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Koszul duality for operadic categories

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The aim of this paper (which is a sequel to *Operadic categories as a natural environment for Koszul duality*) is to set up the cornerstones of Koszul duality and Koszulity in the context of operads over a large class of operadic categories. In particular, for these operadic categories we will study concrete examples of binary quadratic operads, describe their Koszul duals and prove that they are Koszul. This includes operads (for operadic categories) whose algebras are the most important operad- and PROP-like structures such as the classical operads, their variants such as cyclic or modular operads, and also diverse versions of PROPs such as wheeled properads, dioperads, $\frac{1}{2}$PROPs, and still more exotic objects such as permutads and pre-permutads.

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Introduction

Koszul duality is an important ingredient in the theory of algebraic operads. The classical Koszul duality theory for associative algebras goes a long way back to Priddy [23], but a milestone was the 1994 paper *Koszul duality for operads* by Ginzburg and Kapranov [10], where they generalize it to operads. The key examples are the operads $\mathcal{L}ie$ and $\mathcal{C}om$ for Lie, resp. commutative associative algebras that are Koszul dual to each other, whereas the operad $\mathcal{A}ss$ for associative algebras is self-dual. While many aspects of operad theory can be formulated in general symmetric monoidal categories, such as the category of sets or the category of spaces, Koszul duality theory is really

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a feature specific to algebraic operads, meaning operads in the category `Vect` of graded vector spaces.

By [10, Definition 4.1.3], a symmetric quadratic operad $\mathcal{P}$ is Koszul if the dual dg (abbreviating differential graded) operad $\mathbb{D}(\mathcal{P}^!)$ of its Koszul dual resolves $\mathcal{P}$. The operad $\mathbb{D}(\mathcal{P}^!)$ then provides, using a construction of [16], a canonical explicit L_∞-algebra capturing deformations of $\mathcal{P}$-algebras via its moduli space of Maurer–Cartan elements. Moreover, algebras for $\mathbb{D}(P^!)$ are strongly homotopy $\mathcal{P}$-algebras, whose salient feature is the transfer property over weak homotopy equivalences. The operad $\mathbb{D}(\mathcal{P}^!)$ also leads to the canonical (co)homology theory of $\mathcal{P}$-algebras, cf. [21, Section II.3.8]. This explains the prominent position of quadratic Koszul operads in the traditional operad theory.

The aim of the present article is to implant the theory of quadratic operads and their Koszulity into the context of operadic categories. Operadic categories, introduced by the authors in [2] is a general abstract framework to accommodate operad-like structures. Just as algebraic structures of different kinds can fruitfully be interpreted as algebras over operads, there are also various kinds of operads, which in this theory are interpreted as operads over operadic categories: each operadic category has its attending notion of operad. By carrying out the theory at this level of generality, we get unified proofs of the main theorems of Koszul duality theory for various flavors of operads. Not all operadic categories admit this theory. In order for the theory to work it is necessary to impose several further axioms on top of the general axioms of operadic categories. This requires a considerable amount of groundwork which we have carried out in the companion article *Operadic categories as a natural environment for Koszul duality* [3], whose results are really a prerequisite for the present article. Although the additional axioms are briefly recollected below, we have to refer the reader to [3] for more detailed background, also on the general theory of operadic categories, which it is not practical to reproduce here.

As in the classical case, free operads over operadic categories will play the central rôle in our generalized Koszul duality theory. Following the approach of [17], we use a version of operads whose composition laws are binary. Operad-like structures based on these "partial compositions" were later called Markl operads. The theory of these operads in the context of operadic categories was developed in [3]; we recall their definition including the necessary auxiliary material in Subsection 1.2. In the rest of this introduction, by an operad we will mean a Markl operad, typically denoted by $\mathcal{M}$ in contrast to the generic notation $\mathcal{P}$ for the standard operads [2, Definition 1.1] with the compositions along all fibers performed simultaneously.

After describing free operads, we introduce quadratic operads as those isomorphic to free operads quotiented by quadratic relations. As in the classical setup, each quadratic operad $\mathcal{M}$ possesses a Koszul dual $\mathcal{M}^!$. Given an operad $\mathcal{M}$, we define its dual dg operad $\mathbb{D}(\mathcal{M})$ and, for $\mathcal{M}$ quadratic, construct the canonical morphism $can_{\mathcal{M}} : \mathbb{D}(\mathcal{M}^!) \to \mathcal{M}$. A quadratic operad $\mathcal{M}$ will be called Koszul if $can_{\mathcal{M}}$ induces a component-wise isomorphism of homology. We finally prove that the operads whose algebras are the most relevant operad- and PROP-like structures are Koszul. Our method is to show that the dual dg operads of their Koszul duals are isomorphic to their minimal models described in [4]. Here, as in the classical case [17, Proposition 2.6], this characterizes their Koszulity.

Throughout this article, we will be working with operads in the category of dg vector spaces, although the operadic categories themselves are completely combinatorial and do not depend on any linear structure. The reason is that even to write down the algebraic presentations it is necessary to have a linear structure, and that linear duality is a key ingredient, as are the notions of homology and resolutions.

Koszulity of the semi-classical colored operad whose algebras are modular operads was established, in the setup of groupoid-colored operads, in [25]. We believe that the advantage of our approach is that, after some heavy preparatory work has been done, everything is stripped to bare bones, conveniently hiding details that only complicate the picture, such as the groupoid actions and explicit indices, in the way explained in Remark 5.7.

Plan of the paper

In Section 1 we recall from [3] some additional axioms of operadic categories required in the present paper, and Markl operads in the context of operadic categories. While the underlying structure of a classical operad is a collection of spaces equipped with actions of symmetric groups,

for general operadic categories the situation is subtler. The rôle of underlying collections is played by presheaves on the category $\mathtt{QO}_{\mathtt{vrt}}(e)$ of virtual isomorphisms constructed in the first part of Section 2. The second part of that section describes the precise relation of Markl operads to the category $\mathtt{QO}_{\mathtt{vrt}}(e)$. Free Markl operads are explicitly described in Section 3. With the notion of free operads available, we introduce quadratic operads and their Koszul duals in Section 4.

The remaining sections are devoted to explicit calculations. In Section 5 we study the constant operad $\mathbf{1}_{\mathtt{ggGrc}}$ whose algebras are modular operads. We show that this operad is binary quadratic and that algebras over its Koszul dual are odd (aka twisted) modular operads. In Section 6 we make a similar analysis for operads whose algebras are ordinary and cyclic operads, and pre-permutads. In Section 7 we continue the analysis for wheeled properads, dioperads, $\frac{1}{2}$PROPs and permutads. With the exception of the operads $\mathfrak{K}_{\mathtt{ggGrc}}$, resp. $\mathfrak{pp}$, whose algebras are odd modular operads, resp. pre-permutads, all these operads are terminal operads over an appropriate operadic category.

Section 8 describes the cobar construction and Section 9 the dual dg operad of a Markl operad, needed for the definition of Koszulity. Theorem 9.6 then establishes Koszulity of the operads whose algebras are modular, cyclic and ordinary operads, and wheeled PROPs.

In Section 5 we need to refer to concrete axioms of modular and odd modular operads. Since the only source we are aware of where these axioms are listed in a concise and itemized form is the recent monograph [6], we decided to recall them in the Appendix. For the same reasons we also included itemized axioms of the classical Markl operads so that the reader need not consult the ancient paper [17]. To help the reader navigate through the paper, we included an index of terminology and notation.

Conventions. Operadic categories and related notions were introduced in [2]; some basic concepts of that paper are recalled in [3, Section 1]. We will freely use the terminology and notation from there and, when necessary, refer to concrete definitions, diagrams, results or formulas in those sources. If not stated otherwise, ϕ^* will denote the image of a morphism ϕ under some presheaf.

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1 Recollections

In the first part of this section we recall from [3] some additional requirements on the base operadic category $\mathtt{O}$, formulated in the form of axioms, guaranteeing that free operads admit a nice explicit description. In the second part we recall Markl operads. The material is taken from [3] almost verbatim.

1.1 The Axioms

Intuitively, the axioms below express that an operadic category $\mathtt{O}$ is of combinatorial nature, close to various categories of graphs. The first axiom involves the subcategory $\mathtt{O}_{\mathtt{qb}} \subset \mathtt{O}$ of *quasibijections*, i.e. morphisms in $\mathtt{O}$ all of whose fibers are the chosen local terminal objects; in [2] we called them "trivial."

Invertibility of quasibijections - QBI. *All quasibijections in* $\mathtt{O}$ *are invertible.*

In the next axiom, $\mathtt{O}_{\mathtt{ord}}$ will denote the subcategory of $\mathtt{O}$ with the objects of $\mathtt{O}$, and morphisms $f : S \to T$ of $\mathtt{O}$ such that $|f| : |S| \to |T|$ is order preserving.

Weak blow-up axiom - WBU. *For any $f' : S' \to T$ in $\mathtt{O}_{\mathtt{ord}}$ and morphisms $\pi_i : {f'}^{-1}(i) \to F''_i$ in $\mathtt{O}$, $i \in |T|$, there exists a unique factorization of f'*

$$\begin{array}{ccc} S' & \xrightarrow{\omega} & S'' \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f''} & \\ & T & \end{array}$$

such that $f'' \in \mathtt{O}_{\mathtt{ord}}$ and the maps ω_i between the fibers induced by ω satisfy $\omega_i = \pi_i$ for all $i \in |T|$.

Before we recall the strong version of the above axiom, we need to remind the reader of the notation introduced in a lemma of [3]:

Lemma 1.1 (Lemma 2.4 of [3])**.** *Consider the commutative diagram in an operadic category*

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S'' \\ {\scriptstyle f'}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f''} \\ T' & \xrightarrow{\sigma} & T''. \end{array} \tag{1}$$

Let $j \in |T''|$ and $|\sigma|^{-1}(j) = \{i\}$ for some $i \in |T'|$. Diagram (1) determines:

(i) the map $f'_j : f^{-1}(j) \to \sigma^{-1}(j)$ whose unique fiber equals ${f'}^{-1}(i)$, and

(ii) the induced map $\pi_j : f^{-1}(j) \to {f''}^{-1}(j)$.

If $\sigma^{-1}(j)$ is trivial, in particular if σ is a quasibijection, then π induces a map

$$\pi_{(i,j)} : {f'}^{-1}(i) \to {f''}^{-1}(j) \tag{2}$$

which is a quasibijection if π is.

In the situation of Lemma 1.1 with σ a quasibijection, the *derived sequence* is the sequence of morphisms

$$\left\{\pi_{(i,j)} : {f'}^{-1}(i) \to {f''}^{-1}(j), j = |\sigma|(i)\right\}_{i \in |T'|} \tag{3}$$

consisting of quasibijections if π is a quasibijection. The derived sequence is featured in the following axiom.

Blow-up axiom - BU. *Consider a corner in the operadic category $\mathtt{O}$*

$$\begin{array}{ccc} S' & & \\ {\scriptstyle f'}\downarrow & & \\ T' & \xrightarrow[\sim]{\sigma} & T'' \end{array} \tag{4}$$

in which σ is a quasibijection and $f' \in \mathtt{O}_{\mathtt{ord}}$. Assume we are given objects F''_j, $j \in |T''|$ together with a collection of maps

$$\left\{\pi_{(i,j)} : {f'}^{-1}(i) \to F''_j,\ j = |\sigma|(i)\right\}_{i \in |T'|}. \tag{5}$$

Then the corner (4) can be completed uniquely into the commutative square

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f''} \\ T' & \xrightarrow[\sim]{\sigma} & T'' \end{array} \tag{6}$$

in which $f'' \in \mathtt{O}_{\mathtt{ord}}$, ${f''}^{-1}(j) = F''_j$ for $j \in |T''|$, and such that derived sequence (3) induced by f'' coincides with (5).

One of the axioms of operadic categories says that the fiber of the unique morphism $T \to U$ to a chosen local terminal object U is T. The following axiom requires that this property characterizes the chosen local terminal objects.

Unique fiber condition - UFib. *If the fiber of the unique morphism* $! : T \to t$ *to a local terminal object is* T*, then* t *is a chosen local terminal object.*

The following axiom refers to the categories $\mathtt{O}_{\mathtt{ord}}$ and $\mathtt{O}_{\mathtt{qb}}$ recalled above.

Factorizability - Fac. *An operadic category* $\mathtt{O}$ *is* factorizable *if each morphism* $f \in \mathtt{O}$ *decomposes, not necessarily uniquely, as* $\phi\sigma$ *for some* $\phi \in \mathtt{O}_{\mathtt{ord}}$ *and* $\sigma \in \mathtt{O}_{\mathtt{qb}}$ *or, symbolically,* $\mathtt{O} = \mathtt{O}_{\mathtt{ord}}\,\mathtt{O}_{\mathtt{qb}}$.

Rigidity - Rig. *An operadic category* $\mathtt{O}$ *is* rigid *if for each* $\phi \in \mathtt{O}_{\mathtt{ord}}$ *the only isomorphism* σ *that makes*

$$\begin{array}{ccc} S & = & S \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ T & \xrightarrow[\cong]{\sigma} & T \end{array} \tag{7}$$

commutative is the identity $\mathbb{1}_T : T \to T$.

The last axiom recalled here involves a *grading*, defined as a map $e : \mathrm{Objects}(\mathtt{O}) \to \mathbb{N}$ with the property that

$$e(T) + e(F_1) + \cdots + e(F_k) = e(S)$$

for each $f : S \to T$ with fibers $F_1, \ldots, F_k$. In this situation, the *grade* $e(f)$ of f is the difference $e(f) := e(S) - e(T)$.

Strict grading - SGrad. *A graded operadic category* $\mathtt{O}$ *is* strictly graded *if a morphism* $f \in \mathtt{O}$ *is an isomorphism if and only if* $e(f) = 0$.

Finally, all operadic categories are assumed to be *constant-free*, which by [3, Definition 2.18] requires that $|f| : |T| \to |S|$ is surjective for each morphism $f : T \to S$. With $\mathtt{Fin}$ the operadic category of finite ordinals $\bar{n} = \{1, \ldots, n\}$, $n \in \mathbb{N}$, and their set-theoretic maps, this means that the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$ factorizes through the operadic category $\mathtt{Fin}_{\mathrm{semi}}$ of nonempty finite sets and their surjections.

The above axioms will be imposed at many places of the paper including the rest of this section. The operadic category $\mathtt{O}$ will therefore fulfill the following assumptions:

Assumptions 1.2. The operadic category $\mathtt{O}$ is a strictly graded and factorizable, all quasibijections are invertible, the blow-up axiom and unique fiber condition are fulfilled, and a morphism f is an isomorphism if and only if $e(f) = 0$. In brief, we require

$$\text{Fac} \;\&\; \text{BU} \;\&\; \text{QBI} \;\&\; \text{UFib} \;\&\; \text{SGrad}.$$

1.2 Markl operads

Composition laws of Markl operads are labeled by morphisms which are elementary in the sense of the following definition.

Definition 1.3. A morphism $\phi : T \to S \in \mathtt{O}_{\mathtt{ord}}$ in a graded operadic category $\mathtt{O}$ is *elementary* if all its fibers are trivial (= chosen local terminal) except precisely one whose grade is ≥ 1.

When $i \in |S|$ is the unique element with nontrivial fiber $\phi^{-1}(i)$, we will sometimes write ϕ as the pair (ϕ, i) and say that ϕ is *i-elementary*. If we want to name the unique nontrivial fiber $F := \phi^{-1}(i)$ explicitly, we will write $F \triangleright_i T \xrightarrow{\phi} S$, or $F \triangleright T \xrightarrow{\phi} S$ when the concrete $i \in |S|$ is not important.

Definition 1.4. Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms. If $|\psi|(j) = i$ we say that the fibers of ϕ and ψ are *joint*. If $|\psi|(j) \neq i$ we say that ϕ and ψ have *disjoint fibers* or, more specifically, that the fibers of ϕ and ψ are *(i,j)-disjoint*. Denoting $k := |\psi|(i)$, we call (ψ, ϕ) a *(k,i)-pair*.

An example of a configuration with disjoint fibers is portrayed in the picture following Definition 5.4 of [3]. We need also to recall from [3] the following lemma and its corollary.

