and only if $\psi = u$. On the other hand, the commutativity of the upper triangle in (31) implies that $u = \psi\alpha$. Since ψ is a monomorphism, one sees that $\alpha = 1\!\!1$, thus t is the chosen local terminal object. $\square$

Since the assumptions of Lemma 2.19 are satisfied by the operadic category of ordered graphs, one has:

Corollary 3.21. *The category* $\mathtt{Gr}$ *is rigid.*

In the second paper [7] of the series we introduce the concept of quadraticity of operads over an operadic category $\mathtt{O}$. For this purpose we add the following definitions.

Definition 3.22. A *grading* on an operadic category $\mathtt{O}$ is a map $e : \mathrm{Objects}(\mathtt{O}) \to \mathbb{N}$ of sets with the property that

$$e(T) + e(F_1) + \cdots + e(F_k) = e(S) \tag{32}$$

for each $f : S \to T$ with fibers $F_1, \ldots, F_k$. In this situation we define the *grade* $e(f)$ of f by $e(f) := e(S) - e(T)$.

Example 3.23. Each constant-free operadic category $\mathtt{O}$ bears the *canonical grading* given by $e(T) := |T| - 1$.

Definition 3.24. A graded operadic category $\mathtt{O}$ is *strictly graded* if a morphism $f \in \mathtt{O}$ is an isomorphism if and only if $e(f) = 0$.

We are grateful to our anonymous referee for the idea and the proof of the following

Lemma 3.25. *Let* $\mathtt{O}$ *be a graded operadic category. Then*

(i) for any local terminal object t *of* $\mathtt{O}$ *its grade* $e(t) = 0$.

(ii) For any isomorphism f *in* $\mathtt{O}$ *the grade* $e(f) = 0$.

Conversely, if $\mathtt{O}$ *is factorizable, all quasibijections in* $\mathtt{O}$ *are isomorphisms, the weak blow-up axiom is fulfilled and the only objects of grade* 0 *are local terminals, then* $e(f) = 0$ *implies that* f *is an isomorphism, so* $\mathtt{O}$ *is strictly graded.*

Proof. Observe that the identity morphism $\mathbb{1} : U \to U$ of a trivial object has a unique fiber U, hence $2e(U) = e(U)$ by (32) and $e(U) = 0$. Let t be a local terminal object and F be the fiber of the unique morphism $U \to t$. Then $e(t) + e(F) = e(U) = 0$. So both $e(t)$ and $e(F)$ must be 0. This proves (i).

According to the first part of Lemma 2.12, every isomorphism f has local terminals as its fibers, hence $e(f) = 0$. This proves (ii). The converse statement follows from the second part of Lemma 2.12. □

The grading can be transferred along operadic functors.

Lemma 3.26. *if $F : \mathtt{O} \to \mathtt{P}$ is an operadic functor and $\mathtt{P}$ is graded, then $\mathtt{O}$ has a transferred grading given by the formula*

$$e(T) := e(F(T)),\ T \in \mathtt{O}. \tag{33}$$

Remark 3.27. It is easy to see that a grading on $\mathtt{O}$ is the same as an $\mathtt{O}$-operad in the discrete symmetric monoidal category $(\mathbb{N}, +, 0)$. The transfer of the grading amounts to the restriction functor between the category of operads.

Lemma 3.28. *The operadic category $\mathtt{Gr}$ is graded by the number of internal edges of a graph.*

Example 3.29. The grading of $\mathtt{Gr}$ is not strict. Indeed, consider a unique morphism $!_{p+q}$ of the non-connected graph obtained as the disjoint union of corollas $c_p \cup c_q$ to c_{p+q}; recall that $c_n, n \geq 0$, are the trivial objects of $\mathtt{Gr}$. This is not an isomorphism, but $e(!_{p+q}) = 0$. On the other hand the operadic subcategory of *connected* graphs $\mathtt{Grc}$, cf. Example 4.6, is strictly graded for the transferred grading since the only connected graphs without internal edges are ordered corollas, which are local terminals (but not necessary trivial) in $\mathtt{Grc}$.

4 Discrete operadic (op)fibrations

In this section we focus on discrete operadic fibrations $p : \mathtt{O} \to \mathtt{P}$. We show that the operadic category $\mathtt{O}$ retains some useful properties of $\mathtt{P}$. Since, as we know from [6, page 1647], each set-valued $\mathtt{P}$-operad determines a discrete operadic fibration $p : \mathtt{O} \to \mathtt{P}$, this gives a method to obtain new operadic categories with controlled properties from the old ones. In the second part of this section we formulate similar statements for opfibrations and cooperads.

4.1 Discrete operadic fibrations

We start by recalling Definition 2.1 of [6]:

Definition 4.1. An operadic functor $p : \mathtt{O} \to \mathtt{P}$ is a *discrete operadic fibration* if

(i) p induces a surjection $\pi_0(\mathtt{O}) \twoheadrightarrow \pi_0(\mathtt{P})$ and

(ii) for any morphism $f : T \to S$ in $\mathtt{P}$ and any list of objects $t_1, \ldots, t_k, s \in \mathtt{O}$, where $k = |S|$, such that

$$p(s) = S \text{ and } p(t_i) = f^{-1}(i) \text{ for all } i \in |S|,$$

there exists a unique $\sigma : t \to s$ in $\mathtt{O}$ such that

$$p(\sigma) = f \text{ and } t_i = \sigma^{-1}(i) \ \text{ for all } i \in |S|.$$

Lemma 4.2. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration and $f : T \xrightarrow{\sim} S$ a quasibijection in $\mathtt{P}$. Let $s \in \mathtt{O}$ be such that $p(s) = S$. Then there exists a unique quasibijection σ in $\mathtt{O}$ with codomain s such that $p(\sigma) = f$.*

Proof. We invoke [6, Lemma 2.2] saying that a discrete operadic fibration induces an isomorphism of π_0's, plus the fact that operadic functors are required to send trivial objects to trivial ones. Therefore p establishes a bijection between the sets of trivial objects of the categories $\mathtt{O}$ and $\mathtt{P}$. Hence, we can uniquely complete the data for s by a list of trivial objects in place of the prescribed fibers and construct σ as the unique lift of these data. □

Lemma 4.3. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration. If in $\mathtt{P}$ all quasibijections are invertible, the same is true also for quasibijections in $\mathtt{O}$. In this case we also have that, for any quasibijection $f : T \xrightarrow{\sim} S$ in $\mathtt{P}$ and $t \in \mathtt{O}$ such that $p(t) = T$, there exists a unique quasibijection $\sigma : t \xrightarrow{\sim} s$ such that $p(\sigma) = f$.*

Proof. Let $\sigma : t \xrightarrow{\sim} s$ be a quasibijection in $\mathtt{O}$. Consider the inverse $g : p(s) \to p(t)$ to the quasibijection $p(\sigma) : p(t) \xrightarrow{\sim} p(s)$. Notice that g is a quasibijection by Corollary 2.3. Using Lemma 4.2, we lift g to a unique quasibijection $\eta : s' \xrightarrow{\sim} t$. The composite $\sigma\eta$ is the lift of the identity $p(s) \to p(s)$ so, by uniqueness, it is the identity as well, in particular, $s = s'$. The composite $\eta\sigma$ is the identity for the same reason.

The second part can be established as follows. Let $g : S \xrightarrow{\sim} T$ be the inverse quasibijection to f. We lift it to a quasibijection $\tilde{g} : s \xrightarrow{\sim} t$ in $\mathtt{O}$ and define $\sigma : t \xrightarrow{\sim} s$ to be the inverse of this lift. The uniqueness of the lifting guarantees that σ is a lift of f. □

Proposition 4.4. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration. If $\mathtt{P}$ is a factorizable operadic category in which all quasibijections are invertible, then also $\mathtt{O}$ is factorizable.*

Proof. Let $\xi : t \to s$ be a morphism in $\mathtt{O}$. Let $T \xrightarrow{f} Z \xrightarrow{g} S$ be the factorization of $p(\xi) : T \to S$ into a quasibijection f followed by an order-preserving $g \in \mathtt{P}_{\mathtt{ord}}$. Let $h : Z \to T$ be the inverse to f. Using Lemma 4.3 we lift f to the unique and invertible quasibijection $\alpha : t \xrightarrow{\sim} z$. Let $\beta : z \to t$ be its inverse. Then the morphism

$$z \xrightarrow{\beta} t \xrightarrow{\xi} s$$

is order preserving since

$$p(\xi\beta) = p(\xi)p(\beta) = p(\xi)h = g$$

is order preserving. Thus $(\xi\beta)\alpha$ is the desired factorization. □

Proposition 4.5. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration.*

(i) If the weak blow-up axiom holds in $\mathtt{P}$, it also holds in $\mathtt{O}$;

(ii) If the blow-up axiom holds in $\mathtt{P}$, it also holds in $\mathtt{O}$.

Proof. For the weak blow-up axiom let $h : T \to S$ be an order preserving morphism in $\mathtt{O}$ with the list of fibers $T_i, i \in |S|$, and let $\tau_i : T_i \to F_i, i \in |S|$ be a family of morphisms. We need to find the unique factorization of h in $\mathtt{O}$

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "h"'] & & R \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

such that g is order preserving and for each $i \in S$ the induced morphism on fibers f_i coincides with τ_i.

We apply p to h and $\tau_i, i \in |S|$. We thus obtain the input data for the weak blow-up axiom in $\mathtt{P}$ and we have the corresponding unique factorization of $p(h)$

$$\begin{tikzcd} p(T) \arrow[rr, "\xi"] \arrow[dr, "p(h)"'] & & \Gamma \arrow[dl, "\gamma"] \\ & p(S) & \end{tikzcd}$$

where ξ acts on fibers as $\xi_i = p(\tau_i) : p(T_i) \to p(F_i)$.

Invoking the lifting property of discrete operadic fibrations, we lift γ to $g : R \to S$ with $g^{-1}(i) = F_i$. Since γ is order preserving in $\mathtt{P}$ the morphism g is also order preserving in $\mathtt{O}$.

Observe that the fibers of ξ are given by $\xi^{-1}(j) = \xi^{-1}_{|\gamma|(j)}(j), j \in |\Gamma| = |R|$ by Axiom (iv) of operadic categories, and hence, are equal to the fibers $p(\tau_{|\gamma|(j)})^{-1}(j)$. We now use the lifting property of the operadic fibration for the second time to lift ξ to a morphism $f : Q \to R$ in $\mathtt{O}$ whose fibers are exactly $\tau^{-1}_{|\gamma|(j)}(j), j \in |R|$.

Then the fibers of $f_i : (gf)^{-1}(i) \to g^{-1}(i) = F_i$ are $\tau_i^{-1}(j), j \in |\gamma|^{-1}(i)$. But $\tau_i : T_i \to F_i$ has exactly the same fibers, and both morphisms are liftings of ξ_i. Hence, by uniqueness of lifting, we have $f_i = \tau_i$ and $(gf)^{-1}(i) = T_i$. But now we see that both h and gf are liftings of $p(h)$ and have the same fibers. By uniqueness of lifting again $Q = S$ and $h = gf$ and we obtained the required factorization.