Lemma 1.5 (Lemma 5.5 of [3]). *If the fibers of elementary morphisms ϕ and ψ in Definition 1.4 are joint, then the composite $\xi = \psi \circ \phi$ is elementary as well, with nontrivial fiber over i, and the induced morphism $\phi_i : \xi^{-1}(i) \to \psi^{-1}(i)$ is elementary with the nontrivial fiber over j equal to $\phi^{-1}(j)$. For $l \neq i$ the morphism ϕ_l equals the identity $U_c \to U_c$ of trivial objects.*

If the fibers of ϕ and ψ are (i,j)-disjoint then the morphism $\xi = \psi \circ \phi$ has exactly two nontrivial fibers and these are fibers over i and $k := |\psi|(j)$. Moreover, there is a canonical induced quasibijection

$$\phi_i : \xi^{-1}(i) \to \psi^{-1}(i) \in \mathtt{O}_{\mathtt{ord}} \tag{8a}$$

and the equality

$$\xi^{-1}(k) = \phi^{-1}(j). \tag{8b}$$

Definition 1.6. We will call the pair $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ of morphisms in Definition 1.4 with disjoint fibers *harmonic* if $\xi^{-1}(i) = \psi^{-1}(i)$ and the map ϕ_i in (8a) is the identity.

Corollary 1.7 (Corollary 5.8 of [3]). *Assume that*

$$\begin{array}{ccccc} & & P' & & \\ & \overset{(\phi',j)}{\nearrow} & & \overset{(\psi',i)}{\searrow} & \\ T & & & & S \\ & \underset{(\phi'',l)}{\searrow} & & \underset{(\psi'',k)}{\nearrow} & \\ & & P'' & & \end{array} \tag{9}$$

is a commutative diagram of elementary morphisms. Assume that $|\psi''|(l) = i$, $|\psi'|(j) = k$ and $i \neq k$. Let F', F'', G', G'' be the only nontrivial fibers of $\phi', \phi'', \psi', \psi''$, respectively. Then one has canonical quasibijections

$$\sigma' : F' \longrightarrow G'' \quad \text{and} \quad \sigma'' : F'' \longrightarrow G'. \tag{10}$$

If both pairs in (9) are harmonic, then $F' = G''$, $F'' = G'$ and σ', σ'' are the identities.

In the following definition of a Markl operad, $\mathtt{V}$ is a strict symmetric monoidal category with a strict monoidal unit $\Bbbk$ and symmetry τ, and $\mathtt{O}_{\mathtt{iso}} \subset \mathtt{O}$ the subcategory of isomorphisms. Axiom (ii) of that definition is given in the simplified form assuming the BU axiom that guarantees, by [3, Corollary 5.7], that all pairs of elementary morphisms with disjoint fibers are harmonic. A general form can be found in [3].

Definition 1.8. A *Markl $\mathtt{O}$-operad* in $\mathtt{V}$ is a presheaf $\mathcal{M} : \mathtt{O}_{\mathtt{iso}}^{\mathrm{op}} \to \mathtt{V}$ with values in $\mathtt{V}$ equipped, for each elementary morphism $F \triangleright T \xrightarrow{\phi} S$, with a "circle product"

$$\circ_\phi : \mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T). \tag{11}$$

These operations must satisfy the following set of axioms.

(i) Let $T \xrightarrow{(\phi,j)} S \xrightarrow{(\psi,i)} P$ be elementary morphisms such that $|\psi|(j) = i$ and let $\xi : T \to P$ be the composite $\psi\phi$. Then the diagram

$$\begin{array}{ccccc} & & \mathcal{M}(P) \otimes \mathcal{M}(\xi^{-1}(i)) & & \\ & \overset{\mathbb{1}\otimes\circ_{\phi_i}}{\nearrow} & & \overset{\circ_\xi}{\searrow} & \\ \mathcal{M}(P) \otimes \mathcal{M}(\psi^{-1}(i)) \otimes \mathcal{M}(\phi^{-1}(j)) & & & & \mathcal{M}(T) \\ & \underset{\circ_\psi\otimes\mathbb{1}}{\searrow} & & \underset{\circ_\phi}{\nearrow} & \\ & & \mathcal{M}(S) \otimes \mathcal{M}(\phi^{-1}(j)) & & \end{array} \tag{12}$$

commutes.

(ii) Consider the diagram

$$\begin{array}{ccccc} & & P' & & \\ & {\scriptstyle(\phi',j)}\nearrow & & \searrow{\scriptstyle(\psi',i)} & \\ T & & & & S \\ & {\scriptstyle(\phi'',l)}\searrow & & \nearrow{\scriptstyle(\psi'',k)} & \\ & & P'' & & \end{array} \tag{13}$$

of elementary morphisms with disjoint fibers as in Corollary 1.7. By the harmonicity implied by BU, the morphisms σ', σ'' in (10) are the identities; denote $F := F' = G''$ and $G := G' = F''$. With this notation, the diagram

$$\begin{array}{ccc} \mathcal{M}(S)\otimes\mathcal{M}(G)\otimes\mathcal{M}(F) & \xrightarrow{\circ_{\psi'}\otimes\mathbb{1}} & \mathcal{M}(P')\otimes\mathcal{M}(F) \\ {\scriptstyle\mathbb{1}\otimes\tau}\uparrow & & \downarrow{\scriptstyle\circ_{\phi'}} \\ & & \mathcal{M}(T) \\ & & \uparrow{\scriptstyle\circ_{\phi''}} \\ \mathcal{M}(S)\otimes\mathcal{M}(F)\otimes\mathcal{M}(G) & \xrightarrow{\circ_{\psi''}\otimes\mathbb{1}} & \mathcal{M}(P'')\otimes\mathcal{M}(G) \end{array} \tag{14}$$

commutes.

(iii) For every commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow[\cong]{\omega} & T'' \\ \downarrow{\scriptstyle\phi'} & & \downarrow{\scriptstyle\phi''} \\ S' & \xrightarrow[\sim]{\sigma} & S'' \end{array}$$

where ω is an isomorphism, σ a quasibijection, and $F'\triangleright_i T' \xrightarrow{\phi'} S'$, $F''\triangleright_j T'' \xrightarrow{\phi''} S''$, the diagram

$$\begin{array}{ccc} \mathcal{M}(F'')\otimes\mathcal{M}(S'') & \xrightarrow{\circ_{\phi''}} & \mathcal{M}(T'') \\ {\scriptstyle\omega^*_{(i,j)}\otimes\sigma^*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\omega^*} \\ \mathcal{M}(F')\otimes\mathcal{M}(S') & \xrightarrow[\circ_{\phi'}]{} & \mathcal{M}(T') \end{array} \tag{15}$$

in which $\omega_{(i,j)} : F' \to F''$ is the induced map (2) of fibers, commutes.

Markl operad $\mathcal{M}$ is *unital* if one is given, for each trivial (i.e. chosen local terminal) object U of $\mathtt{O}$, a map $\eta_U : \Bbbk \to \mathcal{M}(U)$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}(U)\otimes\mathcal{M}(T) & \xrightarrow{\circ_!} & \mathcal{M}(T) \\ {\scriptstyle\eta_U\otimes\mathbb{1}}\uparrow & & \| \\ \Bbbk\otimes\mathcal{M}(T) & \xrightarrow{\cong} & \mathcal{M}(T) \end{array} \tag{16}$$

commutes whenever T is such that $e(T) \geq 1$ and $T \triangleright T \xrightarrow{!} U$ is the unique map.

Let $\mathtt{O}_{\mathtt{ltrm}}$ be the operadic subcategory of $\mathtt{O}$ consisting of its local terminal objects. Denote by $\mathbf{1}_{\mathtt{trm}} : \mathtt{O}_{\mathtt{ltrm}} \to \mathtt{V}$ the constant functor. The collection $\{\eta_U : \Bbbk \to \mathcal{M}(U)\}$ of unit maps extends uniquely into a transformation

$$\eta : \mathbf{1}_{\mathtt{trm}} \to \iota^*\mathcal{M} \tag{17}$$

from $\mathbf{1}_{\mathtt{trm}}$ to the restriction of $\mathcal{M}$ along the inclusion $\iota : \mathtt{O}_{\mathtt{ltrm}} \hookrightarrow \mathtt{O}$. Transformation (17) amounts to a family of maps $\eta_u : \Bbbk \to \mathcal{M}(u)$ given for each local terminal $u \in \mathtt{O}$, such that the diagram

$$\begin{array}{ccc} \mathcal{M}(u) & \xrightarrow{!^*} & \mathcal{M}(v) \\ {\scriptstyle\eta_u}\uparrow & & \uparrow{\scriptstyle\eta_v} \\ \Bbbk & = & \Bbbk \end{array} \tag{18}$$

commutes for each (unique) map $! : v \to u$ of local terminal objects. We will call the components $\eta_u : \Bbbk \to \mathcal{M}(u)$ of the transformation (17) the *extended units*.

For each T with $e(T) \geq 1$ and $F \triangleright T \xrightarrow{!} u$ with u a local terminal object, one has a map $\vartheta(T,u) : \mathcal{M}(F) \to \mathcal{M}(T)$ defined by the diagram

$$\begin{array}{ccc} \mathcal{M}(u) \otimes \mathcal{M}(F) & \xrightarrow{\circ_!} & \mathcal{M}(T) \\ {\scriptstyle \eta_u \otimes \mathbb{1}} \uparrow & & \uparrow {\scriptstyle \vartheta(T,u)} \\ \Bbbk \otimes \mathcal{M}(F) & \xrightarrow{\cong} & \mathcal{M}(F)\,. \end{array} \tag{19}$$

The unitality offers a generalization of Axiom (iii) of Markl operads which postulates for each commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow[\cong]{\omega} & T'' \\ {\scriptstyle \phi'} \downarrow & \overset{\phi}{\searrow} & \downarrow {\scriptstyle \phi''} \\ S' & \xrightarrow[\cong]{\sigma} & S'' \end{array} \tag{20}$$

where the horizontal maps are isomorphisms and the vertical maps are elementary, with $F' \triangleright_i T' \xrightarrow{\phi'} S'$, $F'' \triangleright_j T'' \xrightarrow{\phi''} S''$, the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{M}(F) \otimes \mathcal{M}(S'') & \xleftarrow[\cong]{\omega_j^* \otimes \mathbb{1}} & \mathcal{M}(F'') \otimes \mathcal{M}(S'') & \xrightarrow{\circ_{\phi''}} & \mathcal{M}(T'') \\ {\scriptstyle \vartheta(F,\sigma^{-1}(j)) \otimes \mathbb{1}} \uparrow & & & & \downarrow {\scriptstyle \cong\, \omega^*} \\ \mathcal{M}(F') \otimes \mathcal{M}(S'') & \xrightarrow[\cong]{\mathbb{1} \otimes \sigma^*} & \mathcal{M}(F') \otimes \mathcal{M}(S') & \xrightarrow[\circ_{\phi'}]{} & \mathcal{M}(T')\,, \end{array} \tag{21}$$

in which $F := \phi^{-1}(j)$ and $\omega_j : F \to F''$ is the induced map of fibers. Notice that if σ is a quasibijection, (21) implies (15).

Definition 1.9. A Markl operad $\mathcal{M}$ is *strictly unital* if all the maps $\vartheta(T,u)$ in (19) are identities. It is 1-*connected* if the unit maps $\eta_U : \Bbbk \to \mathcal{M}(U)$ are isomorphisms for each trivial U.

If $\mathcal{M}$ is strictly unital, one has $\mathcal{M}(F) = \mathcal{M}(F')$ in (21), so this diagram assumes a particularly simple form, namely

$$\begin{array}{ccc} \mathcal{M}(F'') \otimes \mathcal{M}(S'') & \xrightarrow{\circ_{\phi''}} & \mathcal{M}(T'') \\ {\scriptstyle \omega_j^* \otimes \sigma^*} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong\, \omega^*} \\ \mathcal{M}(F') \otimes \mathcal{M}(S') & \xrightarrow{\circ_{\phi'}} & \mathcal{M}(T')\,. \end{array} \tag{22}$$

2 Markl operads and virtual isomorphisms

We introduce the category of virtual isomorphisms $\mathtt{O}_{\mathtt{vrt}}$ related to an operadic category $\mathtt{O}$, its extension $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$, and the quotient $\mathtt{QO}_{\mathtt{vrt}}$ of $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$ modulo virtual isomorphisms. In the presence of a grading e on $\mathtt{O}$ we will further consider the subgroupoid $\mathtt{O}_{\mathtt{vrt}}(e)$ of objects of grade ≥ 1, the extension $\mathtt{O}_{\mathtt{vrt}}(e) \int \mathtt{O}_{\mathtt{iso}}$ and the related quotient $\mathtt{QO}_{\mathtt{vrt}}(e)$. Presheaves on $\mathtt{QO}_{\mathtt{vrt}}(e)$ will then serve as the underlying collections for Markl operads.

The operadic category $\mathtt{O}$ will be required to fulfill *Assumptions 1.2* although the grading (which need not even be strict) will be used only in the second half of this section. All definitions and results of this section hold also for operadic categories which are not constant-free. As before we denote by $\mathtt{O}_{\mathtt{ltrm}}$ the groupoid of local terminal objects in $\mathtt{O}$ and by $\mathtt{O}_{\mathtt{iso}} \subset \mathtt{O}$ the subcategory with the same objects as $\mathtt{O}$, and morphisms the isomorphisms of $\mathtt{O}$.

Let $T \in \mathtt{O}$ and let $t \in \mathtt{O}$ be a local terminal object in the connected component of T. We therefore have a unique morphism $T \to t$ with a unique fiber F, which will be expressed by the

shorthand $F \blacktriangleright T \to t$. In this situation we write $F \rightsquigarrow T$ or $F \overset{t}{\rightsquigarrow} T$ if t needs to be specified, and speak about a *virtual morphism* from F to T. Notice that $F \rightsquigarrow T$ does not represent an "actual" morphism $F \to T$; in fact there may not be any morphisms from F to T in $\mathtt{O}$, cf. Example 2.4 below. Since we assume BU and UFib, there exists at most one virtual morphism between two given objects of $\mathtt{O}$ by [3, Lemma 2.15]. In fact, "$\rightsquigarrow$" interpreted as a relation on $\mathtt{O}$ is a preorder.

The notation $F \blacktriangleright T \to t$ shall not be confused with $F \triangleright T \to t$ used before. The map $T \to t$ need not be an elementary morphism even when $\mathtt{O}$ is graded, since we do not demand $e(T) \geq 1$ or any analog of this. The following statement shows that one may define a composition rule for virtual morphisms so that they form a groupoid.

Lemma 2.1. *Virtual morphisms in the operadic category* $\mathtt{O}$ *form a groupoid* $\mathtt{O}_{\mathtt{vrt}}$.

Proof. The lemma will follow from the following facts: each object of $\mathtt{O}$ possesses a virtual endomorphism; virtual endomorphisms can be composed; a virtual morphism $S \rightsquigarrow T$ exists if and only if there exist a virtual morphism $T \rightsquigarrow S$. Since there is at most one virtual morphism between two given objects of $\mathtt{O}$ by [3, Lemma 2.15], all properties of a groupoid follow automatically.

The virtual endomorphism of $T \in \mathtt{O}$ is $T \overset{U}{\rightsquigarrow} T$, associated to the unique morphism $T \to U$ to the chosen local terminal object U in the connected component of T. The composition of virtual morphisms is defined as follows.

Let $S \overset{t}{\rightsquigarrow} T \overset{r}{\rightsquigarrow} R$ be a chain of virtual morphisms. This means that S is the fiber of the unique morphism $\phi : T \to t$, and T is the fiber of $\psi : R \to r$, i.e. $S \blacktriangleright T \overset{\phi}{\to} t$ and $T \blacktriangleright R \overset{\psi}{\to} r$, $t \in \mathtt{O}_{\mathtt{ltrm}}$. By the weak blow-up axiom there exists a unique factorization of ψ as in the diagram

$$\begin{array}{ccc} R & \xrightarrow{\xi} & s \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & r & \end{array} \tag{23}$$

such that ξ_1, the induced map between the unique fibers, equals ϕ. From Axiom (iv) of an operadic category, one has $\xi^{-1}(1) = \xi_1^{-1}(1) = \phi^{-1}(1) = S$, that is $S \blacktriangleright R \overset{\xi}{\to} s$. We take the related virtual morphism $S \overset{s}{\rightsquigarrow} R$ as the composite of $S \overset{t}{\rightsquigarrow} T$ and $T \overset{r}{\rightsquigarrow} R$.