The proof of the second part of the proposition is similar. We only have to twist indices by the effect of the quasibijection σ which is a part of the input data of the blow-up axiom. $\square$

Important examples of discrete operadic fibrations are provided by the *operadic Grothendieck construction* introduced in [6, page 1647]. Assume that one is given a set-valued $\mathtt{P}$-operad $\mathcal{O}$. One then has the operadic category $\int_{\mathtt{P}} \mathcal{O}$ whose objects are pairs (T, t) where $T \in \mathtt{P}$ and $t \in \mathcal{O}(T)$. A morphism $\sigma : (T, t) \to (S, s)$ for $t \in \mathcal{O}(T)$ and $s \in \mathcal{O}(S)$ is a pair (ε, f) consisting of a morphism $f : T \to S$ in $\mathtt{P}$ and of some $\varepsilon \in \prod_{i \in |S|} \mathcal{O}\big(f^{-1}(i)\big)$ such that

$$\gamma_f(\varepsilon, s) = t,$$

where γ is the composition law of the operad $\mathcal{O}$. Composition of morphisms is defined in the obvious manner. The category $\int_{\mathtt{P}} \mathcal{O}$ thus constructed is an operadic category such that the functor $p : \int_{\mathtt{P}} \mathcal{O} \to \mathtt{P}$ given by

$$p(t) := T \text{ for } t \in \mathcal{O}(T) \text{ and } p(\varepsilon, f) := f \tag{34}$$

is a discrete operadic fibration. The trivial objects are given by the operad units $1_c \in \mathcal{O}(U_c)$. By [6, Proposition 2.5], the above construction establishes an equivalence between the category of set-valued $\mathtt{P}$-operads and the category of discrete operadic fibrations over $\mathtt{P}$.

Example 4.6. Consider the $\mathtt{Gr}$-operad C in $\mathtt{Set}$ such that

$$C(\Gamma) := \begin{cases} 1 \text{ (one point set)} & \text{if } \Gamma \text{ is connected} \\ \emptyset & \text{otherwise.} \end{cases}$$

There is a unique way to extend this construction to a $\mathtt{Gr}$-operad. The Grothendieck construction of C produces a discrete operadic fibration $\mathtt{Grc} \to \mathtt{Gr}$. We call $\mathtt{Grc}$ the operadic category of *connected ordered graphs.*

Example 4.7. A construction similar to the one in Example 4.6 produces the operadic category $\mathtt{Tr}$ of trees. We consider the operad Π with

$$\Pi(\Gamma) := \begin{cases} 1 & \text{if the geometric realization } \mathrm{B}(\Gamma) \text{ of } \Gamma \text{ is contractible} \\ \emptyset & \text{otherwise.} \end{cases}$$

The Grothendieck construction gives a discrete operadic fibration $\mathtt{Tr} \to \mathtt{Gr}$.

Example 4.8. Let us orient edges of a tree $T \in \mathtt{Tr}$ so that they point to the leg which is the smallest in the global order. We say that T is *rooted* if the outgoing half-edge of each vertex is the smallest in the local order at that vertex. Now define

$$R(T) := \begin{cases} 1 & \text{if } T \text{ is rooted} \\ \emptyset & \text{otherwise.} \end{cases}$$

The Grothendieck construction associated to the operad R gives the operadic category $\mathtt{RTr}$ of *rooted trees.*

Example 4.9. There is a unique isotopy class of embeddings of $T \in \mathtt{Tr}$ into the plane such that the local orders are compatible with the orientation of the plane. This embedding in turn determines a cyclic order of the legs of T. We say that T is *planar* if this cyclic order coincides with the cyclic order induced by the global order of the legs. The operad

$$P(T) := \begin{cases} 1 & \text{if } T \text{ is planar} \\ \emptyset & \text{otherwise} \end{cases}$$

gives rise to the operadic category $\mathtt{PTr}$ of planar trees. In a similar manner we obtain the operadic category $\mathtt{PRTr}$ of *planar rooted trees.*

All the above constructions fall into the situation captured by the following lemma whose proof is obvious.

Lemma 4.10. *Let $i : \mathtt{C} \subset \mathtt{P}$ be a full operadic subcategory such that*

(i) the set of local chosen terminal objects of $\mathtt{P}$ coincides with the set of local chosen terminal objects of $\mathtt{C}$, and

(ii) for any morphism f in $\mathtt{P}$ whose codomain and all fibers are in $\mathtt{C}$, the domain of f is also in $\mathtt{C}$.

Then i is a discrete operadic fibration.

Remark 4.11. For a discrete operadic fibration $p : \mathtt{O} \to \mathtt{P}$, it is not true in general that the unique fiber condition is satisfied in $\mathtt{O}$ if it is satisfied in $\mathtt{P}$. Thus it has to be verified separately in each concrete case.

Example 4.12. Consider the one-object, one-morphism operadic category $\mathtt{1}$ whose set-valued operads are monoids. The operadic category $\mathtt{1}$ obviously satisfies the unique fiber condition. Let $(\mathcal{M}, \cdot, e)$ be a monoid. The operadic Grothendieck construction $\int_{\mathtt{1}} \mathcal{M}$ is fibered over $\mathtt{1}$ and has pairs $(1, t) =: \mathbf{t},\ t \in \mathcal{M}$, as objects. A morphism from $\mathbf{x}$ to $\mathbf{y}$ is given by an element $a \in \mathcal{M}$ such that $ay = x$. The fiber of such a morphism is $\mathbf{a}$. The category $\int_{\mathtt{1}} \mathcal{M}$ is connected with the trivial object $\mathbf{e} = (1, e)$. Notice that $t \in \mathcal{M}$ is invertible if and only if $\mathbf{t}$ is a local terminal object in $\int_{\mathtt{1}} \mathcal{M}$. Indeed, if t is invertible, then the equation $at = x$ has a unique solution $a = xt^{-1}$, hence there is a unique morphism from $\mathbf{x}$ to $\mathbf{t}$ in $\int_{\mathtt{1}} \mathcal{M}$. The opposite implication is also clear.

On the other hand, the equation $xt = x$ with invertible t does not force, in general, the equality $t = e$ unless x is invertible as well. For example, in the monoid $\mathcal{M}_2(\mathbb{Z})$ of 2×2 integer-valued matrices under the standard matrix multiplication there are always $t \neq e$ and x which satisfy this equation, for instance

$$t := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ x := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since t is invertible, we have that $\mathbf{t}$ is local terminal and the fiber of $\mathbf{x} \to \mathbf{t}$ is $\mathbf{x}$, but $\mathbf{t}$ is not the chosen local terminal. Thus $\int_{\mathtt{1}} \mathcal{M}_2(\mathbb{Z})$ does not fulfill the unique fiber condition.

We close this subsection with the following useful statement.

Proposition 4.13. *Let $\mathcal{O}$ be a set-valued $\mathtt{P}$-operad, $\int_{\mathtt{P}} \mathcal{O} \to \mathtt{P}$ its operadic Grothendieck construction and $1_{\int_{\mathtt{P}} \mathcal{O}}$ the terminal set-valued $\int_{\mathtt{P}} \mathcal{O}$-operad. Then the categories of $\mathcal{O}$-algebras and $1_{\int_{\mathtt{P}} \mathcal{O}}$-algebras are isomorphic, i.e.*

$$\mathcal{O}\text{-}\mathtt{Alg} \cong 1_{\int_{\mathtt{P}} \mathcal{O}}\text{-}\mathtt{Alg}.$$

Proof. The sets of connected components of the categories $\mathtt{P}$ and $\int_{\mathtt{P}} \mathcal{O}$ are canonically isomorphic via the correspondence

$$U_c \longleftrightarrow 1_c \in \mathcal{O}(U_c)$$

of the chosen local terminal objects. We use this isomorphism to identify $\pi_0(\mathtt{P})$ with $\pi_0(\int_{\mathtt{P}} \mathcal{O})$. Under this convention, the sets $\pi_0(s(T))$ of connected components of the sources of an object $T \in \mathtt{P}$ and the sets $\pi_0(s(t))$ of $t \in \mathcal{O}(T)$ representing an object of $\int_{\mathtt{P}} \mathcal{O}$ are the same, and similarly $\pi_0(T) = \pi_0(t)$. The structure operations of an $\mathcal{O}$-algebra are by Definition 1.10

$$\alpha_T : \mathcal{O}(T) \times \bigtimes_{c\, \in\, \pi_0(s(T))} A_c \longrightarrow A_{\pi_0(T)},\ T \in \mathtt{P}, \tag{35a}$$

which can be interpreted as families

$$\alpha_t : \bigtimes_{c\, \in\, \pi_0(s(T))} A_c \longrightarrow A_{\pi_0(T)},\ t \in \mathcal{O}(T),\ T \in \mathtt{P},$$

of maps parametrized by $t \in \mathcal{O}(T)$. Using the above identifications, we rewrite the above display as

$$\alpha_t : \bigtimes_{c\, \in\, \pi_0(s(t))} A_c \longrightarrow A_{\pi_0(t)},\ t \in \mathcal{O}(T),\ T \in \mathtt{P}, \tag{35b}$$

which are precisely the structure operations of an $1_{\int_{\mathtt{P}} \mathcal{O}}$-algebra. It is simple to verify that the correspondence between (35a) and (35b) extends to an isomorphism of the categories of algebras. $\square$

4.2 Discrete operadic opfibrations

In Subsection 4.1 we recalled how set-valued operads produce discrete operadic fibrations. We are going to present a dual construction for cooperads.

The notion of a cooperad over an operadic category is obtained from that of an operad by reversing the arrows. A set-valued $\mathtt{P}$-*cooperad* is thus a collection $\mathcal{C} = \{\mathcal{C}(T)\}_{T\in\mathtt{P}}$ of sets together with structure maps

$$\Delta_f : \mathcal{C}(T) \longrightarrow \mathcal{C}(S) \times \mathcal{C}(F_1) \times \cdots \times \mathcal{C}(F_s) \tag{36}$$

defined for an arbitrary $f : T \to S$ with fibers $F_1, \ldots, F_s$. The rôle of counits is played by the unique maps

$$\mathcal{C}(U_c) \to *, \;\; c \in \pi_0(\mathtt{O}),$$

to a terminal one-point set $*$. These operations are required to satisfy axioms dual to those in [6, Definition 1.11].

A set-valued $\mathtt{P}$-cooperad $\mathcal{C}$ leads to an operadic category $\int^{\mathtt{P}}\mathcal{C}$ via a dual version of the Grothendieck construction recalled in Subsection 4.1. The objects of $\int^{\mathtt{P}}\mathcal{C}$ are pairs (T, t), where $T \in \mathtt{P}$ and $t \in \mathcal{C}(T)$. A morphism $\sigma : (T, t) \to (S, s)$ is a morphism $f : T \to S$ in $\mathtt{P}$ such that

$$\Delta_f(t) = (s, \varepsilon)$$

for some, necessarily unique, $\varepsilon \in \prod_{i\in|S|} \mathcal{C}\big(f^{-1}(i)\big)$, where Δ_f is the structure map (36).

The category $\int^{\mathtt{P}}\mathcal{C}$ is an operadic category equipped with a functor $p : \int^{\mathtt{P}}\mathcal{C} \to \mathtt{P}$ defined by (34). The trivial objects are all objects of the form $u \in \mathcal{C}(U_c)$, $c \in \pi_0(\mathtt{P})$. It turns out that the functor $p : \int^{\mathtt{P}}\mathcal{C} \to \mathtt{P}$ is a standard discrete opfibration:

Definition 4.14. A *discrete operadic opfibration* is an operadic functor $p : \mathtt{O} \to \mathtt{P}$ which, as a functor, is a discrete opfibration. That is, for any morphism $f : T \to S$ in $\mathtt{P}$ and any $t \in \mathtt{O}$ such that $p(t) = T$, there exists a unique $\sigma : t \to s$ in $\mathtt{O}$ such that $p(\sigma) = f$.

Dualizing the steps in the proof of [6, Proposition 2.5] one can show that the dual Grothendieck construction is an equivalence between the category of set-valued $\mathtt{P}$-cooperads and the category of discrete operadic opfibrations over $\mathtt{P}$. As the following statement shows, discrete operadic opfibrations behave nicely with respect to chosen local terminal objects.

Lemma 4.15. *Operadic functors preserve local terminal objects. If $p : \mathtt{O} \to \mathtt{P}$ is a discrete operadic opfibration, then $t \in \mathtt{O}$ is trivial if and only if $p(t)$ is trivial.*

Proof. Operadic functors send trivial objects to trivial objects. Let t be a local terminal in $\mathtt{O}$ and let $! : t \to U$ be the unique isomorphism to a trivial object. Then $p(!) : p(u) \to P(U)$ is an isomorphism to a trivial object, and hence $p(t)$ is a local terminal.

Suppose that p is a discrete operadic opfibration. For $t \in \mathtt{O}$ let $! : t \to U$ be the unique map to a trivial object. If $p(t)$ is trivial in $\mathtt{P}$, the map $p(!) : p(t) \to p(U)$ is the identity, so its lifts $!$ and $\mathbb{1}_t$ are two lifts of the identity $\mathbb{1}_{p(t)}$ with the common domain t. Hence, $! = \mathbb{1}_t$. $\square$

The next property of opfibrations has to be compared to Remark 4.11.

Lemma 4.16. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic opfibration. If the unique fiber condition holds in $\mathtt{P}$, then it also holds in $\mathtt{O}$.*

Proof. Suppose we have a situation $T \blacktriangleright T \to t$ in $\mathtt{O}$, with t local terminal. By the first part of Lemma 4.15, we have $p(T) \blacktriangleright p(T) \to p(t)$ in $\mathtt{P}$ with $p(t)$ local terminal. By the unique fiber condition for $\mathtt{P}$, $p(t)$ is a chosen local terminal object in $\mathtt{P}$, so t is a chosen local terminal object in $\mathtt{O}$ by the second part of Lemma 4.15. $\square$

It turns out that analogs of Lemmas 4.2, 4.3 and Propositions 4.4, 4.5 hold also for discrete operadic opfibrations. As an example, we prove the following variant of Lemma 4.2.

Lemma 4.17. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic opfibration and $f : T \stackrel{\sim}{\to} S$ a quasibijection in $\mathtt{P}$. Let $t \in \mathtt{O}$ be such that $p(t) = T$. Then there exists a unique quasibijection σ in $\mathtt{O}$ with domain t such that $p(\sigma) = f$.*

Proof. By the lifting property of opfibrations, f lifts to a unique σ so we only need to prove that σ is a quasibijection. Since p is an operadic functor, it maps the fibers of σ to the fibers of f. Since the latter are trivial in $\mathtt{P}$, the former must be trivial in $\mathtt{O}$ by Lemma 4.15. So σ is a quasibijection. $\square$

An analog of Lemma 4.3 for a discrete operadic opfibration $p : \mathtt{O} \to \mathtt{P}$ reads as follows:

Lemma 4.18. *Let $p : \mathtt{O} \to \mathtt{P}$ be a discrete operadic opfibration. If all quasibijections in $\mathtt{P}$ are invertible, then all quasibijections in $\mathtt{O}$ are also invertible. Moreover, for each quasibijection $f : T \stackrel{\sim}{\to} S$ in $\mathtt{P}$ and $s \in \mathtt{O}$ such that $p(s) = S$, there exists a unique quasibijection $\sigma : t \stackrel{\sim}{\to} s$ such that $p(\sigma) = f$.*

We leave the proof of this lemma as an exercise, as well as the verification that Propositions 4.4 and 4.5 hold verbatim for discrete operadic opfibrations as well.

Example 4.19. In Example 4.6 we constructed the operadic category $\mathtt{Grc}$ of connected ordered graphs. We introduce a set-valued $\mathtt{Grc}$-cooperad G as follows. For $\Gamma = (V, F) \in \mathtt{Grc}$ we put

$$G(\Gamma) := \mathrm{Map}(V, \mathbb{N}) = \{g(v) \in \mathbb{N} \mid v \in V\}.$$

The cooperad structure operations

$$\Delta_\Phi : G(\Gamma') \longrightarrow G(\Gamma'') \times G(\Gamma_1) \times \cdots \times G(\Gamma_s)$$

are, for a map $\Phi : \Gamma' = (V', F') \to \Gamma'' = (V'', F'')$ with fibers $\Gamma_i = (V_i, F_i)$ over $i \in V''$, given as $\Delta_\Phi(g') := (g'', g_1, \ldots, g_s)$, where g_i is the restriction of g' to $V_i \subset V'$ and

$$g''(i) := \sum_{v \in V_i} g_i(v) + \dim\left(H^1(\mathrm{B}(\Gamma_i); \mathbb{Z})\right),\ i \in V'',$$

where $\mathrm{B}(\Gamma_i)$ is the geometric realization of Γ_i.

The Grothendieck construction applied to G produces the operadic category $\mathtt{ggGrc}$ of *genus-graded* connected ordered graphs. The morphisms in this category coincide with the morphisms of graphs as introduced in [20, Section 2], modulo the orders which we used to make $\mathtt{ggGrc}$ an operadic category.

Example 4.20. We say that an ordered graph $\Gamma \in \mathtt{Gr}$ is *oriented* if

(i) each internal edge in Γ is oriented, meaning that one of the half-edges forming this edge is marked as the input one, and the other as the output, and

(ii) also the legs of Γ are marked as either input or output ones.

We will call the above data an *orientation* and denote the set of all orientations of Γ by $\mathrm{Or}(\Gamma)$. It is easy to see that Or is a cooperad over $\mathtt{Grc}$. The operadic category $\mathtt{Whe}$ resulting from the Grothendieck construction applied to Or consists of *oriented* ordered connected graphs. We choose the notation $\mathtt{Whe}$ because algebras of the terminal $\mathtt{Whe}$-operads are the *wheeled PROPs* introduced in [32].

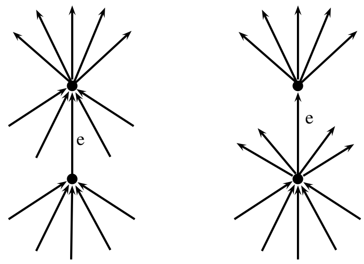
Example 4.21. Let C be an obvious modification of the operad of Example 4.7 to the category $\mathtt{Whe}$. The Grothendieck construction associated to this modified C produces the operadic category $\mathtt{Dio}$ of *simply-connected* oriented ordered graphs. The notation expresses that algebras of the terminal $\mathtt{Dio}$-operad are *dioperads* [16].

The *valency* of a vertex u in a graph Γ is the number of half-edges adjacent to u. For any $v \geq 2$, all operadic categories mentioned above that consist of simply connected graphs, i.e. $\mathtt{Tr}$, $\mathtt{PTr}$, $\mathtt{RTr}$, $\mathtt{PRTr}$ and $\mathtt{Dio}$, possess full operadic subcategories $\mathtt{Tr}_v$, $\mathtt{PTr}_v$, $\mathtt{RTr}_v$, $\mathtt{PRTr}_v$ and $\mathtt{Dio}_v$ of graphs all of whose vertices have valency $\geq v$.

Example 4.22. We call an ordered simply-connected graph $\Gamma \in \mathtt{Dio}$ a $\frac{1}{2}$*graph* if each internal edge e of Γ satisfies the following condition:

- either e is the unique outgoing edge of its initial vertex, or
- e is the unique incoming edge of its terminal vertex.

Edges allowed in a $\frac{1}{2}$graph are portrayed in the picture:



borrowed from [29]. For $\Gamma \in \mathtt{Dio}$, let us define

$$\tfrac{1}{2}(\Gamma) := \begin{cases} 1 & \text{if } \Gamma \text{ is a } \tfrac{1}{2}\text{graph} \\ \emptyset & \text{otherwise.} \end{cases}$$

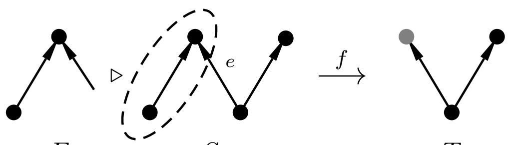
It is easy to verify that the restriction of $\frac{1}{2}$ to $\mathtt{Dio}_3 \subset \mathtt{Dio}$ is an operad. The Grothendieck construction applied to $\frac{1}{2}$ produces the operadic category $\frac{1}{2}\mathtt{Gr}_3$ of $\frac{1}{2}$graphs whose vertices have valency ≥ 3. Algebras for the terminal $\frac{1}{2}\mathtt{Gr}_3$-operad are $\frac{1}{2}$PROPs as in [30, Definition 4].

The constructions above are summarized in the diagram

$$\begin{tikzcd}
 & \mathtt{RTr} \arrow[dr, "\mathtt{fib}"] & & \mathtt{ggGrc} \arrow[d, "\mathtt{opfib}"] & & & \\
\mathtt{PRTr} \arrow[ur, "\mathtt{fib}"] \arrow[dr, "\mathtt{fib}"'] & & \mathtt{Tr} \arrow[r, hook, "\mathtt{fib}"] & \mathtt{Grc} \arrow[rrr, hook, "\mathtt{fib}"] & & & \mathtt{Gr} \\
 & \mathtt{PTr} \arrow[ur, "\mathtt{fib}"'] & & \mathtt{Whe} \arrow[u, "\mathtt{opfib}"'] & \mathtt{Dio} \arrow[l, hook', "\mathtt{fib}"'] & \mathtt{Dio}_3 \arrow[l, hook', "\mathtt{opfib}"'] & \tfrac{1}{2}\mathtt{Gr}_3 \arrow[l, hook', "\mathtt{bifib}"']
\end{tikzcd} \tag{37}$$

in which $\mathtt{fib}$ denotes fibrations, $\mathtt{opfib}$ opfibrations and $\mathtt{bifib}$ the inclusion that is both a fibration and an opfibration.

Example 4.23. The inclusion $\frac{1}{2}\mathtt{Gr}_3 \to \mathtt{Dio}_3$ is a discrete operadic fibration. The same is however not true for the inclusion $\frac{1}{2}\mathtt{Gr} \to \mathtt{Dio}$ of the categories of graphs with vertices of arbitrary valencies. While condition (i) of Lemma 4.10 is satisfied, condition (ii) is violated e.g. by the map $f : S \to T$ depicted below



F S T

given by contracting the subgraph in the dashed oval to the gray vertex of the rightmost graph. Both the target T and the fiber F are $\frac{1}{2}$-graphs, but the domain S is not, since it contains the edge e that is neither the unique outgoing, nor the unique incoming one. The inclusion $\frac{1}{2}\mathtt{Gr}_3 \to \mathtt{Dio}$ is not a discrete operadic fibration either; this time it is item (i) of Lemma 4.10 that is violated.

Example 4.24. Let Δ_{semi} be the subcategory of $\mathtt{Fin}_{\mathrm{semi}}$ consisting of order-preserving surjections. It is an operadic category whose operads are the classical constant-free non-symmetric operads [6, Example 1.15]. One has the Δ_{semi}-cooperad $\mathcal{S}$ with components

$$\mathcal{S}(\bar{n}) := \coprod_{m \geq n} \mathrm{Surj}(\bar{m}, \bar{n}), \ \ n \geq 1,$$

where $\mathrm{Surj}(\bar{m}, \bar{n})$ denotes the set of all (not necessarily order-preserving) surjections. Its structure map $\Delta_f : \mathcal{S}(\bar{n}) \to \mathcal{S}(\bar{s}) \times \mathcal{S}(f^{-1}(1)) \times \cdots \times \mathcal{S}(f^{-1}(s))$ is, for $f : \bar{n} \to \bar{s}$, given by

$$\Delta_f(\alpha) := \beta \times \alpha_1 \times \cdots \times \alpha_s, \ \ \alpha \in \mathcal{S}(\bar{n}),$$

property of $\mathtt{P}$	p is fibration then $\mathtt{O}$ satisfies	p is opfibration then $\mathtt{O}$ satisfies
Fac	?	Fac
Fac & QBI	Fac & QBI	Fac & QBI
BU	BU	BU
UFib	?	UFib
Rig	?	Rig
$\mathtt{P}$ is graded	$\mathtt{O}$ is graded	$\mathtt{O}$ is graded
SGrad	?	SGrad

Table 1: How discrete operadic (op)fibrations $p : \mathtt{O} \to \mathtt{P}$ interact with properties of operadic categories.

where $\beta := f\alpha$ and $\alpha_i : (f\alpha)^{-1}(i) \to f^{-1}(i)$ is the restriction of α, $i \in \bar{s}$. The Grothendieck construction of the cooperad $\mathcal{S}$ leads to the operadic category $\mathtt{Per}$ whose operads are permutads, introduced in [31].