Consider a virtual morphism $S \overset{t}{\rightsquigarrow} T$ given by some $S \blacktriangleright T \overset{\phi}{\to} t \in \mathtt{O}$. The morphism ϕ has a unique factorization $T \to U \overset{\delta}{\to} t$ through a chosen local terminal object U. Let s be the unique fiber of δ, i.e. $s \blacktriangleright U \overset{\delta}{\to} t$. The diagram

$$\begin{array}{ccc} T & \longrightarrow & U \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & t & \end{array}$$

induces a morphism of fibers $S \to s$ whose fiber is T, giving rise to the required virtual morphism $T \overset{s}{\rightsquigarrow} S$ in the opposite direction. □

As we already noticed, if there exists a morphism $S \rightsquigarrow T$ in $\mathtt{O}_{\mathtt{vrt}}$ then it is unique. Together with Lemma 2.1 this implies that $\mathtt{O}_{\mathtt{vrt}}$ is equivalent as a category to a discrete set. The notation $T \rightsquigarrow S$ will denote the unique isomorphism from T to S, tacitly assuming its existence, and $S \rightsquigarrow T$ its inverse.

As the next step towards our construction of $\mathtt{QO}_{\mathtt{vrt}}$, we extend $\mathtt{O}_{\mathtt{vrt}}$ to a category $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$ which has the same objects as $\mathtt{O}$ but whose morphisms $T \to R$ are sequences

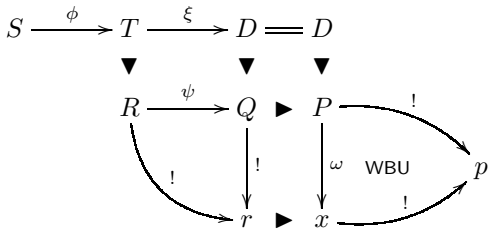
$$S \overset{\phi}{\to} T \rightsquigarrow R$$

where $\phi : S \to T \in \mathtt{O}_{\mathtt{iso}}$ is an isomorphism in $\mathtt{O}$. To define the composition, consider the sequence

$$S \overset{\phi}{\to} T \rightsquigarrow R \overset{\psi}{\to} Q \rightsquigarrow P.$$

The virtual morphism $T \rightsquigarrow R$ is related to a morphism $T \blacktriangleright R \to r$ with a unique $r \in \mathtt{O}_{\mathtt{ltrm}}$ and the virtual morphism $Q \rightsquigarrow P$ to $Q \blacktriangleright P \to p$ with $p \in \mathtt{O}_{\mathtt{ltrm}}$, The objects R and Q live in the same

connected component, so there is a unique $D \blacktriangleright Q \to r$. We may therefore construct the diagram



(24)

in which ω is the unique map such that the induced map between the fibers is $Q \xrightarrow{!} r$, and $\xi := \psi_1$ is the induced map between the fibers. We then define the composite

$$(R \xrightarrow{\psi} Q \rightsquigarrow P)(S \xrightarrow{\phi} T \rightsquigarrow R)$$

as the sequence

$$S \xrightarrow{\xi\phi} D \rightsquigarrow P, \tag{25}$$

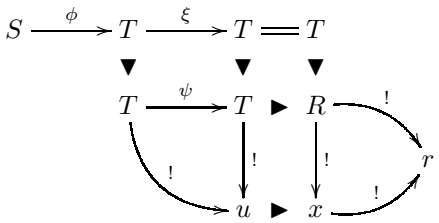
with $D \rightsquigarrow P$ given by $D \blacktriangleright P \to x$. The identity automorphism of S is $S \xrightarrow{\mathbb{1}} S \rightsquigarrow S$.

One can easily check that the above structure makes $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$ a category which is, in fact, a groupoid: for a morphism $\Phi : S \xrightarrow{\phi} T \rightsquigarrow R$ in $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$ take the inverse $R \rightsquigarrow T$ to $T \rightsquigarrow R$ and the inverse $\psi : T \to S$ of $\phi : S \to T \in \mathtt{O}$. Then the composite

$$(T \xrightarrow{\psi} S \rightsquigarrow S)(R \xrightarrow{\mathbb{1}} R \rightsquigarrow T)$$

in $\mathtt{O}_{\mathtt{vrt}} \int \mathtt{O}_{\mathtt{iso}}$ is the inverse to Φ. We leave the details to the reader.

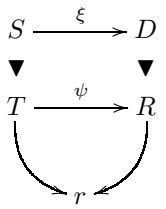
Remark 2.2. The composite $(T \xrightarrow{\mathbb{1}} T \rightsquigarrow R)(S \xrightarrow{\phi} T \rightsquigarrow T)$ equals $S \xrightarrow{\phi} T \rightsquigarrow R$ as expected. The diagram (24) in this particular case becomes



where u is the *chosen* local terminal object. Both ξ and ϕ are the identities by the axioms of operadic categories, while $x = r$ because u is chosen local terminal. The claim follows. One can similarly verify the identity

$$(T \xrightarrow{\psi} R \rightsquigarrow R)(S \xrightarrow{\mathbb{1}} S \rightsquigarrow T) = (S \xrightarrow{\xi} D \rightsquigarrow T)$$

with $S \xrightarrow{\xi} D$ given by the diagram



We may express the result of the above calculation as the distributive law

$$(T \xrightarrow{\psi} R)(S \rightsquigarrow T) \longmapsto (D \rightsquigarrow R)(S \xrightarrow{\xi} D)$$

between $\rightsquigarrow$ and $\xrightarrow{\cong}$, cf. [24].

We now consider the quotient $\mathtt{QO}_{\mathtt{vrt}}$ of $\mathtt{O}_{\mathtt{vrt}}\int\mathtt{O}_{\mathtt{iso}}$ whose objects are classes of objects of $\mathtt{O}_{\mathtt{vrt}}\int\mathtt{O}_{\mathtt{iso}}$ with respect to the relation generated by virtual isomorphisms. That is, two objects are equivalent if there is a virtual isomorphism between them. More precisely, $\mathtt{QO}_{\mathtt{vrt}}$ is defined by the pushout

$$\begin{array}{ccc} \mathtt{O}_{\mathtt{vrt}} & \longrightarrow & \mathtt{O}_{\mathtt{vrt}}\int\mathtt{O}_{\mathtt{iso}} \\ \downarrow & & \downarrow \\ \pi_0(\mathtt{O}_{\mathtt{vrt}}) & \longrightarrow & \mathtt{QO}_{\mathtt{vrt}} \end{array} \tag{26}$$

in the category of groupoids. Since the left vertical functor is an equivalence and the top horizontal functor a cofibration of groupoids in the canonical model structure on the category of groupoids [1], the right vertical functor is an equivalence of groupoids too.

It is easy to see that morphisms between objects in $\mathtt{QO}_{\mathtt{vrt}}$ are equivalence classes of non-virtual isomorphisms in the following sense. Let $\phi' : T' \to S'$ and $\phi'' : T'' \to S''$ be two isomorphisms in $\mathtt{O}$. They are equivalent if there exists a local terminal object t such that ϕ'' is the induced fiber map in the diagram

$$\begin{array}{ccc} T' & \xrightarrow{\phi'} & S' \\ & \searrow \quad \swarrow & \\ & t\,. & \end{array} \tag{27}$$

Example 2.3. Assume that each connected component of $\mathtt{O}$ contains precisely one terminal object, i.e. all local terminal objects are the trivial (chosen) ones. Then $\mathtt{QO}_{\mathtt{vrt}} \cong \mathtt{O}_{\mathtt{iso}}$. This is the case of e.g. the category $\mathtt{Fin}$ of finite ordinals or of the operadic category $\mathtt{Per}$ in Subsection 7.4.

Example 2.4. In the operadic category $\mathtt{Bq}(\mathfrak{C})$ recalled in [3, Example 1.5] two bouquets are virtually equivalent if they differ only in the last color. Notice that an "actual" morphism $b' \to b''$ in $\mathtt{Bq}(\mathfrak{C})$ between virtually equivalent bouquets exists if and only if $b' = b''$. The groupoid $\mathtt{QBq}(\mathfrak{C})_{\mathtt{vrt}}$ is the groupoid of strings $(i_1, \dots, i_k)$, $k \geq 1$, with morphisms arbitrary bijections.

Example 2.5. Two graphs in the operadic category $\mathtt{Gr}$ of [3, Definition 3.13] are virtually equivalent if they differ only in the global orders of their leaves. Morphisms in $\mathtt{QGr}_{\mathtt{vrt}}$ are isomorphisms of graphs which need not preserve the global orders.

Assume that $\mathtt{O}$ possesses a grading $e : \mathrm{Objects}(\mathtt{O}) \to \mathbb{N}$. In this case we denote by $\mathtt{O}_{\mathtt{vrt}}(e) \subset \mathtt{O}_{\mathtt{vrt}}$ the full subgroupoid with objects $T \in \mathtt{O}$ such that $e(T) \geq 1$. We construct $\mathtt{O}_{\mathtt{vrt}}(e)\int\mathtt{O}_{\mathtt{iso}}$ out of $\mathtt{O}_{\mathtt{vrt}}(e)$ and $\mathtt{O}_{\mathtt{iso}}$ as before, and define its quotient $\mathtt{QO}_{\mathtt{vrt}}(e)$ by replacing $\mathtt{O}_{\mathtt{vrt}}$ by $\mathtt{O}_{\mathtt{vrt}}(e)$ in (26). In the following lemma, $\mathtt{V}$ denotes a strict symmetric monoidal category as in Definition 1.8.

Lemma 2.6. *Each unital Markl $\mathtt{O}$-operad $\mathcal{M}$ with values in $\mathtt{V}$ induces a covariant functor $\mathtt{O}_{\mathtt{vrt}}(e) \to \mathtt{V}$, denoted $\mathcal{M}$ again, which acts as $\mathcal{M}$ on objects, and on virtual morphisms is defined by*

$$\mathcal{M}(F \rightsquigarrow T) := \vartheta(T, u),$$

where $\vartheta(T, u)$ is as in (19). Since $\mathtt{O}_{\mathtt{vrt}}(e)$ is a groupoid, all maps $\vartheta(T, u)$ are invertible.

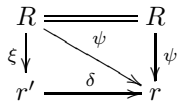
Proof. It follows from the unitality (16) of $\mathcal{M}$ that $\mathcal{M}(T \rightsquigarrow T) = \mathbb{1}_{\mathcal{M}(T)}$. Let us verify the functoriality

$$\mathcal{M}(S \rightsquigarrow R) = \mathcal{M}(T \rightsquigarrow R)\mathcal{M}(S \rightsquigarrow T). \tag{28}$$

To this end we consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{M}(T) \otimes \mathcal{M}(r) & & \xrightarrow{\circ_!} & & \mathcal{M}(R) \\ \uparrow{\scriptstyle \vartheta(T,r)\otimes\mathbb{1}} & & & & \uparrow{\scriptstyle \circ_!} \\ \mathcal{M}(S) \otimes \mathcal{M}(r) & & \xrightarrow{\mathbb{1}\otimes\delta^*} & & \mathcal{M}(S) \otimes \mathcal{M}(r')\,. \\ \uparrow{\scriptstyle \mathbb{1}\otimes\eta_r} & \nearrow{\scriptstyle \mathbb{1}\otimes\eta_{r'}} & & & \\ \mathcal{M}(S) & \xrightarrow{\cong} & \mathcal{M}(S) \otimes \Bbbk & & \end{array} \tag{29}$$

Its upper square is (21) applied to the diagram



in place of (20), in which the symbols have the same meaning as in (23). The commutativity of the bottom triangle follows from the commutativity of (18). It follows from the definition of the maps $\vartheta(T,u)$ that the composite

$$\mathcal{M}(S) \cong \mathcal{M}(S) \otimes \Bbbk \xrightarrow{\mathbb{1}\otimes\eta_{r'}} \mathcal{M}(S) \otimes \mathcal{M}(r') \xrightarrow{\circ_!} \mathcal{M}(R)$$

in (29) equals the left-hand side of (28), while the composite

$$\mathcal{M}(S) \cong \mathcal{M}(S) \otimes \Bbbk \xrightarrow{\mathbb{1}\otimes\eta_{r}} \mathcal{M}(S) \otimes \mathcal{M}(r) \xrightarrow{\vartheta(T,r)\otimes\mathbb{1}} \mathcal{M}(T) \otimes \mathcal{M}(r) \xrightarrow{\circ_!} \mathcal{M}(R)$$

equals the right-hand side of (28). □

Proposition 2.7. *The $\mathtt{O}_{\mathtt{iso}}$-presheaf structure of a Markl $\mathtt{O}$-operad $\mathcal{M}$ in $\mathtt{V}$ combined with the functor $\mathcal{M} : \mathtt{O}_{\mathtt{vrt}}(e) \to \mathtt{V}$ of Lemma 2.6 makes $\mathcal{M}$ an $\mathtt{O}_{\mathtt{vrt}}(e)\int\mathtt{O}_{\mathtt{iso}}$-presheaf via the formula*

$$\mathcal{M}\big(S \xrightarrow{\phi} T \rightsquigarrow R\big) := \phi^*\mathcal{M}(R \rightsquigarrow T). \tag{30}$$

Proof. Since clearly $\mathcal{M}(T \xrightarrow{\mathbb{1}} T \rightsquigarrow T) = \mathbb{1}_{\mathcal{M}(T)}$, we need only to prove that

$$\mathcal{M}\big(S \xrightarrow{\phi} T \rightsquigarrow R\big)\mathcal{M}\big(R \xrightarrow{\psi} Q \rightsquigarrow P\big) = \mathcal{M}\big(S \xrightarrow{\xi\phi} (Q \rightsquigarrow P)(D \rightsquigarrow Q)\big) \tag{31}$$

for the composite

$$(R \xrightarrow{\psi} Q \rightsquigarrow P)(S \xrightarrow{\phi} T \rightsquigarrow R) = S \xrightarrow{\xi\phi} (Q \rightsquigarrow P)(D \rightsquigarrow Q)$$

defined in (25). Evaluating both sides of (31) using (30) gives

$$\phi^*\mathcal{M}(R \rightsquigarrow T)\psi^*\mathcal{M}(P \rightsquigarrow Q) = (\xi\phi)^*\mathcal{M}(Q \rightsquigarrow D)\mathcal{M}(P \rightsquigarrow Q).$$

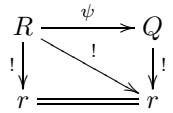
Since all the maps involved are isomorphisms, we easily see that (31) is equivalent to

$$\mathcal{M}(D \rightsquigarrow Q)(\xi^{-1})^* = (\psi^{-1})^*\mathcal{M}(T \rightsquigarrow R). \tag{32}$$

To prove this equality, consider the diagram

$$\begin{array}{ccccc} \mathcal{M}(T) \cong \mathcal{M}(T) \otimes \Bbbk & \xrightarrow{\mathbb{1}\otimes\eta_r} & \mathcal{M}(T) \otimes \mathcal{M}(r) & \xleftarrow{\xi^*\otimes\mathbb{1}} & \mathcal{M}(D) \otimes \mathcal{M}(r) \\ & & \downarrow{\scriptstyle \circ_!} & & \downarrow{\scriptstyle \circ_!} \\ & & \mathcal{M}(R) & \xleftarrow{\psi^*} & \mathcal{M}(Q) \end{array} \tag{33}$$

in which the square is (21) associated to



in place of (20). It follows from the definitions that the composite of the maps

$$\mathcal{M}(T) \cong \mathcal{M}(T) \otimes \Bbbk \xrightarrow{\mathbb{1}\otimes\eta_r} \mathcal{M}(T) \otimes \mathcal{M}(r) \xrightarrow{(\xi^*\otimes\mathbb{1})^{-1}} \mathcal{M}(D) \otimes \mathcal{M}(r) \xrightarrow{\circ_!} \mathcal{M}(Q)$$

in (33) equals the left-hand side of (32), while the composite

$$\mathcal{M}(T) \cong \mathcal{M}(T) \otimes \Bbbk \xrightarrow{\mathbb{1}\otimes\eta_r} \mathcal{M}(T) \otimes \mathcal{M}(r) \xrightarrow{\circ_!} \mathcal{M}(R) \xrightarrow{\psi^{*\,-1}} \mathcal{M}(Q)$$

equals its right-hand side. □

Proposition 2.8. *If $\mathcal{M}$ is a unital Markl $\mathtt{O}$-operad in a cocomplete symmetric monoidal category $\mathtt{V}$, then the $\mathtt{O}_{\mathtt{vrt}}(e)\int\mathtt{O}_{\mathtt{iso}}$-presheaf of Proposition 2.7 associated to $\mathcal{M}$ functorially descends to a $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaf $\mathring{\mathcal{M}}$ by means of the left Kan extension along the equivalence $\big(\mathtt{O}_{\mathtt{vrt}}(e)\int\mathtt{O}_{\mathtt{iso}}\big)^{\mathrm{op}} \to \mathtt{QO}_{\mathtt{vrt}}(e)^{\mathrm{op}}$.*

Proof. We will construct the presheaf $\mathring{\mathcal{M}}$ explicitly. Objects of $\mathtt{QO}_{\mathtt{vrt}}(e)$ are, by definition, the equivalence classes $[T]$ of the objects of $\mathtt{O}$ modulo the relation $[T'] = [T'']$ if $T'' \rightsquigarrow T'$. We define $\mathring{\mathcal{M}}([T])$ as the colimit

$$\mathring{\mathcal{M}}([T]) := \operatorname{colim} \mathcal{M}(S) \tag{34}$$

over the groupoid of all $S \in \mathtt{O}$ virtually isomorphic to T. It is clear that the canonical injection $\iota_T : \mathcal{M}(T) \hookrightarrow \mathring{\mathcal{M}}([T])$ is an isomorphism.