The leftmost column of Table 1 lists properties required in the second paper of the series [7]. Its top four rows record results obtained in this section. The 5th and 6th rows easily follow from the uniqueness of lifts in discrete operadic opfibrations, while the grading in the last row is given by formula (33) and does not require any additional assumptions on $p : \mathtt{O} \to \mathtt{P}$.

Remark 4.25. If $p : \mathtt{O} \to \mathtt{P}$ is a discrete operadic opfibration and $\mathtt{P}$ fulfills the properties listed in the leftmost column of Table 1, then $\mathtt{O}$ shares the same properties. If $p : \mathtt{O} \to \mathtt{P}$ is a discrete operadic fibration, the situation in not so simple. One may however invoke the implication UFib & WBU $\Longrightarrow$ Rig of Lemma 2.19 and conclude that if one "manually" verifies UFib and the presence of a strict grading, then $\mathtt{O}$ satisfies all the properties in the leftmost column also in the case of opfibrations.

Example 4.26. According to Section 3 the operadic category $\mathtt{Gr}$ satisfies all properties in the leftmost column of Table 1 except for being strictly graded. Then the category $\mathtt{Grc}$ being fibered over $\mathtt{Gr}$ satisfies Fac, QBI, BU. The fact that it satisfies UFib follows promptly from the fact that local terminals and chosen local terminals in $\mathtt{Gr}$ and $\mathtt{Grc}$ coincide and $\mathtt{Gr}$ satisfies UFib. Hence, $\mathtt{Grc}$ also satisfies Rig. The grading on $\mathtt{Grc}$ is strict, see Example 3.29. It now follows from Example 4.19 that the operadic category $\mathtt{ggGrc}$ is opfibered over $\mathtt{Grc}$, and hence, satisfies all the properties listed in the rightmost column of the Table 1.

5 Elementary morphisms

While composition laws $\gamma_f : \mathcal{P}(f) \otimes \mathcal{P}(S) \to \mathcal{P}(T)$ of an operad $\mathcal{P}$ over an operadic category $\mathtt{O}$ are associated to an arbitrary morphism $f : T \to S$ in $\mathtt{O}$, cf. [6, Definition 1.11], composition laws of Markl operads introduced in Section 6 are associated to morphisms with only one nontrivial fiber. The precise definition and properties of this class of morphisms are the subject of this section. From now on all operadic categories will be *graded.*

Definition 5.1. A morphism $\phi : T \to S \in \mathtt{O}_{\mathtt{ord}}$ in an operadic category $\mathtt{O}$ is *elementary* if all its fibers are trivial (= chosen local terminal) except precisely one whose grade is ≥ 1. If $\phi^{-1}(i)$ is, for $i \in |S|$, the unique nontrivial fiber, we will sometimes write ϕ as the pair (ϕ, i). If we want to name the unique nontrivial fiber $F := \phi^{-1}(i)$ explicitly, we will write $F \triangleright_i T \xrightarrow{\phi} S$, or $F \triangleright T \xrightarrow{\phi} S$ when the concrete $i \in |S|$ is not important.

Notation. In the setup of Lemma 2.4 with σ a quasibijection, assume that the morphisms f', f'' are elementary, ${f'}^{-1}(a)$ is the only nontrivial fiber of f', and ${f''}^{-1}(b)$ with $b := |\sigma|(a)$ the only nontrivial fiber of f''. In this situation, we denote by

$$\overline{\pi} := \pi_{(a,b)} : {f'}^{-1}(a) \to {f''}^{-1}(b) \tag{38}$$

the only nontrivial part of the derived sequence (8).

Remark 5.2. If π is a quasibijection, the only nontrivial fiber of f'' *must be* $f''^{-1}(b)$ with $b := |\sigma|(a)$. Indeed, the maps in (7) are quasibijections, so their fibers are, by definition, the chosen local terminal objects. When $f''^{-1}(j)$ is the chosen local terminal object, then the (unique) fiber of $\pi_{(i,j)}$ is $f'^{-1}(i)$, so it must be, by Axiom (iii) of an operadic category, a chosen local terminal object too.

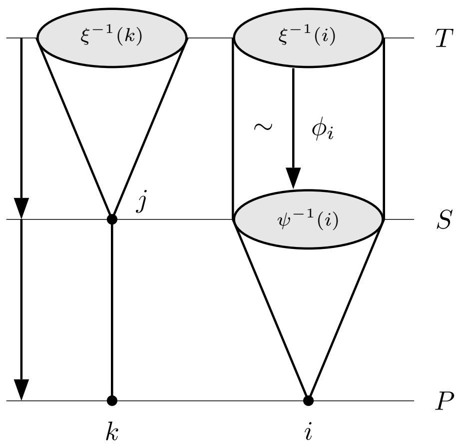
Corollary 5.3. *Assume the blow-up axiom and suppose that in the corner for blow-up as on the left of the display*

$$\begin{tikzcd} S' \arrow[d, "f'"'] & & & S' \arrow[r, "\pi"] \arrow[d, "f'"'] & S'' \arrow[d, "f''"] \\ T' \arrow[r, "\sigma", "\sim"'] & T'' & & T' \arrow[r, "\sigma", "\sim"'] & T'' \end{tikzcd} \tag{39}$$

the map f' is elementary, with the unique nontrivial fiber over $a \in |T'|$. Let $b := |\sigma|(a)$ and assume we are given a map $\overline{\pi} : f'^{-1}(a) \to F$, where $e(F) \geq 1$. Then the corner in (39) can be uniquely completed into a commutative square as on the right in which f'' is elementary with the unique nontrivial fiber $f''^{-1}(b) = F$ and the nontrivial part of the derived sequence is $\overline{\pi}$.

Proof. By BU, (11) is uniquely determined by the maps between the fibers. The only map between nontrivial fibers is $\overline{\pi}$, while all maps between trivial ones are unique by the terminality of trivial objects, thus there is no room for choices of the induced maps between fibers. ◻

Definition 5.4. Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms. If $|\psi|(j) = i$ we say that the fibers of ϕ and ψ are *joint*. If $|\psi|(j) \neq i$ we say that ϕ and ψ have *disjoint fibers* or, more specifically, that the fibers of ϕ and ψ are (i,j)-*disjoint*, cf. the following picture.



Lemma 5.5. *If the fibers of elementary morphisms ϕ and ψ in Definition 5.4 are joint, then the composite $\xi = \psi(\phi)$ is elementary as well, with the nontrivial fiber over i, and the induced morphism $\phi_i : \xi^{-1}(i) \to \psi^{-1}(i)$ is elementary with the nontrivial fiber over j that equals $\phi^{-1}(j)$. For $l \neq i$ the morphism ϕ_l equals the identity $U_c \to U_c$ of trivial objects.*

If the fibers of ϕ and ψ are (i,j)-disjoint then the morphism $\xi = \psi(\phi)$ has exactly two non-trivial fibers and these are fibers over i and $k := |\psi|(j)$. Moreover, there is a canonical induced quasibijection

$$\phi_i : \xi^{-1}(i) \to \psi^{-1}(i) \in \mathtt{O}_{\mathtt{ord}} \tag{40a}$$

and we have the equality

$$\xi^{-1}(k) = \phi^{-1}(j). \tag{40b}$$

Proof. By Axiom (iv) of an operadic category, we have $\phi_i^{-1}(j) = \phi^{-1}(j)$, thus $e(\phi_i^{-1}(k)) \geq 1$. If $k \in |\psi|^{-1}(i)$ is such that $k \neq j$, then $\phi_i^{-1}(k) = \phi^{-1}(k) = U_c$. Therefore ϕ_i is an elementary morphism.

Let us prove that ξ is elementary as well. For $i = k \in |P|$, we have $\phi_i : \xi^{-1}(i) \to \psi^{-1}(i)$, hence the grade of $\xi^{-1}(i)$ must be greater than or equal to the grade of $\phi_i^{-1}(j) = \phi^{-1}(j)$, which is greater than or equal to 1. For $k \neq i$, $\phi_k : \xi^{-1}(k) \to \psi^{-1}(k) = U'$ has the unique fiber equal to $\xi^{-1}(k)$. On the other hand for the unique l such that $|\psi|(l) = k$,

$$\phi_k^{-1}(l) = \phi^{-1}(l) = U'',$$

hence $\xi^{-1}(k) = U''$, so ξ is elementary.

Let us prove the second part of the lemma. If $l \neq i, k$ then $\phi_l : \xi^{-1}(l) \to \psi^{-1}(l) = U'$, where U' is a trivial object. So the unique fiber of ϕ_l equals $\xi^{-1}(l)$. Since $|\psi|$ is surjective, there exists $l' \in |S|$ such that $|\psi|(l') = l$, and such an l' is unique because ψ is elementary. Hence $\phi_l^{-1}(l') = \phi^{-1}(l') = U'$ and so $\xi^{-1}(l) = U'$. This proves that the only nontrivial fibers of ξ can be those over i and k. Their grades are clearly ≥ 1.

Let us prove that ϕ_i is a quasibijection. If $l \in |\psi^{-1}(i)|$ then $\phi_i^{-1}(l) = \phi^{-1}(l)$. Since the fibers of ϕ and ψ are (i,j)-disjoint by assumption, we have $|\psi|(l) = i \neq |\psi|(j)$, hence $l \neq j$. Since the only nontrivial fiber of ϕ is $\phi^{-1}(j)$, we conclude that $\phi^{-1}(l)$ and therefore also $\phi_i^{-1}(l)$ is trivial. To prove that $\phi_i \in \mathtt{O}_{\mathtt{ord}}$, notice that by Axiom (iii), $|\phi_i|$ is the map of sets $|\xi|^{-1} \to |\psi|^{-1}$ induced by the diagram

$$\begin{tikzcd} {|T|} \arrow[rr, "|\phi|"] \arrow[dr, "|\xi|"'] & & {|S|} \arrow[dl, "|\psi|"] \\ & {|P|\,.} & \end{tikzcd}$$

Regarding (40b), by Axiom (iv) we have $\phi^{-1}(j) = \phi_k^{-1}(j)$. But $\phi_k : \xi^{-1}(k) \to \psi^{-1}(k) = U''$ and hence its unique fiber is equal to $\xi^{-1}(k)$. So, $\phi^{-1}(j) = \xi^{-1}(k)$. ◻

Definition 5.6. We will call the pair $T \overset{(\phi,j)}{\longrightarrow} S \overset{(\psi,i)}{\longrightarrow} P$ of morphisms in Definition 5.4 with disjoint fibers *harmonic* if $\xi^{-1}(i) = \psi^{-1}(i)$ and the map ϕ_i in (40a) is the identity.

Corollary 5.7. *If the blow-up axiom is satisfied then all pairs with disjoint fibers are harmonic.*

Proof. The map ϕ_i in (40a) is a quasibijection in $\mathtt{O}_{\mathtt{ord}}$, so it is the identity by Corollary 2.6. ◻

Corollary 5.8. *Assume that*

$$\begin{tikzcd} & P' \arrow[dr, "{(\psi', i)}"] & \\ T \arrow[ur, "{(\phi', j)}"] \arrow[dr, "{(\phi'', l)}"'] & & S \\ & P'' \arrow[ur, "{(\psi'', k)}"'] & \end{tikzcd} \tag{41}$$

is a commutative diagram of elementary morphisms. Assume that $|\psi''|(l) = i, |\psi'|(j) = k$ and $i \neq k$. Let F', F'', G', G'' be the only nontrivial fibers of $\phi', \phi'', \psi', \psi''$, respectively. Then one has canonical quasibijections

$$\sigma' : F' \longrightarrow G'' \quad \textit{and} \quad \sigma'' : F'' \longrightarrow G'. \tag{42}$$

If both pairs in (41) are harmonic, then $F' = G''$, $F'' = G'$ and σ', σ'' are the identities.

Proof. Let $\xi : T \to S$ be the composite $\psi'\phi' = \psi''\phi''$. One has $G' = \psi'^{-1}(j)$, $G'' = \psi''^{-1}(k)$ and, by Lemma 5.5, $F' = \phi'^{-1}(j) = \xi^{-1}(k)$ and $F'' = \phi''^{-1}(l) = \xi^{-1}(i)$. We define

$$\sigma' : F' = \xi^{-1}(k) \xrightarrow{\phi''_k} \psi''^{-1}(k) = G'' \quad \sigma'' : F'' = \xi^{-1}(i) \xrightarrow{\phi'_i} \psi''^{-1}(i) = G'.$$

These maps are quasibijections by Lemma 5.5. The second part of the corollary follows directly from the definition of harmonicity. ◻

6 Markl operads

The aim of this section is to introduce Markl operads and their algebras in the context of operadic categories, and formulate assumptions under which these notions agree with the standard ones introduced in [6].

Assumptions. We assume that $\mathtt{O}$ is a strictly graded factorizable operadic category in which all quasibijections are invertible, the blow-up axiom and unique fiber condition are fulfilled, and a morphism f is an isomorphism if and only if $e(f)=0$; recall that by Lemma 2.12 this happens if and only if all fibers of f are local terminal. In brief, we require

$$\mathsf{Fac}\ \&\ \mathsf{BU}\ \&\ \mathsf{QBI}\ \&\ \mathsf{UFib}\ \&\ \mathsf{SGrad}.$$

Denoting by $\mathtt{O}_{\mathtt{iso}}$ the subcategory of $\mathtt{O}$ consisting of all isomorphisms we therefore have by SGrad

$$\mathtt{O}_{\mathtt{iso}} = \{f : S \to T;\ e(f) = 0\} = \{f : S \to T;\ e(F) = 0 \text{ for each fiber } F \text{ of } f\}.$$

Another consequence of the strict grading assumption is that $T \in \mathtt{O}$ is local terminal if and only if $e(T) = 0$.

Definition 6.1. A *Markl* $\mathtt{O}$*-operad* in a symmetric monoidal category $\mathtt{V}$ is a presheaf $\mathcal{M} : \mathtt{O}_{\mathtt{iso}}^{\mathrm{op}} \to \mathtt{V}$ with values in $\mathtt{V}$ equipped, for each elementary morphism $F \triangleright T \xrightarrow{\phi} S$ as in Definition 5.1, with a "circle product"

$$\circ_\phi : \mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T). \tag{43}$$

These operations must satisfy the following set of axioms.

(i) Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms such that $|\psi|(j) = i$ and let $\xi : T \to P$ be the composite $\psi\phi$. Then the diagram

$$\begin{tikzcd} & \mathcal{M}(P) \otimes \mathcal{M}(\xi^{-1}(i)) \arrow[dr, "\circ_\xi"] & \\ \mathcal{M}(P) \otimes \mathcal{M}(\psi^{-1}(i)) \otimes \mathcal{M}(\phi^{-1}(j)) \arrow[ur, "\mathbb{1}\otimes\circ_{\phi_i}"] \arrow[dr, "\circ_\psi\otimes\mathbb{1}"'] & & \mathcal{M}(T) \\ & \mathcal{M}(S) \otimes \mathcal{M}(\phi^{-1}(j)) \arrow[ur, "\circ_\phi"'] & \end{tikzcd} \tag{44}$$

commutes.

(ii) Let us consider the diagram

$$\begin{tikzcd} & P' \arrow[dr, "{(\psi', i)}"] & \\ T \arrow[ur, "{(\phi', j)}"] \arrow[dr, "{(\phi'', l)}"'] & & S \\ & P'' \arrow[ur, "{(\psi'', k)}"'] & \end{tikzcd} \tag{45}$$

of elementary morphisms with disjoint fibers as in Corollary 5.8. Then the diagram

$$\begin{tikzcd} \mathcal{M}(S) \otimes \mathcal{M}(G') \otimes \mathcal{M}(F') \arrow[r, "\circ_{\psi'}\otimes\mathbb{1}"] & \mathcal{M}(P') \otimes \mathcal{M}(F') \arrow[d, "\circ_{\phi'}"] \\ \mathcal{M}(S) \otimes \mathcal{M}(F'') \otimes \mathcal{M}(G'') \arrow[u, "\mathbb{1}\otimes(\sigma''^{-1})^*\otimes\sigma'^*"] & \mathcal{M}(T) \\ \mathcal{M}(S) \otimes \mathcal{M}(G'') \otimes \mathcal{M}(F'') \arrow[u, "\mathbb{1}\otimes\tau"] \arrow[r, "\circ_{\psi''}\otimes\mathbb{1}"] & \mathcal{M}(P'') \otimes \mathcal{M}(F'') \arrow[u, "\circ_{\phi''}"'] \end{tikzcd} \tag{46}$$

commutes whenever $F' \triangleright T \xrightarrow{\phi'} P'$, $F'' \triangleright T \xrightarrow{\phi''} P''$, $G' \triangleright P' \xrightarrow{\psi'} S$ and $G'' \triangleright P'' \xrightarrow{\psi''} S$, and the maps $(\sigma''^{-1})^*$ and σ'^* are induced by quasibijections (42).

(iii) For every commutative diagram

$$\begin{tikzcd} T' \arrow[r, "\omega", "\cong"'] \arrow[d, "\phi'"] & T'' \arrow[d, "\phi''"] \\ S' \arrow[r, "\sigma", "\sim"'] & S'' \end{tikzcd}$$

where ω is an isomorphism, σ a quasibijection, and $F' \triangleright_i T' \xrightarrow{\phi'} S'$, $F'' \triangleright_j T'' \xrightarrow{\phi''} S''$, the diagram

$$\begin{tikzcd} \mathcal{M}(F'') \otimes \mathcal{M}(S'') \arrow[r, "\circ_{\phi''}"] \arrow[d, "\bar{\omega}^* \otimes \sigma^*"', "\cong"] & \mathcal{M}(T'') \arrow[d, "\cong"', "\omega^*"] \\ \mathcal{M}(F') \otimes \mathcal{M}(S') \arrow[r, "\circ_{\phi'}"'] & \mathcal{M}(T')\,, \end{tikzcd} \tag{47}$$

in which $\bar{\omega} : F' \to F''$ is the induced map (7) of fibers, commutes.

A Markl operad $\mathcal{M}$ is *unital* if one is given, for each trivial U, a map $\eta_U : \Bbbk \to \mathcal{M}(U)$ such that the diagram

$$\begin{tikzcd} \mathcal{M}(U) \otimes \mathcal{M}(T) \arrow[r, "\circ_!"] & \mathcal{M}(T) \arrow[d, equal] \\ \Bbbk \otimes \mathcal{M}(T) \arrow[u, "\eta_U \otimes 1\!\!1"] \arrow[r, equal, "\cong"] & \mathcal{M}(T) \end{tikzcd} \tag{48}$$

where $T \triangleright T \xrightarrow{!} U$ is the unique map, commutes whenever T is such that $e(T) \geq 1$.

Remark 6.2. The definition above is more general than we need in the rest of the paper but we believe it will be useful in a future. Since we assume the strong blow-up axiom, all pairs of morphisms with disjoint fibers are harmonic by Corollary 5.7. Thus in Axiom (ii) the morphisms σ' and σ'' are the identities. Denoting $F := F' = G''$ and $G := G' = F''$, diagram (46) takes the form

$$\begin{tikzcd} \mathcal{M}(S) \otimes \mathcal{M}(G) \otimes \mathcal{M}(F) \arrow[r, "\circ_{\psi'} \otimes 1\!\!1"] & \mathcal{M}(P') \otimes \mathcal{M}(F) \arrow[d, "\circ_{\phi'}"] \\ & \mathcal{M}(T) \\ \mathcal{M}(S) \otimes \mathcal{M}(F) \otimes \mathcal{M}(G) \arrow[uu, "1\!\!1 \otimes \tau"] \arrow[r, "\circ_{\psi''} \otimes 1\!\!1"] & \mathcal{M}(P'') \otimes \mathcal{M}(G)\,. \arrow[u, "\circ_{\phi''}"'] \end{tikzcd}$$

Let $\mathtt{O}_{\mathtt{LTrm}}$ be the operadic subcategory of $\mathtt{O}$ consisting of all local terminal objects of $\mathtt{O}$. Denote by $\mathbf{1}_{\mathtt{Term}} : \mathtt{O}_{\mathtt{LTrm}} \to \mathtt{V}$ the constant functor, i.e. the functor such that $\mathbf{1}_{\mathtt{Term}}(u) = \Bbbk$ for each local terminal $u \in \mathtt{O}$. Since $\mathtt{O}_{\mathtt{LTrm}}$ is equivalent, as a category, to the discrete groupoid of trivial objects, for a unital Markl operad $\mathcal{M}$ the collection $\{\eta_U : \Bbbk \to \mathcal{M}(U)\}$ of unit maps extends uniquely into a transformation

$$\eta : \mathbf{1}_{\mathtt{Term}} \to \iota^* \mathcal{M} \tag{49}$$

from the constant functor $\mathbf{1}_{\mathtt{Term}}$ to the restriction of $\mathcal{M}$ along the inclusion $\iota : \mathtt{O}_{\mathtt{LTrm}} \hookrightarrow \mathtt{O}$. The component $\eta_u : \Bbbk \to \mathcal{M}(u)$ of that extension is, for u local terminal, given by $\eta_u := !^* \eta_U$, where $! : u \to U$ is the unique map to a trivial U. Transformation (49) of course amounts to a family of maps $\eta_u : \Bbbk \to \mathcal{M}(u)$ given for each local terminal $u \in \mathtt{O}_{\mathtt{LTrm}}$, such that the diagram

$$\begin{tikzcd} \mathcal{M}(u) \arrow[r, "!^*"] & \mathcal{M}(v) \\ \Bbbk \arrow[u, "\eta_u"] \arrow[r, equal] & \Bbbk \arrow[u, "\eta_v"'] \end{tikzcd} \tag{50}$$

commutes for each (unique) map $! : v \to u$ of local terminal objects. We will call the components $\eta_u : \Bbbk \to \mathcal{M}(u)$ of the transformation (49) the *extended units*.

For each T with $e(T) \geq 1$ and $F \triangleright T \xrightarrow{!} u$ with u a local terminal object, one has a map $\vartheta(T,u) : \mathcal{M}(F) \to \mathcal{M}(T)$ defined by the diagram

$$\begin{tikzcd}
\mathcal{M}(u) \otimes \mathcal{M}(F) \arrow[r, "\circ_!"] & \mathcal{M}(T) \\
\mathcal{M}(U) \otimes \mathcal{M}(F) \arrow[u, "!^* \otimes \mathbb{1}"] & \\
\Bbbk \otimes \mathcal{M}(F) \arrow[u, "\eta_U \otimes \mathbb{1}"] \arrow[r, equal, "\cong"] & \mathcal{M}(F) \,. \arrow[uu, "\vartheta(T,u)"']
\end{tikzcd} \tag{51}$$

Notice that the composite of the maps in the left column equals $\eta_u \otimes \mathbb{1}$, where η_u is a component of the extension (49).

The unitality offers a generalization of Axiom (iii) of Markl operads which postulates for each commutative diagram

$$\begin{tikzcd}
T' \arrow[r, "\omega", "\cong"'] \arrow[d, "\phi'"] \arrow[dr, "\phi"] & T'' \arrow[d, "\phi''"] \\
S' \arrow[r, "\sigma", "\cong"'] & S''
\end{tikzcd} \tag{52}$$

where the horizontal maps are isomorphisms and the vertical maps are elementary, with $F' \triangleright_i T' \xrightarrow{\phi'} S'$, $F'' \triangleright_j T'' \xrightarrow{\phi''} S''$, the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{M}(F) \otimes \mathcal{M}(S'') & \mathcal{M}(F'') \otimes \mathcal{M}(S'') \arrow[l, "\omega_j^* \otimes \mathbb{1}"', "\cong"] \arrow[r, "\circ_{\phi''}"] & \mathcal{M}(T'') \arrow[d, "\omega^*", "\cong"'] \\
\mathcal{M}(F') \otimes \mathcal{M}(S'') \arrow[u, "\vartheta(F,\sigma^{-1}(j)) \otimes \mathbb{1}"] \arrow[r, "\mathbb{1} \otimes \sigma^*", "\cong"'] & \mathcal{M}(F') \otimes \mathcal{M}(S') \arrow[r, "\circ_{\phi'}"'] & \mathcal{M}(T')
\end{tikzcd} \tag{53}$$

in which $F := \phi^{-1}(j)$ and $\omega_j : F \to F''$ is the induced map of fibers. Notice that if σ is a quasibijection, (53) implies (47).