Consider a morphism $[\phi] : [T] \to [S]$ in $\mathtt{QO}_{\mathtt{vrt}}(e)$ given by an isomorphism $\phi : T \to S$. We define $\mathring{\mathcal{M}}([\phi]) : \mathring{\mathcal{M}}([S]) \to \mathring{\mathcal{M}}([T])$ by the diagram

$$\begin{array}{ccc} \mathring{\mathcal{M}}([S]) & \xrightarrow{\mathring{\mathcal{M}}([\phi])} & \mathring{\mathcal{M}}([T]) \\ {\scriptstyle\iota_S}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\iota_T} \\ \mathcal{M}(S) & \xrightarrow{\phi^*} & \mathcal{M}(T) \end{array}$$

in which ϕ^* refers to the $\mathtt{O}_{\mathtt{iso}}$-presheaf structure of $\mathcal{M}$.

We need to show that $\mathring{\mathcal{M}}([\phi])$ does not depend on the choice of a representative of the map $[\phi]$ under the equivalence that identifies ϕ' as in (27) with the induced map ϕ'' between the fibers over t. To this end, consider the commutative diagram

$$\begin{array}{ccccccc} & & & \overset{\phi'^*}{\frown} & & & \\ \mathcal{M}(T') & = & \Bbbk \otimes \mathcal{M}(T') & \xleftarrow{\mathbb{1}\otimes\phi'^*} & \Bbbk \otimes \mathcal{M}(S') & = & \mathcal{M}(S') \\ {\scriptstyle\vartheta(T',t)}\downarrow & & {\scriptstyle\eta_u\otimes\mathbb{1}}\downarrow & & {\scriptstyle\eta_u\otimes\mathbb{1}}\downarrow & & {\scriptstyle\vartheta(S',t)}\downarrow \\ \mathcal{M}(T'') & \xleftarrow{\circ_!} & \mathcal{M}(u)\otimes\mathcal{M}(T') & \xleftarrow{\mathbb{1}\otimes\phi''^*} & \mathcal{M}(u)\otimes\mathcal{M}(S') & \xrightarrow{\circ_!} & \mathcal{M}(S'') \\ & & & \underset{\phi''^*}{\smile} & & & \end{array} \tag{35}$$

in which the leftmost and rightmost squares are instances of (19). The commutativity of the central square and of the upper part is clear. Finally, the commutativity of the lower part follows from axiom (15) of Markl operads. An easy diagram chase shows that the commutativity of (35) implies the commutativity of the middle square in

$$\begin{array}{ccccccc} \mathring{\mathcal{M}}([T']) & \xrightarrow[\cong]{\iota_{T'}} & \mathcal{M}(T') & \xleftarrow{\phi'^*} & \mathcal{M}(S') & \xleftarrow[\cong]{\iota_{S'}} & \mathring{\mathcal{M}}([S']) \\ \| & & {\scriptstyle\vartheta(T',t)}\downarrow & & {\scriptstyle\vartheta(S',t)}\downarrow & & \| \\ \mathring{\mathcal{M}}([T'']) & \xrightarrow[\cong]{\iota_{T''}} & \mathcal{M}(T'') & \xleftarrow{\phi''^*} & \mathcal{M}(S'') & \xleftarrow[\cong]{\iota_{S''}} & \mathring{\mathcal{M}}([S'']) \,. \end{array}$$

The independence of $\mathring{\mathcal{M}}([\phi])$ on the choice of a representative of $[\phi]$ is now clear. □

Remark 2.9. If $\mathcal{M}$ is strictly unital, the definition in (34) via a colimit can be replaced by $\mathring{\mathcal{M}}([T]) := \mathcal{M}(T)$.

3 Free Markl operads

This section is devoted to our construction of free strictly unital 1-connected Markl operads over $\mathtt{O}$ generated by $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaves. In the light of [3, Theorem 6.5] this will also provide free (standard) $\mathtt{O}$-operads. We require $\mathtt{O}$ to satisfy *Assumptions 1.2*. The base symmetric monoidal category is assumed to be monoidally cocomplete.

3.1 Chains of elementary morphisms

In this subsection we introduce the cornerstones of free Markl operads. We will need the following result.

Proposition 3.1. *Consider a diagram*

$$\begin{array}{ccc} T' & \xrightarrow[\sim]{\sigma_T} & T'' \\ {\scriptstyle(\phi',j')}\downarrow & & \downarrow{\scriptstyle(\phi'',j'')} \\ P' & \xrightarrow[\sim]{\sigma_P} & P'' \\ {\scriptstyle(\psi',i')}\downarrow & & \downarrow{\scriptstyle(\psi'',i'')} \\ S' & \xrightarrow[\sim]{\sigma_S} & S'' \end{array} \tag{36}$$

whose vertical maps are elementary with disjoint fibers as indicated, and where the horizontal maps are quasibijections. Denoting $k' := |\psi'|(j')$, $k'' := |\psi''|(j'')$, *one has*

$$|\sigma_S|(i') = i'' \quad |\sigma_S|(k') = k''. \tag{37}$$

If we are given a subdiagram of (36) consisting only of the morphisms $\phi', \phi'', \psi', \psi'', \sigma_T$ *and* σ_S, *i.e.*

$$\begin{array}{ccc} T' & \xrightarrow[\sim]{\sigma_T} & T'' \\ {\scriptstyle(\phi',j')}\downarrow & & \downarrow{\scriptstyle(\phi'',j'')} \\ P' & & P'' \\ {\scriptstyle(\psi',i')}\downarrow & & \downarrow{\scriptstyle(\psi'',i'')} \\ S' & \xrightarrow[\sim]{\sigma_S} & S'', \end{array} \tag{38}$$

then the conditions (37) are also sufficient for the existence of a unique quasibijection σ_P *as in (36).*

Proof. The only nontrivial fiber of ψ' is ${\psi'}^{-1}(i')$ and the only nontrivial fiber of ψ'' is ${\psi''}^{-1}(i'')$ so, by [3, Remark 5.2], we have $|\sigma_S|(i') = i''$. By the same argument, $|\sigma_P|(j') = j''$. Since $|-|$ is a functor, we have

$$k'' = |\psi''||\sigma_P|(j') = |\sigma_S||\psi'|(j') = |\sigma_S|(k')$$

proving the first part of the proposition.

To prove the second part, denote by ξ' (resp. by ξ'') the composite of the maps in the left (resp. right) column of (38). Since the left column of (38) is harmonic by [3, Corollary 5.7], we may define a map $(\sigma_P)_{(i',i'')}$ by the commutativity of the diagram

$$\begin{array}{ccc} {\xi'}^{-1}(i') & \xrightarrow{(\sigma_T)_{(i',i'')}} & {\xi''}^{-1}(i'') \\ {\scriptstyle\phi'_{i'}=\mathbb{1}}\| & & \downarrow{\scriptstyle\phi''_{i''}} \\ {\psi'}^{-1}(i') & \xrightarrow{(\sigma_P)_{(i',i'')}} & {\psi''}^{-1}(i''). \end{array} \tag{39}$$

The blow-up axiom produces a commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow[\sim]{\sigma_P} & \tilde{P}'' \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\tilde{\psi}''} \\ S' & \xrightarrow[\sim]{\sigma_S} & S'' \end{array}$$

in which, by construction, $\tilde{\psi}''$ is elementary with the only nontrivial fiber ${\psi''}^{-1}(i'')$ over i'', and the map between nontrivial fibers induced by σ_P is $(\sigma_P)_{(i',i'')}$. Consider now two commutative diagrams

$$\begin{array}{ccc} T' & \xrightarrow[\sim]{\sigma_P\phi'} & \tilde{P}'' \\ {\scriptstyle\xi'}\downarrow & & \downarrow{\scriptstyle\tilde{\psi}''} \\ S' & \xrightarrow[\sim]{\sigma_S} & S'' \end{array} \qquad \begin{array}{ccc} T' & \xrightarrow[\sim]{\phi''\sigma_T} & P'' \\ {\scriptstyle\xi'}\downarrow & & \downarrow{\scriptstyle\psi''} \\ S' & \xrightarrow[\sim]{\sigma_S} & S''. \end{array} \tag{40}$$

In both diagrams, the right vertical map is elementary, with the only nontrivial fiber $\psi''^{-1}(i'')$. We will show that both $\sigma_P\phi'$ and $\phi''\sigma_T$ induce the same map between nontrivial fibers. One has

$$(\sigma_P\phi')_{(i',i'')} = (\sigma_P)_{(i',i'')}\phi'_{i'}$$

while

$$(\phi''\sigma_T)_{(i',i'')} = \phi''_{i''}(\sigma_T)_{(i',i'')}.$$

By the defining diagram (39), the right-hand sides of both equations coincide. By BU, the diagrams in (40) are the same, therefore both squares in (36) with σ_P constructed above commute. This finishes the proof. □

For the validity of the following lemma and Lemma 3.4 below, only the weak blow-up axiom WBU is required.

Lemma 3.2. *Let $\rho : S \to T \in \mathtt{O}_{\mathtt{ord}}$ be elementary with the unique fiber F over $a \in |T|$. Suppose that we are given a chain of elementary morphisms*

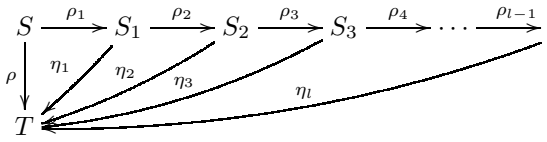
$$F \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_2} F_2 \xrightarrow{\varphi_3} F_3 \xrightarrow{\varphi_4} \cdots \xrightarrow{\varphi_{l-1}} F_{l-1}. \tag{41a}$$

Then there exists a unique factorization

$$S \xrightarrow{\rho_1} S_1 \xrightarrow{\rho_2} S_2 \xrightarrow{\rho_3} S_3 \xrightarrow{\rho_4} \cdots \xrightarrow{\rho_{l-1}} S_{l-1} \xrightarrow{\rho_l} T \tag{41b}$$

of ρ into elementary morphisms such that $(\rho_l \cdots \rho_s)^{-1}(a) = F_{s-1}$ for each $2 \le s \le l$, and $(\rho_s)_a = \varphi_s$ for each $1 \le s < l$.

Proof. We will inductively construct maps in the commutative diagram



(42)

The weak blow-up axiom implies that the maps

$$\varphi_1 : F = \rho^{-1}(a) \to F_1,\ \ \mathbb{1} : \rho^{-1}(i) = U_i \to U_i \text{ for } i \neq a,$$

uniquely determine a decomposition $\rho = \eta_1\rho_1$. Clearly, η_1 is elementary with the unique fiber F_1 and we may apply the same reasoning to η_1 in place of ρ. The result will be a unique decomposition $\eta_1 = \eta_2\rho_2$. Repeating this process $(l-1)$ times and defining $\rho_l := \eta_l$ finishes the proof. □

Remark 3.3. In the situation of (41a), assume that the pair $(\varphi_t, \varphi_{t+1})$ has (i,j)-disjoint fibers for some $1 \le t \le l-2$. Then the corresponding pair (ρ_t, ρ_{t+1}) in (41b) has $(i+a-1, j+a-1)$-disjoint fibers. This is an immediate consequence of Axiom (iv) of an operadic category.

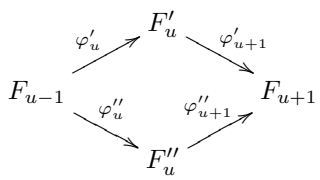
Lemma 3.4. *With the notation of Lemma 3.2, suppose that we are given two chains of elementary morphisms as in (41a) of the form*

$$F \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{u-1}} F_{u-1} \xrightarrow{\varphi'_u} F'_u \xrightarrow{\varphi'_{u+1}} F_{u+1} \xrightarrow{\varphi_{u+2}} \cdots \xrightarrow{\varphi_{l-1}} F_{l-1} \tag{43a}$$

and

$$F \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{u-1}} F_{u-1} \xrightarrow{\varphi''_u} F''_u \xrightarrow{\varphi''_{u+1}} F_{u+1} \xrightarrow{\varphi_{u+2}} \cdots \xrightarrow{\varphi_{l-1}} F_{l-1} \tag{43b}$$

such that the diagram



commutes. Then the corresponding decompositions (41b) are of the form

$$S \xrightarrow{\rho_1} S_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{u-1}} S_{u-1} \xrightarrow{\rho'_u} S'_u \xrightarrow{\rho'_{u+1}} S_{u+1} \xrightarrow{\rho_{u+2}} \cdots \xrightarrow{\rho_{l-1}} S_{l-1} \xrightarrow{\rho_l} T, \tag{44a}$$

respectively

$$S \xrightarrow{\rho_1} S_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{u-1}} S_{u-1} \xrightarrow{\rho''_u} S''_u \xrightarrow{\rho''_{u+1}} S_{u+1} \xrightarrow{\rho_{u+2}} \cdots \xrightarrow{\rho_{l-1}} S_{l-1} \xrightarrow{\rho_l} T, \tag{44b}$$

and the diagram

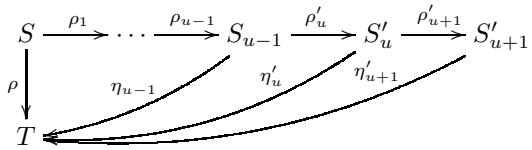
$$\begin{array}{ccccc} & & S'_u & & \\ & \overset{\rho'_u}{\nearrow} & & \overset{\rho'_{u+1}}{\searrow} & \\ S_{u-1} & & & & S_{u+1} \\ & \underset{\rho''_u}{\searrow} & & \underset{\rho''_{u+1}}{\nearrow} & \\ & & S''_u & & \end{array} \tag{45}$$

commutes.

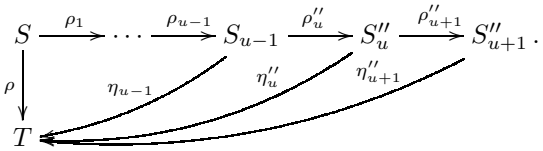
Proof. We will rely on the notation used in the proof of Lemma 3.2. It is clear from the inductive construction described there that the initial parts of the canonical decompositions corresponding to (43a) resp. (43b) coincide and are equal to

$$S \xrightarrow{\rho_1} S_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{u-1}} S_{u-1}.$$

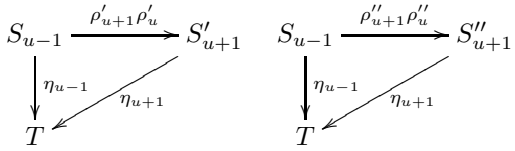
Consider the following two stages of the inductive construction in the proof of Lemma 3.2:



and



The maps η_{u-1}, η'_{u+1} and η''_{u+1} are elementary, with the nontrivial fibers F_{u-1} resp. F_{u+1}. By construction, the horizontal maps in the factorizations



induce the same map between these nontrivial fibers, namely $\varphi'_{u+1}\varphi'_u = \varphi''_{u+1}\varphi''_u$. By the uniqueness of the weak blow-up, the diagrams in the above display coincide, so diagram (45) with $S_{u+1} = S'_{u+1} = S''_{u+1}$ commutes. It is obvious that the remaining parts of (44a) and (44b) are the same. □

3.2 Free operads

Let us proceed to our description of free Markl operads. In this subsection, $\mathtt{V}$ will be a cocomplete strict symmetric monoidal category and 0 its initial object.

Definition 3.5. $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaves in $\mathtt{V}$ will be called 1-*connected* $\mathtt{O}$-*collections* in $\mathtt{V}$. We will denote by $\mathtt{Coll}_1^{\mathtt{V}}(\mathtt{O})$, or simply $\mathtt{Coll}_1^{\mathtt{V}}$ when $\mathtt{O}$ is understood, the corresponding category.