Definition 6.3. A Markl operad $\mathcal{M}$ is *strictly unital* if all the maps $\vartheta(T,u)$ in (51) are identities. It is 1-*connected* if the unit maps $\eta_U : \Bbbk \to \mathcal{M}(U)$ are isomorphisms for each trivial U.

If $\mathcal{M}$ is strictly unital, then $\mathcal{M}(F) = \mathcal{M}(F')$ in (53), so this diagram takes a particularly simple form, namely

$$\begin{tikzcd}
\mathcal{M}(F'') \otimes \mathcal{M}(S'') \arrow[r, "\circ_{\phi''}"] \arrow[d, "\omega_j^* \otimes \sigma^*"', "\cong"] & \mathcal{M}(T'') \arrow[d, "\omega^*", "\cong"'] \\
\mathcal{M}(F') \otimes \mathcal{M}(S') \arrow[r, "\circ_{\phi'}"] & \mathcal{M}(T') \,.
\end{tikzcd} \tag{54}$$

The following lemma is an easy exercise on definitions.

Lemma 6.4. *A* 1*-connected Markl operad is strictly unital if and only if, for each* $F \triangleright T \to u$ *with* u *a local terminal object, one has* $\mathcal{M}(F) = \mathcal{M}(T)$ *and the diagram*

$$\begin{tikzcd}
\mathcal{M}(u) \otimes \mathcal{M}(F) \arrow[rr, "\circ_!"] & & \mathcal{M}(T) \arrow[d, equal] \\
\mathcal{M}(U) \otimes \mathcal{M}(F) \arrow[u, "!^* \otimes \mathbb{1}"'] \arrow[r, "(\eta_U \otimes \mathbb{1})^{-1}"] & \Bbbk \otimes \mathcal{M}(F) \arrow[r, "\cong"] & \mathcal{M}(F)
\end{tikzcd} \tag{55}$$

commutes. Here ! *denotes the corresponding unique map to local terminal objects.*

Let us introduce similar terminology for "standard" $\mathtt{O}$-operads. In this framework, $\mathbf{1}_{\mathtt{O}_{\mathtt{LTrm}}}$ will denote the constant $\mathtt{O}_{\mathtt{LTrm}}$-operad. As for Markl operads, the collection $\{\eta_U : \Bbbk \to \mathcal{P}(U)\}$ of unit maps of an $\mathtt{O}$-operad $\mathcal{P}$ extends uniquely to a transformation

$$\eta : \mathbf{1}_{\mathtt{O}_{\mathtt{LTrm}}} \to \iota^* \mathcal{P} \tag{56}$$

of $\mathtt{O}_{\mathtt{LTrm}}$-operads. One has an obvious analog of diagram (51), and the strict unitality and 1-connectedness for $\mathtt{O}$-operads is defined analogously. The main result of this section reads:

Theorem 6.5. *There is a natural forgetful functor from the category of strictly unital* $\mathtt{O}$*-operads to the category of strictly unital Markl* $\mathtt{O}$*-operads, which restricts to an isomorphism of the subcategories of* 1*-connected operads.*

Example 6.6. Constant-free May operads recalled in the introduction are operads over the operadic category $\mathtt{Fin}_{\rm semi}$ of non-empty finite ordinals and their surjections. Let us analyze the meaning of the above definitions and results in this particular case. With respect to the canonical grading, cf. Section 2, elementary morphisms in the operadic category of finite ordinals $\mathtt{Fin}_{\rm semi}$ are precisely order-preserving surjections

$$\pi(m,i,n) : \overline{m+n-1} \twoheadrightarrow \overline{m},\ m \geq 1,\ n \geq 2,$$

uniquely determined by the property that

$$|\pi(m,i,n)^{-1}(j)| = \begin{cases} 1 \text{ if } j \neq i \\ n \text{ if } j = i. \end{cases} \tag{57}$$

Since $\overline{1}$ is the only local terminal object of $\mathtt{Fin}_{\rm semi}$, the strict unitality is the same as the ordinary one and all isomorphisms are quasibijections. A $(\mathtt{Fin}_{\rm semi})_{\mathtt{iso}}$-presheaf turns out to be a collection $\{\mathcal{M}(n)\}_{n\geq 1}$ of Σ_n-modules, and elementary maps (57) induce operations

$$\circ_i := \circ_{\pi(m,i,n)} : \mathcal{M}(m) \otimes \mathcal{M}(n) \to \mathcal{M}(m+n-1),\ n \geq 2,\ 1 \leq i \leq m,$$

which satisfy the standard axioms listed e.g. in [28, Definition 1.1]. Theorem 6.5 in this case states the well-known fact that the category of unital May operads with $\mathcal{P}(1) = \Bbbk$ is isomorphic to the category of unital Markl operads with $\mathcal{M}(1) = \Bbbk$.

Proof of Theorem 6.5. Let $\mathcal{P}$ be a strictly unital $\mathtt{O}$-operad with composition laws γ_f. If $\omega : T' \to T''$ is an isomorphism, we define $\omega^* : \mathcal{P}(T'') \to \mathcal{P}(T')$ by the diagram

$$\begin{tikzcd}
\mathcal{P}(T'') \otimes \mathcal{P}(\omega) \arrow[r, "\gamma_\omega"] & \mathcal{P}(T') \\
\mathcal{P}(T'') \otimes \Bbbk \arrow[u, "1\!\!1 \otimes \eta_\omega"] \arrow[r, equal, "\cong"] & \mathcal{P}(T'') \arrow[u, "\omega^*"']
\end{tikzcd} \tag{58}$$

in which $\mathcal{P}(\omega)$ denotes the product $\mathcal{P}(u_1) \otimes \cdots \otimes \mathcal{P}(u_s)$ over the fibers $u_1, \ldots, u_s$ of ω and, likewise, $\eta_\omega := \eta_{u_1} \otimes \cdots \otimes \eta_{u_s}$. It is simple to show that this construction is functorial, making $\mathcal{P}$ an $\mathtt{O}_{\mathtt{iso}}$-presheaf in $\mathtt{V}$. In particular, ω^* is an isomorphism. For an elementary $F \triangleright_i T \xrightarrow{\phi} S$ we define $\circ_\phi : \mathcal{P}(S) \otimes \mathcal{P}(F) \to \mathcal{P}(T)$ by the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{P}(S) \otimes \mathcal{P}(U_1) \otimes \cdots \otimes \mathcal{P}(U_{i-1}) \otimes \mathcal{P}(F) \otimes \mathcal{P}(U_{i+1}) \otimes \cdots \otimes \mathcal{P}(U_{|S|}) \arrow[r, "\gamma_\phi"] & \mathcal{P}(T) \\
\mathcal{P}(S) \otimes \Bbbk^{\otimes(i-1)} \otimes \mathcal{P}(F) \otimes \Bbbk^{\otimes(|T|-i)} \arrow[u] \arrow[r, equal, "\cong"] & \mathcal{P}(S) \otimes \mathcal{P}(F) \arrow[u, "\circ_\phi"']
\end{tikzcd}$$

in which the left vertical map is induced by the unit morphisms of $\mathcal{P}$ and the identity automorphism of $\mathcal{P}(F)$. We claim that the $\mathtt{O}_{\mathtt{iso}}$-presheaf $\mathcal{P}$ with operations $\circ_\phi$ defined above is a Markl operad.

It is simple to check that these $\circ_\phi$'s satisfy the associativities (i) and (ii) of a Markl operad. To prove Axiom (iii), consider diagram (52) and invoke Axiom (i) of an operad over an operadic category, see Definition 1.1, once for $\phi = \sigma\phi'$ and once for $\phi = \phi''\omega$ in place of $h = fg$. We will get two commutative squares sharing the edge γ_ϕ. Putting them side-by-side as in

$$\begin{tikzcd}
\bigotimes_k \mathcal{P}(\phi'_k) \otimes \mathcal{P}(\sigma) \otimes \mathcal{P}(S'') \arrow[r, "\otimes_k \gamma_{\phi'_k} \otimes 1\!\!1"] \arrow[d, "1\!\!1 \otimes \gamma_\sigma"'] & \mathcal{P}(\phi) \otimes \mathcal{P}(S'') \arrow[d, "\gamma_\phi"] & \mathcal{P}(\omega) \otimes \mathcal{P}(\phi'') \otimes \mathcal{P}(S'') \arrow[l, "\bigotimes_i \gamma_{\omega_i} \otimes 1\!\!1"'] \arrow[d, "1\!\!1 \otimes \gamma_{\phi''}"] \\
\mathcal{P}(\phi') \otimes \mathcal{P}(S') \arrow[r, "\gamma_{\phi'}"] & \mathcal{P}(T') & \mathcal{P}(\omega) \otimes \mathcal{P}(T'') \arrow[l, "\gamma_\omega"']
\end{tikzcd}$$

produces the central hexagon in the diagram

$$\begin{tikzcd}
 & \boxed{\mathcal{P}(\phi'')\otimes\mathcal{P}(S'')} \arrow[rrr, equal] \arrow[d, "\bigotimes_k \omega_i^*\otimes\mathbb{1}"', "\cong"] & & & \Bbbk\otimes\mathcal{P}(\phi'')\otimes\mathcal{P}(S'') \arrow[d, "\eta_\omega\otimes\mathbb{1}\otimes\mathbb{1}"] & \\
 & \boxed{\mathcal{P}(\phi)\otimes\mathcal{P}(S'')} \arrow[ddrr, dashed, "\gamma_\phi"] & & & \mathcal{P}(\omega)\otimes\mathcal{P}(\phi'')\otimes\mathcal{P}(S'') \arrow[lll] \arrow[d, "\mathbb{1}\otimes\gamma_{\phi''}"'] & \\
 & \bigotimes_k \mathcal{P}(\phi'_k)\otimes\mathcal{P}(\sigma)\otimes\mathcal{P}(S'') \arrow[u, "\bigotimes_k \gamma_{\phi'_k}\otimes\mathbb{1}"] \arrow[dr, "\mathbb{1}\otimes\gamma_\sigma"] & & & \boxed{\mathcal{P}(\omega)\otimes\mathcal{P}(T'')} \arrow[dl, "\gamma_\omega"'] & \\
\mathcal{P}(\phi')\otimes\Bbbk\otimes\mathcal{P}(S'') \arrow[ur, "\mathbb{1}\otimes\eta_\sigma\otimes\mathbb{1}"] \arrow[d, equal] & & \boxed{\mathcal{P}(\phi')\otimes\mathcal{P}(S')} \arrow[r, "\gamma_{\phi'}"] & \boxed{\mathcal{P}(T')} & & \Bbbk\otimes\mathcal{P}(T'') \arrow[ul, "\eta_\omega\otimes\mathbb{1}"'] \arrow[d, equal] \\
\boxed{\mathcal{P}(\phi')\otimes\mathcal{P}(S'')} \arrow[urr, "\mathbb{1}\otimes\sigma^*", "\cong"'] & & & & & \mathcal{P}(T'')\,. \arrow[ull, "\omega^*"', "\cong"]
\end{tikzcd}$$