For $E \in \mathtt{Coll}_1^{\mathtt{V}}$, we will often write simply $E[T]$ instead of $E([T])$. Notice that a 1-connected $\mathtt{O}$-collection can equivalently be defined as a $\mathtt{QO}_{\mathtt{vrt}}$-presheaf E such that $E(T) = 0$ if $e(T) = 0$. It follows that a 1-connected $\mathtt{O}$-collection in $\mathtt{V}$ is the same as an $\mathtt{O}_{\mathtt{iso}}$-presheaf $\mathcal{E}$ with values in $\mathtt{V}$ such that

(i) $\mathcal{E}(T) = 0$ if $e(T) = 0$ (1-connectivity),

(ii) $\mathcal{E}(T) = \mathcal{E}(F)$ whenever $F \blacktriangleright T \xrightarrow{!} u$, and

(iii) $\phi'^* = \phi''^*$, where ϕ' is as in (27) and ϕ'' is the induced map between the fibers.

Example 3.6. It follows from Example 2.3 that the category $\mathtt{Coll}_1^{\mathtt{V}}(\mathtt{Fin})$ is isomorphic to the category of 1-connected Σ-modules, i.e. sequences $\{E(n) \in \mathtt{V}\}_{n\geq 2}$, with actions of the symmetric groups Σ_n.

Proposition 3.7. *One has a forgetful functor* $\square : \mathtt{SUMOp}_1^{\mathtt{V}}(\mathtt{O}) \to \mathtt{Coll}_1^{\mathtt{V}}(\mathtt{O})$ *from the category of* 1*-connected strictly unital Markl* $\mathtt{O}$*-operads to the category of* 1*-connected* $\mathtt{O}$*-collections in a cocomplete symmetric monoidal category* $\mathtt{V}$ *defined, on objects by*

$$\square\mathcal{M}\big([T]\big) := \begin{cases} \mathcal{M}(T) & \text{if } e(T) \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Proof. The $\mathtt{O}_{\mathtt{iso}}$-presheaf structure induces on $\mathcal{M}$ an $\mathtt{O}_{\mathtt{vrt}}(e)\int\mathtt{O}_{\mathtt{iso}}$-presheaf structure by Proposition 2.7. The functor $\square$ is then the composite of the functor $\mathcal{M} \mapsto \mathring{\mathcal{M}}$ of Proposition 2.8, cf. Remark 2.9, with the functor that replaces the values of the presheaf $\mathring{\mathcal{M}}$ by 0 on objects of grade zero. □

In the rest of this section we construct a left adjoint $E \mapsto \mathbb{F}(E)$ to the forgetful functor of Proposition 3.7. Our strategy will be to construct a Markl $\mathtt{O}$-operad $\mathtt{lTw}$ with values in the category of groupoids $\mathtt{Grp}$, extend E to a functor $E : \mathtt{lTw} \to \mathtt{V}$ and define $\mathbb{F}(E)$ as the colimit of this functor. The building blocks of the operad $\mathtt{lTw}$ will be the towers

$$\mathfrak{T} := T \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_2} T_2 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{k-1}} T_{k-1} \xrightarrow{\tau_k} U_c \tag{47}$$

of elementary morphisms as in Definition 1.3, with τ_k the unique morphism to a chosen local terminal object U_c. Since τ_k bears no information, we will sometimes write the tower as

$$\mathfrak{T} := T \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_2} T_2 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{k-1}} T_{k-1}. \tag{48}$$

Let $t_1, \ldots, t_k$ be the unique nontrivial fibers of $\tau_1, \ldots, \tau_k$; notice that $t_k = T_{k-1}$. We will call $t_1, \ldots, t_k$ the *fiber sequence* of the tower $\mathfrak{T}$. The number k is the *height* of the tower $\mathfrak{T}$.

We will denote by $Tw(T)$ the set of all towers with the initial term T. A *morphism* $\boldsymbol{\sigma} : \mathfrak{T}' \to \mathfrak{T}''$ of towers (47) is an array $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ of isomorphisms as in

$$\begin{array}{ccc} T' & \xrightarrow[\cong]{\sigma_1} & T'' \\ {\scriptstyle\tau'_1}\downarrow & & \downarrow{\scriptstyle\tau''_1} \\ T'_1 & \xrightarrow[\cong]{\sigma_2} & T''_1 \\ {\scriptstyle\tau'_2}\downarrow & & \downarrow{\scriptstyle\tau''_2} \\ \vdots & & \vdots \\ {\scriptstyle\tau'_{k-1}}\downarrow & & \downarrow{\scriptstyle\tau''_{k-1}} \\ T'_{k-1} & \xrightarrow[\cong]{\sigma_k} & T''_{k-1}\,. \end{array} \tag{49}$$

Definition 3.8. A *labeled tower* is a pair $(\ell, \mathfrak{T})$ consisting of a tower $\mathfrak{T}$ as in (47) together with an isomorphism (the *labeling*) $\ell : X \to T$. We denote by $lTw(X)$ the set of all labeled towers of this form.

We will equip $lTw(X)$ with the structure of a groupoid generated by morphisms of two types, modulo the commutativity relations specified below. Each morphism $\boldsymbol{\sigma} : \mathfrak{T}' \to \mathfrak{T}''$ of towers (47) determines a morphism $(\ell, \boldsymbol{\sigma}) : (\ell, \mathfrak{T}') \to (\sigma_1 \ell, \mathfrak{T}'')$ of the *first type*. These morphisms compose as follows. Suppose that $(\ell, \boldsymbol{\sigma}') : (\ell, \mathfrak{T}') \to (\sigma'_1 \ell, \mathfrak{T}'')$ and $(\sigma'_1 \ell, \boldsymbol{\sigma}'') : (\sigma'_1 \ell, \mathfrak{T}'') \to (\sigma''_1 \sigma'_1 \ell, \mathfrak{T}''')$ are two such morphisms, then

$$(\sigma'_1 \ell, \boldsymbol{\sigma}'')(\ell, \boldsymbol{\sigma}') := (\sigma''_1 \sigma'_1 \ell, \boldsymbol{\sigma}'' \boldsymbol{\sigma}').$$

To define morphisms of the second type, consider two towers of elementary morphisms,

$$\mathfrak{T}' := T \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_{u-1}} T_{u-1} \xrightarrow{\tau'_u} T'_u \xrightarrow{\tau'_{u+1}} T_{u+1} \xrightarrow{\tau_{u+2}} \cdots \xrightarrow{\tau_{k-1}} T_{k-1}$$

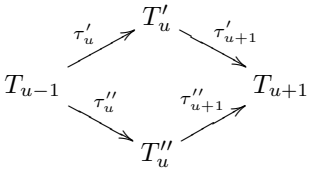
and

$$\mathfrak{T}'' := T \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_{u-1}} T_{u-1} \xrightarrow{\tau''_u} T''_u \xrightarrow{\tau''_{u+1}} T_{u+1} \xrightarrow{\tau_{u+2}} \cdots \xrightarrow{\tau_{k-1}} T_{k-1},$$

as in (48). Their associated fiber sequences are clearly of the form

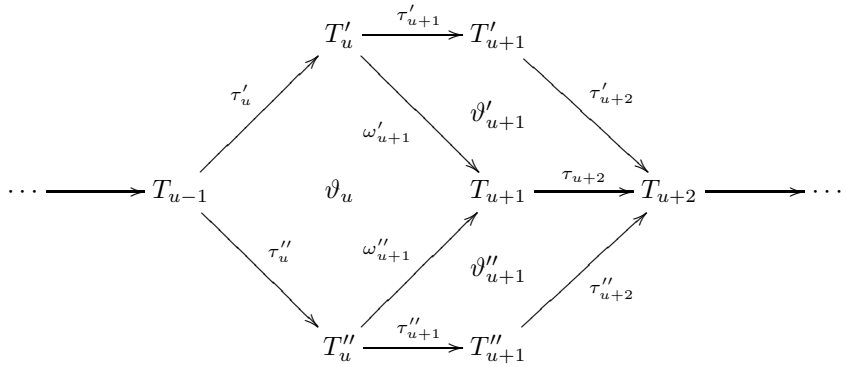
$$t_1, \ldots, t_{u-1}, t'_u, t'_{u+1}, t_{u+2}, \ldots, t_k \quad \text{resp.} \quad t_1, \ldots, t_{u-1}, t''_u, t''_{u+1}, t_{u+2}, \ldots, t_k.$$

Assume that the diagram



is as in (9) from Corollary 1.7, with $\phi' = \tau'_u$, $\phi'' = \tau''_u$, $\psi' = \tau'_{u+1}$ and $\psi'' = \tau''_{u+1}$, and thus $t'_u = t''_{u+1}$ and $t'_{u+1} = t''_u$. The above situation, by definition, determines an invertible morphism $\vartheta_u : (\ell, \mathfrak{T}') \to (\ell, \mathfrak{T}'')$ of the *second type*. The resulting groupoid will be denoted $\mathtt{lTw}(X)$.

Morphisms of both types are subject to relations. They are easy to figure out, so we address only the most complicated case of morphisms ϑ_u, ϑ_v of the second type with $|v-u| = 1$. To this end, consider the diagram

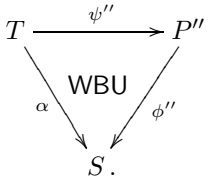


(50)

representing the composition $\vartheta''_{u+1} \vartheta_u \vartheta'_{u+1}$, such that τ'_u is r-elementary, τ'_{u+1} is j-elementary, τ'_{u+2} is i-elementary, and $i, |\tau'_{u+2}|(j)$ $|\tau'_{u+2}\tau'_{u+1}|(r)$ are mutually distinct elements of $|T_{u+2}|$ – the case of three disjoint fibers. Notice that the maps τ'_u, τ'_{u+1} and τ'_{u+2} uniquely determine the remaining ones, by the following

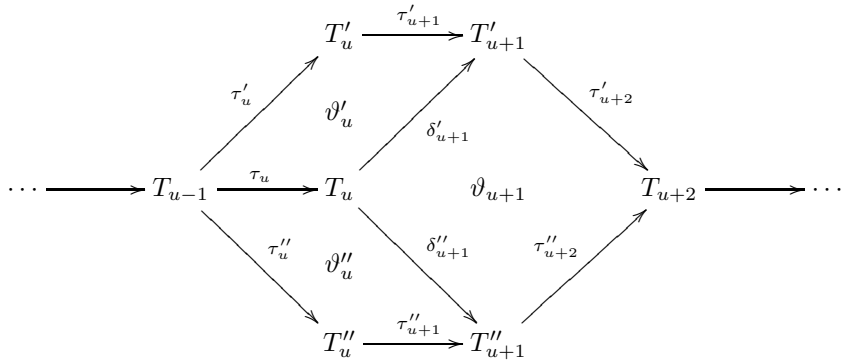
Lemma 3.9. *There exists a natural reflection $(\phi', \psi') \leftrightarrow (\phi'', \psi'')$ between (k,i)-pairs, cf. Definition 1.4, and (i,k)-pairs such that $\phi'\psi' = \phi''\psi''$.*

Proof. Assume that $T \xrightarrow{\psi'} P' \xrightarrow{\phi'} S$ is a (k,i)-pair. Then $\alpha := \phi'\psi'$ has precisely two nontrivial fibers, say F' over k and F'' over i. The weak blow-up axiom produces the commutative diagram



in which ϕ'' is elementary, with its only nontrivial fiber F' over k, under the condition that the prescribed fiber maps ψ''_s, $s \in |T|$, are the unique maps to the chosen local terminal objects except ψ''_k which is the identity $\mathbb{1} : F' \to F'$. Then (ϕ'', ψ'') is an (i,k)-pair. Applying the above construction to (ϕ'', ψ'') clearly produces the original (ϕ', ψ'). □

Returning to (50), Lemma 3.9 with $\phi' = \tau'_{u+2}$ and $\psi' = \tau'_{u+1}$ gives the maps τ_{u+2} and ω'_{u+1}, the maps ω'_{u+1} and τ'_u determine ω''_{u+1} and τ''_u and, finally, τ_{u+2} and ω''_{u+1} determine τ''_{u+2} and τ''_{u+1}. In the same manner, we obtain the diagram



One can prove, by generalizing Lemma 3.9 to the case of three morphisms, that the maps τ''_u, τ''_{u+1} and τ''_{u+2} are the same in both diagrams, giving rise to the relation

$$\vartheta''_{u+1}\vartheta_u\vartheta'_{u+1} = \vartheta''_u\vartheta_{u+1}\vartheta'_u. \tag{51}$$

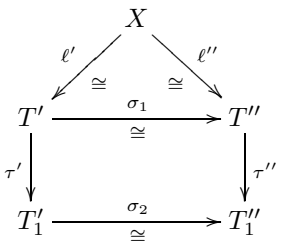
Since all morphisms of the second type are involutions by Lemma 3.9 again, we notice the resemblance of (51) and the relation between the generating transpositions of the symmetric group Σ_3. This is because morphisms of the second type generalize the interchange of adjacent levels in a leveled tree.

Example 3.10. Since morphisms of both types preserve the height of towers, the groupoid $\mathtt{lTw}(X)$ is graded,

$$\mathtt{lTw}(X) = \coprod_{h\geq 1} \mathtt{lTw}^h(X),$$

where $\mathtt{lTw}^h(X)$ is the subgroupoid of labeled towers of height h. It is clear that $\mathtt{lTw}^1(X)$ is the category $X/\mathtt{O}_{\mathtt{iso}}$ of isomorphisms in $\mathtt{O}$ under X.

In $\mathtt{lTw}^2(X)$, only morphisms of the first type exist. Therefore, labeled towers $(\ell', \mathfrak{T}')$ and $(\ell'', \mathfrak{T}'')$ are connected by a morphism if and only if there is a commuting diagram



(52)

with isomorphisms σ_1 and σ_2.

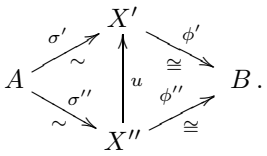
For an isomorphism $\omega : X' \xrightarrow{\cong} X''$ one has the induced map $\omega^* : lTw(X'') \to lTw(X')$ that sends the labeled tower $(\ell'', \mathfrak{T}'') \in lTw(X'')$ to $(\ell''\omega, \mathfrak{T}'') \in lTw(X')$, which clearly extends to a functor (denoted by the same symbol) $\omega^* : \mathtt{lTw}(X'') \to \mathtt{lTw}(X')$. This makes the collection of categories $\mathtt{lTw}(X)$ a $\mathtt{Grp}$-presheaf on $\mathtt{O}_{\mathtt{iso}}$. Our next move will be to construct, for each $G \triangleright W \xrightarrow{\phi} X$, a functor

$$\circ_\phi : \mathtt{lTw}(X) \times \mathtt{lTw}(G) \to \mathtt{lTw}(W). \tag{53}$$

As the first step in this construction we will prove that each labeled tower $(\ell, \mathfrak{T})$ can be functorially replaced by one in which ℓ is a quasibijection. To this end we prove a couple of auxiliary lemmas.

Lemma 3.11. *The factorization $\xi = \phi\sigma$, $\phi \in \mathtt{O}_{\mathtt{ord}}$, $\sigma \in \mathtt{O}_{\mathtt{qb}}$, of an isomorphism $\xi : A \to B$ guaranteed by the factorization axiom is unique, and both ϕ and σ are isomorphisms too.*

Proof. Consider two factorizations, $\phi'\sigma'$ and $\phi''\sigma''$, of ξ. Since σ' and σ'' are quasibijections, they are invertible, so one may define u by the commutativity of the diagram



By the left triangle, u is a quasibijection while it belongs to $\mathtt{O}_{\mathtt{ord}}$ by the right triangle. The uniqueness follows from [3, Corollary 2.6]. The invertibility of ϕ' and ϕ'' is clear. □

Lemma 3.12. *Each corner*

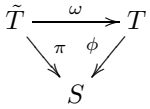
$$\begin{array}{ccc} \tilde{T} & \xrightarrow[\cong]{\omega} & T \\ & & \downarrow{\scriptstyle \phi} \\ & & S \end{array} \tag{54a}$$

in which ω is an isomorphism from $\mathtt{O}_{\mathtt{ord}}$ and ϕ is elementary, can be canonically and functorially completed to the square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow[\cong]{\omega} & T \\ {\scriptstyle \tilde{\phi}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \tilde{S} & \xrightarrow[\cong]{\tilde{\omega}} & S \end{array} \tag{54b}$$

with $\tilde{\omega}$ an isomorphisms from $\mathtt{O}_{\mathtt{ord}}$ and $\tilde{\phi}$ elementary.