The remaining arrows of this diagram are constructed using the $\mathtt{O}_{\mathtt{iso}}$-presheaf structure of $\mathcal{P}$ and the extended units. The boxed terms in the above diagram form the internal hexagon in

$$\begin{tikzcd}
\mathcal{P}(F)\otimes\mathcal{P}(S'') \arrow[dr] & & \mathcal{P}(F'')\otimes\mathcal{P}(S'') \arrow[ll, "\omega_j^*\otimes\mathbb{1}"', "\cong"] \arrow[rr, "\circ_{\phi''}"] \arrow[d] & & \mathcal{P}(T'') \arrow[dl] \arrow[ddd, "\cong"', "\omega^*"] \\
 & \mathcal{P}(\phi)\otimes\mathcal{P}(S'') & \mathcal{P}(\phi'')\otimes\mathcal{P}(S'') \arrow[l, "\cong"'] \arrow[r] & \mathcal{P}(\omega)\otimes\mathcal{P}(T'') \arrow[d] & \\
 & \mathcal{P}(\phi')\otimes\mathcal{P}(S'') \arrow[u] \arrow[r, "\cong"] & \mathcal{P}(\phi')\otimes\mathcal{P}(S') \arrow[r] & \mathcal{P}(T') \arrow[dr, equal] & \\
\mathcal{P}(F')\otimes\mathcal{P}(S'') \arrow[uuu, "\vartheta(F,\sigma^{-1}(j))\otimes\mathbb{1}"] \arrow[ur] \arrow[rr, "\mathbb{1}\otimes\sigma^*", "\cong"'] & & \mathcal{P}(F')\otimes\mathcal{P}(S') \arrow[u] \arrow[rr, "\circ_{\phi'}"] & & \mathcal{P}(T')\,.
\end{tikzcd}$$

The commutativity of the outer hexagon follows from the commutativity of the inner one. We recognize in it diagram (53) with $\mathcal{P}$ in place of $\mathcal{M}$. Since (53) implies (47) for σ a quasibijection, Axiom (iii) is verified. Is is easy to check that the strict extended unit (56) is also the one for $\mathcal{P}$ considered as a Markl operad.

Conversely, let $\mathcal{M}$ be a Markl operad. We are going to define, for each $f : S \to T$ with fibers $F_1, \ldots, F_s$, the composition law

$$\gamma_f : \mathcal{M}(T)\otimes\mathcal{M}(F) \longrightarrow \mathcal{M}(S) \tag{59}$$

where, as several times before, $\mathcal{M}(F)$ denotes $\mathcal{M}(F_1)\otimes\cdots\otimes\mathcal{M}(F_s)$. If f is an isomorphism, then all its fibers are local terminal, so $\mathcal{M}(F)\cong\Bbbk$ by the strict unitality and the 1-connectivity of $\mathcal{M}$. In this case we define γ_f as the composite

$$\mathcal{M}(T)\otimes\mathcal{M}(F)\cong\mathcal{M}(T)\xrightarrow{f^*}\mathcal{M}(S) \tag{60}$$

using the $\mathtt{O}_{\mathtt{iso}}$-presheaf structure of $\mathcal{M}$.

Assume now that $f \in \mathtt{O}_{\mathtt{ord}}$ and that all local terminal fibers of f are trivial. If f is an isomorphism it must be the identity by Corollary 2.6. If it is not the case, at least one fiber of f has grade ≥ 1 and we decompose f, using the strong blow-up axiom, into a chain of elementary morphisms. The operation γ_f will then be defined as the composite of $\circ$-operations corresponding to these elementary morphisms. Let us make this procedure more precise.

To understand the situation better, consider two elementary morphisms ϕ, ψ with (i,j)-disjoint fibers as in Lemma 5.5 and their composite $\xi = \psi(\phi)$. Notice that

$$\mathcal{M}(\xi)\cong\mathcal{M}(\xi^{-1}(i))\otimes\mathcal{M}(\xi^{-1}(k))$$

by the strict unitality and the 1-connectivity of $\mathcal{M}$. In this particular case we define γ_ξ by the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{M}(P) \otimes \mathcal{M}(\xi^{-1}(i)) \otimes \mathcal{M}(\xi^{-1}(k)) \arrow[r, "\cong"] \arrow[d, "1\!\!1 \otimes (\phi_i^*)^{-1} \otimes 1\!\!1"'] & \mathcal{M}(P) \otimes \mathcal{M}(\xi) \arrow[dd, "\gamma_\xi"] \\
\mathcal{M}(P) \otimes \mathcal{M}(\psi^{-1}(i)) \otimes \mathcal{M}(\phi^{-1}(k)) \arrow[d, "1\!\!1 \otimes \circ_\psi"'] & \\
\mathcal{M}(S) \otimes \mathcal{M}(\phi^{-1}(k)) \arrow[r, "\circ_\phi"] & \mathcal{M}(T)\,,
\end{tikzcd}$$

or, in shorthand, by $\gamma_\xi := \circ_\phi(1\!\!1 \otimes \circ_\psi)$.

Now take $f : S \to T \in \mathtt{O}_{\mathtt{ord}}$ whose fibers of grade ≥ 1 are $F_1, \ldots, F_k$ and the remaining fibers are trivial. Using the strong blow-up axiom we factorize f into a chain

$$S = S_1 \xrightarrow{\phi_1} S_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} S_k = T \tag{61}$$

in which each ϕ_i is elementary with the unique nontrivial fiber F_i, $1 \leq i \leq k$; we leave the details on how to obtain such a factorization to the reader. We then define

$$\gamma_f := \circ_{\phi_1}(\circ_{\phi_2} \otimes 1\!\!1) \cdots (\circ_{\phi_k} \otimes 1\!\!1^{\otimes(k-1)}) : \mathcal{M}(T) \otimes \mathcal{M}(F_k) \otimes \cdots \otimes \mathcal{M}(F_1) \longrightarrow \mathcal{M}(S).$$

If $f : S \to T$ is a general morphism in $\mathtt{O}$, we use Lemma 2.13 to factorize it as $f : S \xrightarrow{\omega} X \xrightarrow{\psi} T$ with an isomorphism ω and $\psi \in \mathtt{O}_{\mathtt{ord}}$ all of whose fibers are trivial. Notice that, due to the strict unitality and 1-connectivity, one has $\mathcal{M}(\psi) \cong \mathcal{M}(f)$. We then define γ_f by the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{M}(\psi) \otimes \mathcal{M}(T) \arrow[r, "\cong"] \arrow[d, "\gamma_\psi"] & \mathcal{M}(f) \otimes \mathcal{M}(T) \arrow[d, "\gamma_f"'] \\
\mathcal{M}(X) \arrow[r, "\omega^*"] & \mathcal{M}(S)\,.
\end{tikzcd}$$

The extended units are given by the extended units of $\mathcal{M}$ in the obvious way.

Our definition of the γ_f-operations does not depend on the choices: the commutativity of (53) that holds for unital Markl operads guarantees the independence on the factorization $f = \psi\omega$, while the commutativity of (45) implies the independence on the choice of the decomposition (61). We leave to the reader the tedious but straightforward verification that $\mathcal{M}$ with the above composition laws is a strictly unital $\mathtt{O}$-operad. $\square$

We are going to adapt the notion of algebra over operad, cf. Definition 1.10, to the realm of Markl operads.

Definition 6.7. An *algebra* over a 1-connected Markl operad $\mathcal{M}$ in a symmetric monoidal category $\mathtt{V}$ is a collection $A = \{A_c \mid c \in \pi_0(\mathtt{O})\}$ of objects of $\mathtt{V}$ together with structure maps

$$\Big\{\alpha_T : \mathcal{M}(T) \otimes \bigotimes_{c \in \pi_0(s(T))} A_c \longrightarrow A_{\pi_0(T)}\Big\}_{T \in \mathtt{O}}\,. \tag{62}$$

These operations are required to satisfy the following axioms.

(i) Unitality: for each component $c \in \pi_0(\mathtt{O})$ the diagram

$$\begin{tikzcd}
\mathcal{M}(U_c) \otimes A_c \arrow[r, "\alpha_{U_c}"] & A_c \arrow[d, equal] \\
\Bbbk \otimes A_c \arrow[u, "\eta_{U_c}"] \arrow[r, "\cong"] & A_c
\end{tikzcd}$$

commutes.

(ii) Equivariance: let $f : S \to T$ be an isomorphism with fibers $u_1, \dots, u_s$. For $1 \le i \le s$ put $c_i := \pi_0(s_i(S))$ and $d_i := \pi_0(s_i(T))$. Then the diagram

$$\begin{tikzcd}
\mathcal{M}(T) \otimes A_{c_1} \otimes \cdots \otimes A_{c_s} \arrow[rr, "\cong"] \arrow[d, "f^* \otimes 1\!\!1^{\otimes s}"'] & & \mathcal{M}(T) \otimes \Bbbk \otimes A_{c_1} \otimes \cdots \otimes \Bbbk \otimes A_{c_s} \arrow[d, "1\!\!1 \otimes \eta_{u_1} \otimes \cdots \otimes \eta_{u_s}"] \\
\mathcal{M}(S) \otimes A_{c_1} \otimes \cdots \otimes A_{c_s} \arrow[d, "\alpha_S"'] & & \mathcal{M}(T) \otimes \mathcal{M}(u_1) \otimes A_{c_1} \otimes \cdots \otimes \mathcal{M}(u_s) \otimes A_{c_s} \arrow[d, "1\!\!1 \otimes \alpha_{u_1} \otimes \cdots \otimes \alpha_{u_s}"] \\
A_{\pi_0(S)} \arrow[r, equal] & A_{\pi_0(T)} & \mathcal{M}(T) \otimes A_{d_1} \otimes \cdots \otimes A_{d_s} \arrow[l, "\alpha_T"']
\end{tikzcd}$$

commutes.

(iii) Associativity: for an elementary map $F \triangleright_i S \xrightarrow{\phi} T$, the diagram

$$\begin{tikzcd}
\mathcal{M}(S) \otimes A_{c_1} \otimes \cdots A_{c_{i-1}} \otimes A_{c_i} \otimes \cdots \otimes A_{c_{t+s-1}} \arrow[r, "\alpha_S"] & A_{\pi_0(S)} \arrow[ddd, equal] \\
\mathcal{M}(T) \otimes \mathcal{M}(F) \otimes A_{c_1} \otimes \cdots \otimes A_{c_{i-1}} \otimes A_{c_i} \otimes \cdots \otimes A_{c_{t+s-1}} \arrow[u, "\circ_\phi \otimes 1\!\!1^{\otimes t+s-1}"] \arrow[d, "1\!\!1 \otimes \tau \otimes 1\!\!1^{\otimes t-i}"'] & \\
\mathcal{M}(T) \otimes A_{c_1} \otimes \cdots \otimes A_{c_{i-1}} \otimes \mathcal{M}(F) \otimes A_{c_i} \otimes \cdots \otimes A_{c_{t+s-1}} \arrow[d, "1\!\!1 \otimes 1\!\!1^{\otimes i} \otimes \alpha_F \otimes 1\!\!1^{\otimes t-i}"'] & \\
\mathcal{M}(T) \otimes A_{c_1} \otimes \cdots A_{c_{i-1}} \otimes A_{\pi_0(F)} \otimes \cdots \otimes A_{c_{t+s-1}} \arrow[r, "\alpha_T"] & A_{\pi_0(T)}
\end{tikzcd}$$

commutes, where $s = |S|$, $t = |T|$, $c_j := \pi_0(s_j)$ for $1 \le j \le s+t-1$ and

$$\tau : \mathcal{M}(F) \otimes A_{c_1} \otimes \cdots \otimes A_{c_{i-1}} \longrightarrow A_{c_1} \otimes \cdots \otimes A_{c_{i-1}} \otimes \mathcal{M}(F)$$

the commutativity constraint in $\mathtt{V}$.