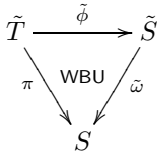
Proof. Since ϕ is elementary, it has one fiber $F = \phi^{-1}(i)$ for some $i \in |S|$ such that $e(F) \geq 1$, and the remaining fibers are the chosen local terminal objects $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{|S|}$. Consider the diagram



in which $\pi := \phi\omega$. Since ω is an isomorphism, by the functoriality of fiber functors, each induced map $\omega_s : \pi^{-1}(s) \to \phi^{-1}(s)$, $s \in |S|$, is an isomorphism too. Using this and the obvious fact that $\pi \in \mathtt{O}_{\mathtt{ord}}$, we see that the ordered list of fibers of π equals

$$v_1, \ldots, v_{i-1}, G, v_{i+1}, \ldots, v_{|S|}$$

where $e(G) \geq 1$, while all the remaining fibers are of grade 0 and cardinality 1. The WBU produces the diagram



such that the ordered list of fibers of $\tilde{\omega}$ equals

$$v_1, \ldots, v_{i-1}, U, v_{i+1}, \ldots, v_{|S|}$$

and all the induced maps between the fibers are the identities except $\tilde{\phi}_i : G \to U$, which is the unique map to the chosen local terminal object U in the connected component of G.

Axiom (iv) of an operadic category, cf. [3, Section 1] or [2, page 1634], identifies the fibers of $\tilde{\phi}$ with the corresponding fibers of the maps induced by $\tilde{\phi}$ between the fibers. Since, again by the axioms of an operadic category, the fibers of the identity map are the chosen local terminal objects,

and the fiber of the unique map $G \to U$ is G, we conclude that $\tilde{\phi}$ is elementary. Finally, $\tilde{\omega}$ is an isomorphism by [3, Lemma 2.12]. This concludes the constructions.

The functoriality means that any isomorphism of corners in (54a), in the usual sense of isomorphisms of diagrams, uniquely extends to an isomorphism of the corresponding squares (54b). This can be established by a standard diagram chase. □

Proposition 3.13. *Each $(\ell, \mathfrak{T}) \in \mathtt{lTw}(X)$ can be functorially replaced within its isomorphism class by some $(\tilde{\ell}, \tilde{\mathfrak{T}})$ in which $\tilde{\ell}$ is a quasibijection.*

Proof. Let $\mathfrak{T}$ be as in (47) and let $\ell : X \to T$ be an isomorphism. We decompose ℓ as $\sigma_1 \tilde{\ell}$, with $\tilde{\ell}$ a quasibijection and σ_1 an isomorphism in $\mathtt{O}_{\mathtt{ord}}$. Lemma 3.12 gives a canonical square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow[\cong]{\sigma_1} & T \\ {\scriptstyle \tilde{\tau}_1}\downarrow & & \downarrow{\scriptstyle \tau_1} \\ \tilde{T}_1 & \xrightarrow[\cong]{\sigma_2} & T_1 \end{array}$$

in which $\tilde{\tau}_1$ is elementary and σ_2 an isomorphism in $\mathtt{O}_{\mathtt{ord}}$. Repeated application of Lemma 3.12 produces a tower $\tilde{\mathfrak{T}}$ labeled by the quasibijection $\tilde{\ell} : X \to \tilde{T}$, as well as a morphism of the first type $(\tilde{\ell}, \tilde{\mathfrak{T}}) \to (\ell, \mathfrak{T})$. □

Let $\widetilde{\mathtt{lTw}}(X)$ be the graded category whose objects are towers $(\ell, \mathfrak{T})$ labeled by a quasibijection. Morphisms of the first type in $\widetilde{\mathtt{lTw}}(X)$ are those $(\ell, \boldsymbol{\sigma}) : (\ell, \mathfrak{T}') \to (\sigma_1 \ell, \mathfrak{T}'')$ in which σ_1 is a quasibijection. Morphisms of the second type are the same as those in $\mathtt{lTw}(X)$. Notice that $\widetilde{\mathtt{lTw}}(X)$ is a full subcategory of $\mathtt{lTw}(X)$. Indeed, if both ℓ and $\sigma_1 \ell$ are quasibijections, then ℓ^{-1} is a quasibijection by QBI, so $\sigma_1 = (\sigma_1 \ell)\ell^{-1}$ is a quasibijection too. This, along with Proposition 3.13, implies that $\widetilde{\mathtt{lTw}}(X)$ is a full reflective graded subcategory of $\mathtt{lTw}(X)$.

Example 3.14. Labeled towers $(\ell', \mathfrak{T}'), (\ell'', \mathfrak{T}'') \in \widetilde{\mathtt{lTw}}^2(X)$ are isomorphic if and only if there is the commuting diagram (52) in which the maps in the upper triangle are quasibijections.

Let $G \triangleright W \xrightarrow{\phi} X$ be elementary. Assume we are given a labeled tower $(\ell', \mathfrak{F}) \in lTw(G)$, where

$$\mathfrak{F} := F \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_2} F_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{l-1}} F_{l-1} \tag{55}$$

is a tower with the associated fibers $f_1, \dots, f_l$, with the labeling $\ell' : G \to F$. Assume we are also given a labeled tower $(\tilde{\ell}, \mathfrak{T}) \in \widetilde{\mathtt{lTw}}(X)$, with $\tilde{\ell}$ a quasibijection. The blow-up axiom gives a unique diagram

$$\begin{array}{ccc} W & \xrightarrow[\cong]{\ell''} & S \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow[\sim]{\tilde{\ell}} & T \end{array}$$

in which $F \triangleright S \xrightarrow{\rho} T$ is elementary and ℓ'' an isomorphism inducing the map $\ell' : G \to F$ between the unique nontrivial fibers of ϕ and ρ, respectively. Lemma 3.2 gives the composite tower

$$\mathfrak{T} \circ_\rho \mathfrak{F} := S \xrightarrow{\rho_1} S_1 \xrightarrow{\rho_2} S_2 \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{l-1}} S_{l-1} \xrightarrow{\rho_l} T \xrightarrow{\tau_1} T_1 \xrightarrow{\tau_2} T_2 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_{k-1}} T_{k-1} \tag{56}$$

whose initial part is (41b), so we have the composite labeled tower

$$(\tilde{\ell}, \mathfrak{T}) \circ_\phi (\ell', \mathfrak{F}) := (\ell'', \mathfrak{T} \circ_\rho \mathfrak{F}) \in lTw(W). \tag{57}$$

The above construction clearly extends to a functor

$$\circ_\phi : \widetilde{\mathtt{lTw}}(X) \times \mathtt{lTw}(G) \to \mathtt{lTw}(W)$$

which, precomposed with the equivalence $\mathtt{lTw}(X) \to \widetilde{\mathtt{lTw}}(X)$ in the first variable, gives (53).

Let $E \in \mathtt{Coll}_1^{\mathtt{V}}$ be a 1-connected collection. For a tower (48) we define

$$E(\mathfrak{T}) := E[t_1] \otimes \cdots \otimes E[t_k] \in \mathtt{V}.$$

We will show how the rule $E(\ell, \mathfrak{T}) := E(\mathfrak{T})$ extends to a functor $E : \mathtt{lTw}(X) \to \mathtt{V}$. Consider a morphism $(\ell, \boldsymbol{\sigma}) : (\ell, \mathfrak{T}') \to (\sigma_1 \ell, \mathfrak{T}'')$ of the first type, with $\boldsymbol{\sigma} : \mathfrak{T}' \to \mathfrak{T}''$ a map of towers as in (49). For each $0 \leq s \leq k$ one has the commutative diagram

$$\begin{array}{ccc} T'_s & \xrightarrow[\cong]{\sigma_{s+1}} & T''_s \\ {\scriptstyle (\tau'_s, i)}\downarrow & \overset{\tau_s}{\searrow} & \downarrow{\scriptstyle (\tau''_s, j)} \\ T'_{s-1} & \xrightarrow[\cong]{\sigma_s} & T''_{s-1} \end{array}$$

in which $\tau_s := \sigma_s \tau'_s = \tau''_s \sigma_{s+1}$ and where $T'_s := T'$, $T''_s := T''$ if $s = 0$. Lemma 1.1 provides us with

$$t'_s \triangleright \tau_s^{-1}(j) \overset{(\tau'_s)_j}{\longrightarrow} \sigma_s^{-1}(j) \quad \text{and} \quad (\sigma_{s+1})_j : \tau_s^{-1}(j) \to t''_s$$

where $\sigma_s^{-1}(j)$ is local terminal by [3, Lemma 2.12], so we can define $\sigma_s^\star : E[t''_s] \to E[t'_s]$ as the composite

$$\sigma_s^\star : E[t''_s] \overset{(\sigma_{s+1})_j^*}{\longrightarrow} E[\tau_s^{-1}(j)] = E[t'_s]$$

where the equality uses the fact that E is constant along virtual isomorphisms. This in turn induces a map

$$\boldsymbol{\sigma}^\star : E(\mathfrak{T}'') = E[t''_1] \otimes \cdots \otimes E[t''_k] \longrightarrow E(\mathfrak{T}') = E[t'_1] \otimes \cdots \otimes E[t'_k] \tag{58}$$

by $\boldsymbol{\sigma}^\star := \sigma_1^\star \otimes \cdots \otimes \sigma_k^\star$. Define finally $E(\ell, \boldsymbol{\sigma}) : E(\ell, \mathfrak{T}'') \to E(\sigma_1 \ell, \mathfrak{T}')$ as $E(\ell, \boldsymbol{\sigma}) := \boldsymbol{\sigma}^\star$.

Let us discuss morphisms of the second type. Corollary 1.7 gives identities

$$t'_u = t''_{u+1} \quad \text{and} \quad t''_u = t'_{u+1}.$$

We define the E-image of this map as the identification of

$$e_1 \otimes \cdots \otimes e''_{u+1} \otimes e'_{u+1} \otimes \cdots \otimes e_k \in E[t_1] \otimes \cdots \otimes E[t'_u] \otimes E[t'_{u+1}] \otimes \cdots \otimes E[t_k]$$

in $E(\mathfrak{T}')$ with

$$e_1 \otimes \cdots \otimes e'_{u+1} \otimes e''_{u+1} \otimes \cdots \otimes e_k \in E[t_1] \otimes \cdots \otimes E[t''_u] \otimes E[t''_{u+1}] \otimes \cdots \otimes E[t_k]$$

in $E(\mathfrak{T}'')$ given by the symmetry constraint in $\mathtt{V}$. It is simple to show that the above definition of the functor E is compatible with the relations between the generating morphisms of $\mathtt{lTw}(X)$. For instance, the compatibility with (51) is guaranteed by the hexagon axiom for the symmetric monoidal category $\mathtt{V}$.

Here and at several places below we use the notation that assumes that the objects of $\mathtt{V}$ have elements. The interested reader can easily rewrite formulas of this type to more general but less intuitive language of diagrams.

Lemma 3.15. *The diagram of functors*

$$\begin{array}{ccc} & & \mathtt{V} \\ & {\scriptstyle E\otimes E}\nearrow \quad \nwarrow{\scriptstyle E} & \\ \widetilde{\mathtt{lTw}}(X) \times \mathtt{lTw}(G) & \xrightarrow{\circ_\phi} & \mathtt{lTw}(W) \end{array}$$

commutes for an arbitrary elementary morphism $G \triangleright W \overset{\phi}{\to} X$.

Proof. Assume that $(\tilde{\ell}, \mathfrak{T}) \in \widetilde{\mathtt{lTw}}(X)$ and $(\ell', \mathfrak{F}) \in \mathtt{lTw}(G)$, with $\mathfrak{T}$ as in (48) and $\mathfrak{F}$ as in (55). Recall that then $(\tilde{\ell}, \mathfrak{T}) \circ_\phi (\ell', \mathfrak{F}) \in \mathtt{lTw}(W)$ is given by formula (57). The crucial fact is that the fiber sequence of $\mathfrak{T} \circ_\rho \mathfrak{F}$ is

$$f_1, \ldots, f_l, t_1, \ldots, t_k,$$

where $f_1, \ldots, f_l$ resp. $t_1, \ldots, t_k$ is the fiber sequence of $\mathfrak{F}$ resp. of $\mathfrak{T}$. The canonical isomorphism

$$E(\ell, \mathfrak{T}) \otimes E(\ell', \mathfrak{F}) \cong E\big((\ell, \mathfrak{T}) \circ_\phi (\ell', \mathfrak{F})\big)$$

then follows immediately from the definition of the functor E as given above. □

Theorem 3.16. *Let* $\mathtt{V}$ *be a cocomplete symmetric monoidal category and let* $E \in \mathtt{Coll}_1^{\mathtt{V}}$ *be a* 1*-connected collection in* $\mathtt{V}$*. Then the formula*

$$\mathbb{F}(E)(X) := \begin{cases} \operatorname*{colim}_{(\ell,\mathfrak{T})\in \mathtt{lTw}(X)} E(\ell,\mathfrak{T}) & \text{if } e(X) \geq 1 \\ \Bbbk & \text{if } e(X) = 0 \end{cases} \tag{59}$$

defines a left adjoint $E \mapsto \mathbb{F}(E)$ *to the forgetful functor* $\square$ *of Proposition 3.7. Therefore* $\mathbb{F}(E)$ *is the free* 1*-connected strictly unital Markl operad generated by* E*.*

Adjoining the monoidal unit in (59) should be compared to adjoining the unit to the free nonunital operad in formula (II.1.58) of [21].

Proof of Theorem 3.16. Assume that $X \in \mathtt{O}$ is such that $e(X) \geq 1$. It is clear that $\mathbb{F}(E)(X)$ is graded by the height k of the underlying towers so that it decomposes as

$$\mathbb{F}(E)(X) \cong \coprod_{k\geq 1} \mathbb{F}^k(E)(X). \tag{60}$$

Elements of $\mathbb{F}^k(E)(X)$ are equivalence classes $[\ell, e]$ consisting of a labeling $\ell : X \xrightarrow{\cong} T$ and of an element $e \in E(\mathfrak{T})$ associated with a labeled tower $(\ell, \mathfrak{T})$ of height k as in Definition 3.8. For an isomorphism $\omega : Y \xrightarrow{\cong} X$ one puts $\omega^*[\ell, e] := [\ell\omega, e] \in \mathbb{F}^k(E)(Y)$. This turns $\mathbb{F}^k(E)$ into an $\mathtt{O}_{\mathtt{iso}}$-presheaf in $\mathtt{V}$. Defining formally $\mathbb{F}^0(E)$ to be the trivial presheaf $\Bbbk$, one thus has a decomposition

$$\mathbb{F}(E) \cong \coprod_{k\geq 0} \mathbb{F}^k(E)$$

of $\mathtt{O}_{\mathtt{iso}}$-presheaves in $\mathtt{V}$.

In particular, this shows that $\mathbb{F}(E)$ is an $\mathtt{O}_{\mathtt{iso}}$-presheaf as required in the definition of Markl operad.

To define the composition, recall that $\widetilde{\mathtt{lTw}}(X)$ is a full reflective subcategory of $\mathtt{lTw}(X)$, therefore one has the canonical isomorphism

$$\mathbb{F}^k(E)(X) \cong \operatorname*{colim}_{(\tilde{\ell},\mathfrak{T})\in \widetilde{\mathtt{lTw}}^k(X)} E(\tilde{\ell},\mathfrak{T})$$

with the colimit taken over towers of height k labeled by quasibijections. The composition law

$$\circ_\phi : \mathbb{F}^k(E)(X) \otimes \mathbb{F}^l(E)(G) \longrightarrow \mathbb{F}^{k+l}(E)(W),\ G \triangleright W \xrightarrow{\phi} X,\ k, l \geq 1, \tag{61}$$

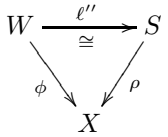
is then defined as the colimit of the natural isomorphism between the functors

$$E \otimes E : \widetilde{\mathtt{lTw}}^k(X) \times \mathtt{lTw}^l(G) \to \mathtt{V} \text{ and } E : \mathtt{lTw}^{k+l}(W) \to \mathtt{V}$$

established in Lemma 3.15. One must also define the composition law in (61) for $k = 0$, i.e., specify a map

$$\circ_\phi : \mathbb{F}^l(E)(G) \cong \mathbb{F}^0(E)(X) \otimes \mathbb{F}^l(E)(G) \longrightarrow \mathbb{F}^l(E)(W). \tag{62}$$

Notice first that the grade of X must be zero, so by our assumptions on $\mathtt{O}$, X is a local terminal object. Consider an element $[\ell', e] \in \mathbb{F}^l(E)(G)$ with $\ell' : G \xrightarrow{\cong} F$ and $e \in E(\mathfrak{F})$ with $\mathfrak{F}$ as in (55). Using the blow-up axiom we include $\phi : W \to X$ to the diagram

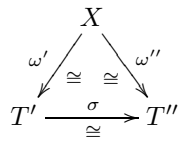


in which ℓ'' induces the map $\ell' : G \xrightarrow{\cong} F$ between the fibers. Let $\mathcal{S}$ be the tower as in (41b) with X in place of T. Then $(\ell'', \mathcal{S}) \in \mathtt{lTw}(W)$. Since by construction the associated fiber sequence of

$\mathfrak{S}$ is the same as the associated fiber sequence of $\mathfrak{F}$, one has $E(\mathfrak{F}) = E(\mathfrak{S})$, thus it makes sense to define $\circ_\phi$ in (62) by $\circ_\phi([\ell', e]) := [\ell'', e]$.

Notice that one cannot have $l = 0$ in (61), since the fiber of an elementary map has always positive grade. We leave to the reader to verify that the above constructions make $\mathbb{F}(E)$ a Markl operad.

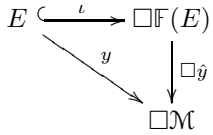
Let us describe $\mathbb{F}^1(E)$ explicitly. As noticed in Example 3.10, $\mathtt{1Tw}^1(X)$ is the category $X/\mathtt{O}_{\mathtt{iso}}$ of isomorphisms in $\mathtt{O}$ under X. Elements of $\mathbb{F}^1(E)(X)$ are equivalence classes $[\omega, e]$ of pairs $\omega : X \xrightarrow{\cong} T$, $e \in E[T]$, modulo the identification $[\sigma\omega', e''] = [\omega', \sigma^* e'']$ for each diagram



of isomorphisms in $\mathtt{O}$. Since $\mathtt{1Tw}^1(X)$ is connected, with a distinguished object $\mathbb{1} : X \to X$, the map $i : E[X] \to \mathbb{F}^1(E)(X)$ given by $i(e) := [\mathbb{1}, e]$ for $e \in E[X]$, is an isomorphism. These isomorphisms assemble into an isomorphism $E \cong \mathbb{F}^1(E)$ of collections. Let us finally denote by $\iota : E \hookrightarrow \square\mathbb{F}(E)$ the composite

$$\iota : E \xrightarrow{\cong} \mathbb{F}^1(E) \hookrightarrow \square\mathbb{F}(E). \tag{63}$$

To establish the freeness of Theorem 3.16 means to prove that, for each 1-connected strictly unital Markl operad $\mathcal{M}$ and a map of collections $y : E \to \square\mathcal{M}$, there exists precisely one map $\hat{y} : \mathbb{F}(E) \to \mathcal{M}$ of strictly unital Markl operads making the diagram



commutative.

Let us assume that such a map $\hat{y} : \mathbb{F}(E) \to \mathcal{M}$ exists and prove that it is unique. To this end consider an arbitrary element $[\ell, e] \in \mathbb{F}(E)(X)$ given by a pair $\ell : X \xrightarrow{\cong} T$, $e \in E(\mathfrak{T})$ for a labeled tower $(\ell, \mathfrak{T}) \in \mathtt{1Tw}(X)$ as in Definition 3.8. For

$$e = e_1 \otimes \cdots \otimes e_k \in E[t_1] \otimes \cdots \otimes E[t_k]$$

it immediately follows from the definition of the operad structure of $\mathbb{F}(E)(X)$ that

$$[\ell, e] = \ell^*\big(e_1 \circ_{\tau_1} (e_2 \circ_{\tau_2} \cdots (e_{k-1} \circ_{\tau_{k-1}} e_k) \cdots)\big),$$

where we used the notation

$$x \circ_{\tau_i} y := (-1)^{|x||y|} \circ_{\tau_i} (y \otimes x)$$

for $x \in E[T_i]$, $y \in E[t_i]$ and $1 \leq i \leq k-1$. We moreover considered $e_1, \ldots, e_k$ as elements of $\mathbb{F}^1(E)$ via the isomorphism $i : E \xrightarrow{\cong} \mathbb{F}^1(E)$. Since $\hat{y}$ is a morphism of operads, we have

$$\hat{y}([\ell, e]) = \ell^*\big(y(e_1) \circ_{\tau_1} (y(e_2) \circ_{\tau_2} \cdots (y(e_{k-1}) \circ_{\tau_{k-1}} y(e_k)) \cdots)\big). \tag{64}$$

On the other hand, one may verify that (64) indeed defines a morphism of operads with the required property, to finish the proof. □

4 Quadratic Markl operads and duality

The goal of this section is to introduce quadratic Markl operads over operadic categories and define their Koszul duals. As customary in this context, the base monoidal category $\mathtt{V}$ here and in the rest of the paper will be the category $\mathtt{Vect}$ of graded vector spaces over a field $\Bbbk$ of characteristic 0. All operads will be tacitly assumed to be strictly unital. The operadic category $\mathtt{O}$ shall fulfill *Assumptions 1.2* plus the rigidity axiom Rig.

Definition 4.1. An *ideal* $\mathcal{I}$ in a Markl operad $\mathcal{M}$ is a sub-$\mathtt{O}_{\mathtt{iso}}$-presheaf of $\mathcal{M}$ which is simultaneously an ideal with respect to the circle products (11), i.e.

$$\circ_\phi(a \otimes b) \in \mathcal{I}(T) \text{ if } a \in \mathcal{I}(S) \text{ or } b \in \mathcal{I}(F) \text{ for } F \triangleright T \xrightarrow{\phi} S.$$

For a sub-$\mathtt{O}_{\mathtt{iso}}$-presheaf R we denote by (R) the component-wise intersection of all ideals containing R. We call (R) the ideal *generated* by R.

Definition 4.2. A *quadratic data* consists of a 1-connected collection $E \in \mathtt{Coll}_1^{\mathtt{V}}$ and an sub-$\mathtt{O}_{\mathtt{iso}}$-presheaf R of $\mathbb{F}^2(E)$. A 1-connected Markl operad $\mathcal{M}$ is *quadratic* if it is of the form

$$\mathcal{M} = \mathbb{F}(E)/(R).$$

It is *binary* if the generating collection E is such that $E[T] \neq 0$ implies that $e(T) = 1$.

Many examples of binary operads will be given in the following sections. Let us proceed to our generalization of the operadic Koszul duality of [10] to operads over general operadic categories.

We start by noticing that the piece $\mathtt{lTw}^k(X)$ of height k of the groupoid $\mathtt{lTw}(X)$ constructed in Section 3 decomposes into the coproduct

$$\mathtt{lTw}^k(X) = \coprod_{c \in \pi_0(\mathtt{lTw}^k(X))} \mathtt{lTw}^k_c(X)$$

over the set $\pi_0(\mathtt{lTw}^k(X))$ of connected components of $\mathtt{lTw}^k(X)$, which is thus also true for the kth piece of the X-component of the free operad

$$\mathbb{F}^k(E)(X) = \bigoplus_{c \in \pi_0(\mathtt{lTw}^k(X))} \mathbb{F}^k_c(E)(X). \tag{65}$$

Choose a labeled tower $(\ell^c, \mathfrak{T}^c)$ in each connected component c of $\mathtt{lTw}^k(X)$ and assume the notation

$$\mathfrak{T}^c := T^c \xrightarrow{\tau_1^c} T_1^c \xrightarrow{\tau_2^c} T_2^c \xrightarrow{\tau_2^c} \cdots \xrightarrow{\tau_{k-1}^c} T_{k-1}^c,$$

with the associated fiber sequence $t_1^c, \ldots, t_k^c$. Since there are no automorphisms of the first type of $(\ell^c, \mathfrak{T}^c)$ in $\mathtt{lTw}^k(X)$ by the rigidity of $\mathtt{O}$, we have

$$\mathbb{F}^k_c(E)(X) \cong E[t_1^c] \otimes \cdots \otimes E[t_k^c],$$

so we have an isomorphism of graded vector spaces

$$\mathbb{F}^k(E)(X) \cong \bigoplus_{c \in \pi_0(\mathtt{lTw}^k(X))} E[t_1^c] \otimes \cdots \otimes E[t_k^c], \tag{66a}$$

cf. the similar presentation [21, formula (II.1.51)] for "ordinary" free operads. In the light of Proposition 3.13, one may assume that the tower $(\ell^c, \mathfrak{T}^c)$ in (65) belongs to $\widetilde{\mathtt{lTw}}^k(X)$, therefore (66a) can be written as the direct sum

$$\mathbb{F}^k(E)(X) \cong \bigoplus_{c \in \pi_0(\widetilde{\mathtt{lTw}}^k(X))} E[t_1^c] \otimes \cdots \otimes E[t_k^c] \tag{66b}$$

over isomorphism classes of objects of $\widetilde{\mathtt{lTw}}^k(X)$.

Let $\uparrow E^*$ be the suspension of the component-wise linear dual of the collection E. With the above preliminaries, it is easy to define a pairing

$$\langle - | - \rangle : \mathbb{F}^2(\uparrow E^*)(X) \otimes \mathbb{F}^2(E)(X) \longrightarrow \Bbbk, \ (\alpha, x) \longmapsto \alpha(x) \tag{67}$$

as follows. If $c' \neq c''$ we declare the subspaces $\mathbb{F}^2_{c'}(\uparrow E^*)(X)$ and $\mathbb{F}^2_{c''}(E)(X)$ of $\mathbb{F}^2(\uparrow E^*)(X)$ resp. $\mathbb{F}^2(E)(X)$ to be orthogonal. If $c := c' = c''$, then

$$\mathbb{F}^2_c(\uparrow E^*)(X) \cong \uparrow E^*[t_1^c] \otimes \uparrow E^*[t_2^c] \ \text{ and } \ \mathbb{F}^2_c(E)(X) \cong E[t_1^c] \otimes E[t_2^c]$$

and the pairing between $\mathbb{F}_c^2(\uparrow E^*)(X)$ and $\mathbb{F}_c^2(E)(X)$ is defined as the canonical evaluation

$$\uparrow E^*[t_1^c] \otimes \uparrow E^*[t_2^c] \otimes E[t_1^c] \otimes E[t_2^c] \longrightarrow \Bbbk.$$

We leave as an exercise to show that this definition does not depend on the choices of the representatives $(\ell^c, \mathfrak{T}^c)$.

Definition 4.3. Let $\mathcal{M}$ be a quadratic Markl operad as in Definition 4.2. Its *Koszul dual* $\mathcal{M}^!$ is the quadratic Markl operad defined as

$$\mathcal{M}^! = \mathbb{F}(\uparrow E^*)/(R^\perp),$$

where $R^\perp$ denotes the component-wise annihilator of R in $\mathbb{F}^2(\uparrow E^*)$ under the pairing (67).

Definition 4.4. A quadratic Markl operad $\mathcal{M}$ is *self-dual* if the associated categories of $\mathcal{M}$- and $\mathcal{M}^!$-algebras in $\mathtt{Vect}$ are isomorphic.

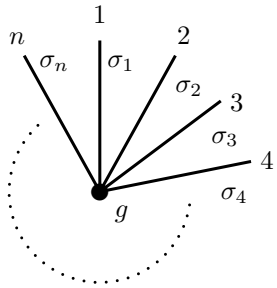
Example 4.5. All assumptions of this section are met by the operadic category $\mathtt{Fin}_{\mathrm{semi}}$ of finite non-empty sets and their surjections. The operads for this category are the classical constant-free operads for which Koszul duality is classical heritage [10]. A similar example is the operadic category Δ_{semi} of non-empty ordered finite sets and their order-preserving surjections. Our theory in this case recovers Koszul duality for nonsymmetric operads. The terminal category $\mathtt{1}$ also satisfies the assumptions of this section. The only 1-connected $\mathtt{1}$-operad, i.e. an associative algebra, is however the ground field $\Bbbk$.

5 Modular and odd modular operads

In this section we analyze binary quadratic operads over the operadic category $\mathtt{ggGrc}$ of connected genus-graded ordered graphs introduced in [3, Example 4.19]. Recall from [3, Example 4.26] that $\mathtt{ggGrc}$ satisfies all of the properties required for Koszul duality, namely, $\mathtt{ggGrc}$ is rigid and fulfills *Assumptions 1.2*.

We will prove that the terminal operad $\mathbf{1}_{\mathtt{ggGrc}}$ in the category $\mathtt{ggGrc}$ is binary quadratic and describe its Koszul dual $\mathfrak{K}_{\mathtt{ggGrc}} := \mathbf{1}^!_{\mathtt{ggGrc}}$. We then show that algebras for $\mathbf{1}_{\mathtt{ggGrc}}$ are modular operads of [9] while algebras for $\mathbf{1}^!_{\mathtt{ggGrc}}$ are their suitably twisted versions. We start by analyzing graphs in $\mathtt{ggGrc}$ with a small number of internal edges.

Example 5.1. The local terminal objects of $\mathtt{ggGrc}$ are genus-graded corollas $c(\sigma)^g$ for a permutation $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma_n$ and a genus $g \in \mathbb{N}$:

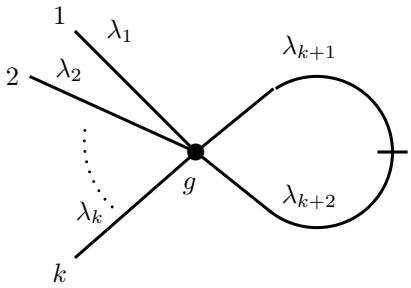


(68)

The chosen local terminal objects are the genus-graded corollas $c_n^g := c(\mathbb{1}_n)^g$ with $\mathbb{1}_n \in \Sigma_n$ the identity permutation.

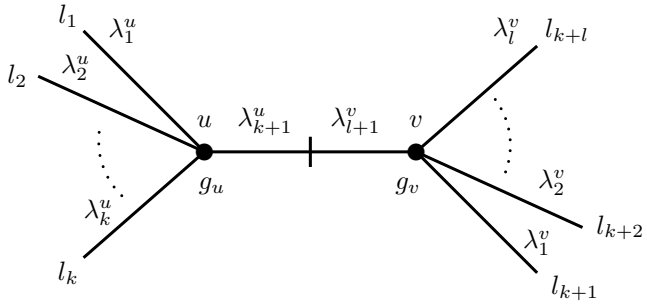
Example 5.2. Any ordered connected genus-graded graph with one internal edge and one vertex

looks like $\xi(\lambda_1,\dots,\lambda_k|\lambda_{k+1},\lambda_{k+2})^g$ in



(69a)

with half-edges labeled by a permutation $\{\lambda_1,\dots,\lambda_{k+2}\}$ of $\{1,\dots,k+2\}$. Its automorphism group equals Σ_2 that interchanges the half-edges forming the loop. Any two graphs of this kind are isomorphic. In



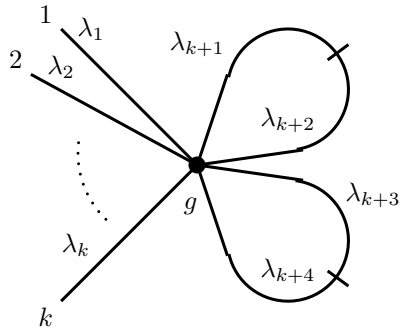
(69b)

we depict a general graph $\nu(\lambda_1^u,\dots,\lambda_k^u|\lambda_{k+1}^u,\lambda_{l+1}^v|\lambda_1^v,\dots,\lambda_l^v)^{g_u|g_v}$ with one internal edge and two vertices labeled by $u,v\in\{1,2\}$ with genera $g_u,g_v\in\mathbb{N}$. Its global order is determined by a (k,l)-shuffle

$$\{l_1<\dots<l_k,\ l_{k+1}<\dots<l_{k+l}\}=\{1,\dots,k+l\}.$$

The half-edges adjacent to u are labeled by a permutation λ^u of $\{1,\dots,k+1\}$; the half-edges adjacent to v by a permutation λ^v of $\{1,\dots,l+1\}$. Two such graphs with the same global orders and the same genera are always isomorphic. There are no nontrivial automorphisms except for the case $k=l=0$ and $g_u=g_v$ when the graph is an interval with no legs. Then one has the automorphism flipping it around its middle.

Example 5.3. A general graph $\xi(\lambda_1,\dots,\lambda_k|\lambda_{k+1},\lambda_{k+2}|\lambda_{k+3},\lambda_{k+4})^g$ with two internal edges and one vertex is depicted in



(70)

Its local order at its single vertex is determined by a permutation λ of $\{1,\dots,k+4\}$. Its automorphism group equals the semidirect product $\Sigma_2\rtimes(\Sigma_2\times\Sigma_2)$. We leave the similar detailed analysis of the remaining graphs with two internal edges as an exercise.