Notice that in the situation of item (ii) of Definition 6.7, one has $s_i(S) = s(u_i)$, $\pi_0(s_i(T)) = \pi_0(u_i)$ and $\pi_0(S) = \pi_0(T)$. Likewise in (iii),

$$\pi_0(s_j(T)) = \begin{cases} \pi_0(s_{|\phi|^{-1}(j)}(S)) & \text{if } j \neq i \\ \pi_0(F) & \text{otherwise.} \end{cases} \tag{63}$$

Proposition 6.8. *The category of algebras of a strictly unital* 1*-connected Markl operad* $\mathcal{M}$ *is isomorphic to the category of algebras of the corresponding operad* $\mathcal{P}$.

Proof. An exercise in the axioms of operads and their algebras. ☐

Let us formulate Definition 6.7 for the particular case of algebras in the category of graded vector spaces.

Definition 6.9. An *algebra* over a 1-connected Markl operad $\mathcal{M}$ in the category $\mathtt{Vect}$ of graded $\Bbbk$-vector spaces is a collection $A = \{A_c \mid c \in \pi_0(\mathtt{O})\}$ together with structure maps

$$\mathcal{M}(T) \otimes \bigotimes_{c \in \pi_0(s(T))} A_c \ni x \otimes a_1 \otimes \cdots \otimes a_s \longmapsto x(a_1, \dots, a_s) \in A_{\pi_0(T)}$$

given for each $T \in \mathtt{O}$. These operations are required to satisfy the following axioms.

(i) Unitality: for a local terminal u, $1 \in \Bbbk \cong \mathcal{M}(u)$ and $a \in A_{\pi_0(s(u))}$, denote $ua := 1(a)$. Then $Ua = a$ for U a chosen local terminal object.

(ii) Equivariance: for an isomorphism $f : S \to T$ with fibers $u_1, \dots, u_s$ and $x \in \mathcal{M}(T)$,

$$f^*(x)(a_1, \dots, a_s) = x(u_1 a_1, \dots, u_s a_s).$$

(iii) Associativity: for an elementary map $F \triangleright_i S \xrightarrow{\phi} T$, $x \in \mathcal{M}(T)$ and $y \in \mathcal{M}(F)$,

$$\circ_\phi(x, y)(a_1, \dots, a_{i-1}, a_i, \dots, a_{t+s-1}) = (-1)^\varepsilon \cdot x(a_1, \dots, a_{i-1}, y(a_i, \dots), \dots a_{t+s-1}),$$

where $\varepsilon := |y|(|a_1| + \cdots + |a_{i-1}|)$, $s = |S|$ and $t = |T|$.

Example 6.10. A Markl operad $\mathcal{M}$ in $\mathtt{Vect}$ such that $\mathcal{M}(T)$ is for each T a 1-dimensional vector space is called a *cocycle* following the terminology of [20]. An important cocycle is the operad $1_{\mathtt{O}}$ such that $1_{\mathtt{O}}(T) := \Bbbk$ for each $T \in \mathtt{O}$, with all composition laws the identities. Slightly imprecisely, we will call $1_{\mathtt{O}}$ the *terminal* $\mathtt{O}$-operad since it is the linearization of the terminal $\mathtt{O}$-operad over the Cartesian monoidal category of sets.

Less trivial cocycles can be constructed as follows. We say that a graded vector space W is *invertible* if $W \otimes W^{-1} \cong \Bbbk$ for some $W^{-1} \in \mathtt{Vect}$. This clearly means that W is an iterated (de)suspension of the ground field $\Bbbk$. Suppose we are given a map $\mathfrak{l} : \pi_0(\mathtt{O}) \to \mathtt{Vect}$ that assigns to each $c \in \pi_0(\mathtt{O})$ an invertible graded vector space $\mathfrak{l}(c)$. With the notation used in (62) we introduce the cocycle $\mathfrak{D}_{\mathfrak{l}}$ by

$$\mathfrak{D}_{\mathfrak{l}}(T) := \mathfrak{l}(\pi_0(T)) \otimes \bigotimes_{c \in \pi_0(s(T))} \mathfrak{l}(c)^{-1}$$

with the trivial action of $\mathtt{O}_{\mathtt{iso}}$. To define, for $F \triangleright_i S \xrightarrow{\phi} T$, the composition laws

$$\circ_\phi : \mathfrak{D}_{\mathfrak{l}}(F) \otimes \mathfrak{D}_{\mathfrak{l}}(T) \to \mathfrak{D}_{\mathfrak{l}}(S)$$

we need to specify a map

$$\mathfrak{l}(\pi_0(F)) \otimes \bigotimes_{c \in \pi_0(s(F))} \mathfrak{l}(c)^{-1} \otimes \mathfrak{l}(\pi_0(T)) \otimes \bigotimes_{c \in \pi_0(s(T))} \mathfrak{l}(c)^{-1} \longrightarrow \mathfrak{l}(\pi_0(S)) \otimes \bigotimes_{c \in \pi_0(s(S))} \mathfrak{l}(c)^{-1}.$$

To do so, we notice that there is an equality of unordered lists

$$\pi_0(s(F)) \sqcup \pi_0(s(T)) = \pi_0(s(S)) \sqcup \{\pi_0(s_i(T))\},$$

and

$$\pi_0(S) = \pi_0(T) \quad \pi_0(F) = \pi_0(s_i(T)),$$

cf. (63). Keeping this in mind, the composition law $\circ_\phi$ is defined as the canonical isomorphism $\mathfrak{D}_{\mathfrak{l}}(F) \otimes \mathfrak{D}_{\mathfrak{l}}(T) \cong \mathfrak{D}_{\mathfrak{l}}(S)$.

Cocycles of the above form are called *coboundaries*. Notice that $1_{\mathtt{O}} = \mathfrak{D}_{\mathfrak{l}}(T)$ with $\mathfrak{l}$ the constant function such that $\mathfrak{l}(c) := \Bbbk$ for each $c \in \pi_0(s(T))$.

Markl operads in $\mathtt{Vect}$ form a symmetric monoidal category, with the monoidal structure given by the level-wise tensor product and $1_{\mathtt{O}}$ the monoidal unit. As an exercise we recommend to prove the following very useful proposition.

Proposition 6.11. *The categories of* $(\mathcal{M} \otimes \mathfrak{D}_{\mathfrak{l}})$*-algebras and of* $\mathcal{M}$*-algebras in* $\mathtt{Vect}$ *are isomorphic. More precisely, there is a natural one-to-one correspondence between*
– $\mathcal{M}$*-algebras with underlying collection* $A = \{A_c \mid c \in \pi_0(\mathtt{O})\}$*, and*
– $(\mathcal{M} \otimes \mathfrak{D}_{\mathfrak{l}})$*-algebras with underlying collection* $A = \{A_c \otimes \mathfrak{l}(c)^{-1} \mid c \in \pi_0(\mathtt{O})\}$.

Proposition 6.11 should be compared to Lemma II.5.49 of [33]. In the classical operad theory, algebras can equivalently be described as morphism to the endomorphism operad. We are going to give similar description also in our setup. While the classical construction assigns the endomorphism operad $\mathcal{E}nd_V$ to a vector space V, here we start with a collection

$$V = \{V_c \mid c \in \pi_0(\mathtt{O})\} \tag{64}$$

of graded vector spaces indexed by the components of $\mathtt{O}$. We moreover assume that to each local terminal object $u \in \mathtt{O}$ we are given a linear map (denoted u again)

$$u : V_{\pi_0(s_1(u))} \to V_{\pi_0(u)} \tag{65}$$

such that, for each map $u \to v$ of local terminal objects with fiber t, the triangle

$$\begin{tikzcd} V_{\pi_0(s_1(u))} \arrow[rr, "u"] \arrow[dr, "t"'] & & V_{\pi_0(u)} \\ & V_{\pi_0(t)} \arrow[ur, "v"'] & \end{tikzcd} \tag{66}$$

commutes. This diagram makes sense since $\pi_0(s_1(u)) = \pi_0(s_1(t))$, by Axiom (iv) of operadic categories applied to $f = \mathbb{1}_u$, $\pi_0(u) = \pi_0(v)$ and $\pi_0(s_1(v)) = \pi_0(t)$. We moreover assume that the maps corresponding to the chosen local terminal objects are the identities. For $T \in \mathtt{O}$ we put

$$\mathcal{E}nd_V(T) := \mathtt{Vect}\big(\textstyle\bigotimes_{c\in\pi_0(s(T))} V_c, V_{\pi_0(T)}\big).$$

We define an action $\mathcal{E}nd_V(T) \ni \alpha \mapsto f^*(\alpha) \in \mathcal{E}nd_V(S)$ of an isomorphism $f : S \to T$ with fibers $u_1, \ldots, u_s$ by

$$f^*(\alpha)(a_1, \ldots, a_s) := \alpha(u_1 a_1, \ldots, u_s a_s),\ a_1 \otimes \cdots \otimes a_s \in \textstyle\bigotimes_{c\in\pi_0(s(T))} V_c. \tag{67}$$

This turns $\mathcal{E}nd_V$ into a functor $\mathtt{O}_{\mathtt{iso}}^{\mathrm{op}} \to \mathtt{Vect}$. The composition law

$$\circ_\phi : \mathcal{E}nd_V(F) \otimes \mathcal{E}nd_V(T) \to \mathcal{E}nd_V(S).$$

is, for an elementary morphism $F \triangleright_i S \xrightarrow{\phi} T$, defined as follows. Assume

$$\alpha : \textstyle\bigotimes_{c\in\pi_0(s(F))} V_c \longrightarrow V_{\pi_0(F)} \in \mathcal{E}nd_V(F),\ \ \beta : \textstyle\bigotimes_{c\in\pi_0(s(T))} V_c \longrightarrow V_{\pi_0(T)} \in \mathcal{E}nd_V(T)$$

and notice that

$$\pi_0(s(S)) = \pi_0(s(F)) \sqcup (\pi_0(s(T)) \setminus \{\pi_0(s_i(T))\}) \ \text{ and } \ \pi_0(F) = \pi_0(s_i(T)).$$

Then

$$\circ_\phi(\alpha\otimes\beta) : \textstyle\bigotimes_{c\in\pi_0(s(S))} V_c \to V_{\pi_0(S)} \in \mathcal{E}nd_V(S)$$

is the map that makes the diagram

$$\begin{tikzcd} \bigotimes_{c\in\pi_0(s(S))} V_c \arrow[r, "\cong"] \arrow[dd, "\circ_\phi(\alpha\otimes\beta)"'] & \bigotimes_{c\in\pi_0(s(F))} V_c \otimes \bigotimes_{c\in\pi_0(s(S))\setminus\{\pi_0(s_i(T))\}} V_c \arrow[d, "\alpha\otimes\mathbb{1}"] \\ & V_{\pi_0(s_i(T))} \otimes \bigotimes_{c\in\pi_0(s(S))\setminus\{\pi_0(s_i(T))\}} V_c \arrow[d, "\cong"] \\ V_{\pi_0(S)} & \bigotimes_{c\in\pi_0(s(S))} V_c \arrow[l, "\beta"'] \end{tikzcd}$$

commute. The result of the above construction is the Markl version of the *endomorphism operad*.

Notice that the components $\eta_u : \Bbbk \to \mathcal{E}nd_V(u)$ of transformation (49) for $\mathcal{M} = \mathcal{E}nd_V$ are given by the maps in (65) as

$$\eta_u(1) := u : V_{\pi_0(s_1(u))} \to V_{\pi_0(u)} \in \mathcal{E}nd_V(u).$$

It is simple to verify that the commutativity of (50) is precisely (66). The induced maps

$$\vartheta(T, u) : \mathcal{E}nd_V(F) \to \mathcal{E}nd_V(T)$$

in (51) are given by the composite, $\vartheta(T, u)(\phi) := u\phi$, with the map (65).

Remark 6.12. The above analysis shows that the morphisms $\vartheta(T,u)$ need not be the identities for a general $\mathcal{E}nd_V$. Endomorphism operads are therefore examples of unital operads that need not be *strictly* unital.

We have the following expected result.

Proposition 6.13. *There is a one-to-one correspondence between $\mathcal{M}$-algebras with underlying collection (64) and morphisms $\mathcal{M} \to \mathcal{E}nd_V$ of Markl operads.*

Proof. Direct verification. □

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