Our next task will be to describe free operads over `ggGrc` using formula (66b). As the first step towards this goal we describe isomorphism classes of labeled towers $(\ell,\mathfrak{T})\in\widetilde{\mathtt{lTw}}^2(X)$ for the ordered graph

$$X:=\xi(1,\dots,k|k+1,k+2|k+3,k+4)^g$$

i.e. for the graph in (70) with λ the identity. As observed in Example 3.14, it suffices to consider diagrams (52) in which all maps in the upper triangle are quasibijections. Since the graphs X, T' and T'' in that triangle have one vertex only, all ℓ', ℓ'' and σ_1 belong to $\mathtt{Gr}_{\mathtt{ord}}$, therefore they are the identities by [3, Corollary 2.6]. Isomorphism classes in $\widetilde{\mathtt{lTw}}^2(X)$ are thus represented by maps

$$\tau : \xi(1,\dots,k|k{+}1,k{+}2|k{+}3,k{+}4)^g \longrightarrow \xi(\nu_1,\dots,\nu_k|\nu_{k+1},\nu_{k+2})^{g+1} \tag{71}$$

modulo the equivalence that identifies τ' with τ'' if and only if there exists an isomorphism σ such that $\tau'' = \sigma\tau'$. Notice that a map as in (71) is automatically elementary, and that all elementary maps from X decreasing the grade by 1 are of this form. Now define the "canonical" maps

$$p_i : \xi(1,\dots,k|k{+}1,k{+}2|k{+}3,k{+}4)^g \longrightarrow \xi(1,\dots,k|k{+}1,k{+}2)^{g+1},\ \ i=1,2,$$

by postulating that p_1 (resp. p_2) contracts the loop $\{k{+}1,k{+}2\}$ (resp. $\{k{+}3,k{+}4\}$) leaving the other loop unaffected. In other words, the injection ψ_1 (resp. ψ_2) of half-edges defining p_1 (resp. p_2) is the order-preserving injection

$$(1,\dots,k{+}2) \hookrightarrow (1,\dots,k{+}4)$$

that misses the subset $\{k{+}1,k{+}2\}$ (resp. $\{k{+}3,k{+}4\}$).

We claim that for each τ in (71) there exist a unique $i \in \{1,2\}$ and a unique isomorphism σ making the diagram

$$\begin{array}{ccc} \xi(1,\dots,k|k{+}1,k{+}2|k{+}3,k{+}4)^g & & \\ {\scriptstyle p_i}\swarrow & \searrow{\scriptstyle \tau} & \\ \xi(1,\dots,k|k{+}1,k{+}2)^{g+1} & \xrightarrow[\cong]{\sigma} & \xi(\nu_1,\dots,\nu_k|\nu_{k+1},\nu_{k+2})^{g+1} \end{array} \tag{72}$$

commutative. Since, by definition, morphisms in $\mathtt{ggGrc}$ preserve global orders, one has for the injections ψ_τ resp. ψ_σ of half-edges defining τ resp. σ,

$$\psi_\tau(\nu_j) = \psi_\sigma(\nu_j) = j \quad \text{for} \quad 1 \le j \le k.$$

Since ψ_τ must further preserve the involutions on the sets of half-edges, there are only two possibilities:

Case 1: $\psi_\tau\{\nu_{k+1},\nu_{k+2}\} = \{k{+}3,k{+}4\}$. In this case we take $i=1$ in (72) and define

$$\psi_\sigma(\nu_{k+1}) := \psi_\tau(\nu_{k+1}) - 2,\ \ \psi_\sigma(\nu_{k+2}) := \psi_\tau(\nu_{k+2}) - 2.$$

It is clear that with this choice the diagram in (72) is commutative and that it is the only such choice.

Case 2: $\psi_\tau\{\nu_{k+1},\nu_{k+2}\} = \{k{+}1,k{+}2\}$. In this case we take $i=2$ and define

$$\psi_\sigma(\nu_{k+1}) := \psi_\tau(\nu_{k+1}),\ \ \psi_\sigma(\nu_{k+2}) := \psi_\tau(\nu_{k+2}).$$

Intuitively, in Case 1 the map τ contracts the loop $\{k{+}1,k{+}2\}$, in Case 2 the loop $\{k{+}3,k{+}4\}$. In both cases the isomorphism σ is uniquely determined by the behavior of τ on the non-contracted edge.

The above calculation shows that there are precisely two isomorphism classes of objects of $\widetilde{\mathtt{lTw}}^2(X)$, namely those of p_1 and p_2. Notice that

$$p_1^{-1}(1) = \xi(1,\dots,k,k{+}3,k{+}4|k{+}1,k{+}2)^g \ \text{ and } \ p_2^{-1}(1) = \xi(1,\dots,k,k{+}1,k{+}2|k{+}3,k{+}4)^g.$$

Let $E \in \mathtt{Coll}_1^{\mathtt{V}}$ be a 1-connected $\mathtt{ggGrc}$-collection as in Definition 3.5. Formula (66a) gives

$$\begin{aligned} \mathbb{F}^2(E)(X) \cong\ & E[\xi(1,\dots,k,k{+}3,k{+}4|k{+}1,k{+}2)^g] \otimes E[\xi(1,\dots,k|k{+}1,k{+}2)^{g+1}] \\ & \oplus E[\xi(1,\dots,k,k{+}1,k{+}2|k{+}3,k{+}4)^g] \otimes E[\xi(1,\dots,k|k{+}1,k{+}2)^{g+1}]\,. \end{aligned}$$

Analogous expressions for $X = \xi(\lambda_1,\dots,\lambda_k|\lambda_{k+1},\lambda_{k+2}|\lambda_{k+3},\lambda_{k+4})^g$ can be obtained from the above ones by substituting $j \mapsto \lambda_j$ for $1 \le j \le k+4$. The result is

$$\begin{aligned} \mathbb{F}^2(E)(X) \cong\ & E[\xi(\lambda_1,\dots,\lambda_k,\lambda_{k+3},\lambda_{k+4}|\lambda_{k+1},\lambda_{k+2})^g] \otimes E[\xi(\lambda_1,\dots,\lambda_k|\lambda_{k+1},\lambda_{k+2})^{g+1}] \\ & \oplus E[\xi(\lambda_1,\dots,\lambda_k,\lambda_{k+1},\lambda_{k+2}|\lambda_{k+3},\lambda_{k+4})^g] \otimes E[\xi(\lambda_1,\dots,\lambda_k|\lambda_{k+1},\lambda_{k+2})^{g+1}]\,. \end{aligned} \tag{73}$$

Example 5.4. The right-hand side of (73) depends only on the virtual isomorphism classes in $\mathtt{QO}_{\mathtt{vrt}}(e)$ of the graphs involved. By the observations made in Example 2.5, these classes do not depend on the global orders. In this particular case, this means that they do not depend on the indices $\lambda_1, \dots, \lambda_k$; we can therefore simplify the exposition by removing them from notation and drawings. We also replace $\lambda_{k+1}, \dots, \lambda_{k+4}$ by less clumsy symbols a, b, c and d. With this convention, we write the two representatives of isomorphism classes in $\widetilde{\mathtt{lTw}}^2(X)$ as

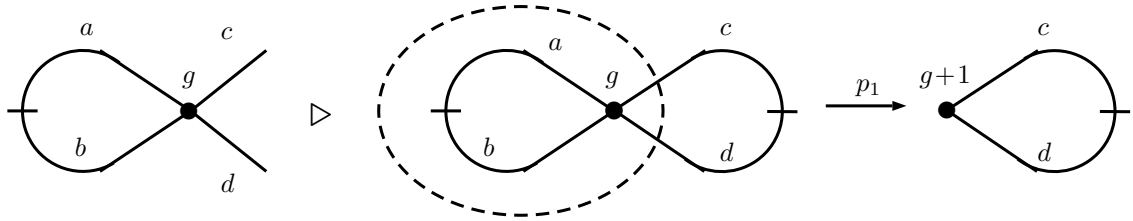
$$\xi(*, c, d|a, b)^g \triangleright \xi(*|a, b|c, d)^g \xrightarrow{p_1} \xi(*|c, d)^{g+1} \text{ and}$$
$$\xi(*, a, b|c, d)^g \triangleright \xi(*|a, b|c, d)^g \xrightarrow{p_2} \xi(*|a, b)^{g+1},$$

where $*$ stands for unspecified labels. The right-hand side of (73) now takes the form

$$\big\{E[\xi(*, c, d|a, b)^g] \otimes E[\xi(*|c, d)^{g+1}]\big\} \oplus \big\{E[\xi(*, a, b|c, d)^g] \otimes E[\xi(*|a, b)^{g+1}]\big\} \tag{74}$$

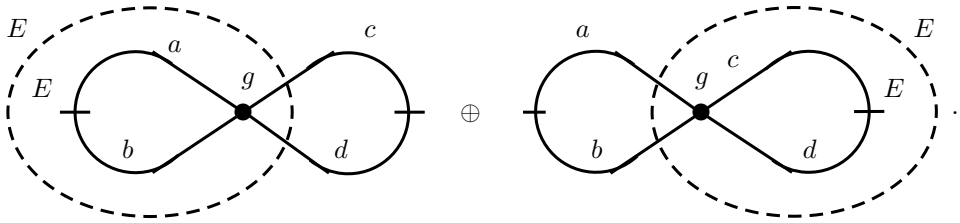
with the first summand corresponding to the class of p_1 and the second to the class of p_2.

We also noticed that the maps p_1 and p_2 are determined by specifying which of the two loops of $\xi(*|a, b|c, d)^g$ they contract. The map p_1 and its unique nontrivial fiber is thus depicted as



where he dashed oval indicates which part of the graph is contracted by p_1. The pictorial expression of p_2 is similar.

We will use similar pictures as a language for free operads over `ggGrc`. As an illustration, here is a pictorial version of (74):



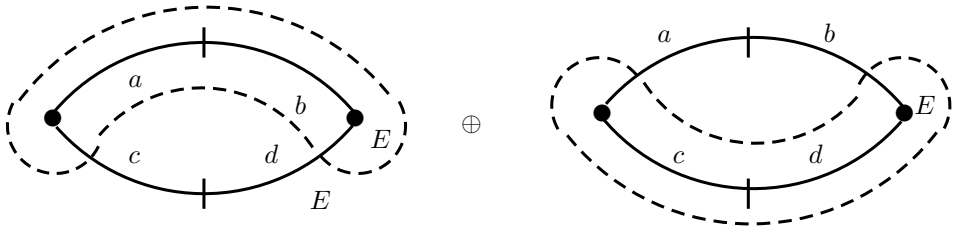
(75)

It features the souls of the relevant graphs in the sense of the following

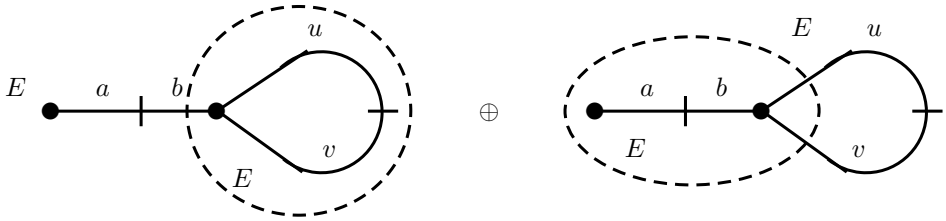
Definition 5.5. The *soul* of a graph Γ is the graph obtained from Γ by amputating its legs.

The E's inside the dashed circles indicate the decoration of the fiber represented by the subgraph inside the circle, while the E's outside the circles indicate the decoration of the images. Thus the object on the left of (75) represents the first summand of (74) and the object on the right the second one. This description should be compared to the description of free "classical" operads in terms of trees with decorated vertices, cf. [21, Section II.1.9]. Here we have graphs instead of trees and "nests" of subgraphs ordered by inclusion in place of vertices.

Example 5.6. Using the same reasoning as in Examples 5.3 and 5.4, we can draw similar pictures describing $\mathbb{F}^2(E)(X)$ for X a graph with two internal edges and two vertices with genera g_1 and g_2. Their souls are shown in (76a)–(76b) below



(76a)



(76b)

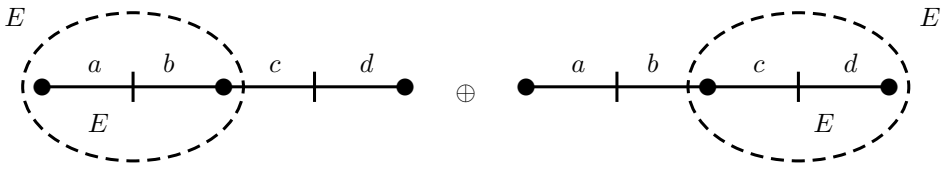
where, in order to ease the interpretation, we did not show the genera of the vertices. The picture in (76a) represents an analog of (74):

$$\begin{aligned}\mathbb{F}^2(E)(X) \cong &\left\{E[\nu(*,c|a,b|d,*)^{g_1|g_2}] \otimes E[\xi(*|c,d)^{g_1+g_2}]\right\}\\ &\oplus \left\{E[\nu(*,a|c,d|b,*)^{g_1|g_2}] \otimes E[\xi(*|a,b)^{g_1+g_2}]\right\}\end{aligned}$$

in which the notation $\nu(*,c|a,b|d,*)$ resp. $\nu(*,a|c,d|b,*)$ refers to the graph in (69b). The picture in (76b) symbolizes

$$\begin{aligned}\mathbb{F}^2(E)(X) \cong &\left\{E[\xi(*,b|u,v)^{g_1}] \otimes E[\nu(*|a,b|*)^{g_1|g_2+1}]\right\}\\ &\oplus \left\{E[\nu(*|a,b|u,v,*)^{g_1|g_2}] \otimes E[\xi(*|u,v)^{g_1+g_2}]\right\}.\end{aligned}$$

The last relevant case is when X is an ordered graph with two internal edges and three vertices with genera g_1, g_2 and g_3. The situation is portrayed in



(77)

where again the genera of the vertices are not shown. The resulting formula is

$$\begin{aligned}\mathbb{F}^2(E)(X) \cong &\left\{E[\nu(*|a,b|c,*)^{g_1|g_2}] \otimes E[\nu(*|c,d|*)^{g_1+g_2|g_3}]\right\}\\ &\oplus \left\{E[\nu(*,b|c,d|*)^{g_2|g_3}] \otimes E[\nu(*|a,b|*)^{g_1|g_2+g_3}]\right\}.\end{aligned}$$

Remark 5.7. In order to appreciate the advantages of our approach, we suggest to compare the simple and self-explaining pictures in (75)–(77) with Figures 2 and 3 of [25] expressing the axioms of modular operads as algebras over a colored operad.

The observations in Examples 5.4 and 5.6 easily generalize to descriptions of isomorphism classes of labeled towers in $\mathtt{lTw}(\Gamma)$ for an arbitrary graph $\Gamma \in \mathtt{ggGrc}$. Since we will be primarily interested in free operads generated by *binary* collections, i.e. 1-connected collections that are trivial on graphs with more than one internal edge, we will consider only towers whose associated fiber sequence consists of graphs with one internal edge. Let

$$\Gamma \xrightarrow{\tau_1} \Gamma_1 \xrightarrow{\tau_2} \Gamma_2 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_{k-1}} \Gamma_{k-1}$$

be such a tower. By the definition of graph morphisms, one has the associated sequence

$$\mathrm{edg}(\Gamma) \supset \mathrm{edg}(\Gamma_1) \supset \cdots \supset \mathrm{edg}(\Gamma_{k-1}) \tag{78}$$

of inclusions of the sets of internal edges. Since the cardinalities of the sets in (78) decrease by one, there is an obvious one-to-one correspondence between sequences (78) and linear orders on $\mathrm{edg}(\Gamma)$ such that $x > y$ if $y \in \mathrm{edg}(\Gamma_i)$, $x \notin \mathrm{edg}(\Gamma_i)$ for some i, $1 \leq i \leq k-1$. We formulate:

Proposition 5.8. *The isomorphism classes of labeled towers in* $l\mathit{Tw}(\Gamma)$ *whose associated fiber sequence consists of graphs with one internal edge are in one-to-one correspondence with linear orders on* $\mathrm{edg}(\Gamma)$ *modulo the relation* $\bowtie$ *that interchanges two adjacent edges without a common vertex in* Γ.

Example 5.9. One has two isomorphism classes of towers for the graph in (70). In the notation of (75), let x be the edge $\{a, b\}$ and y the edge $\{c, d\}$. Then the picture on the left in that display corresponds to the order $x > y$ (x is contracted first), the one on the right to $y > x$.

Proof of Proposition 5.8. Using the same arguments as in Examples 5.4 and 5.6 we show that each tower can be replaced by an isomorphic one all of whose morphisms are pure contractions of internal edges, in the sense of [3, Definition 3.3]. Such towers are determined by the order in which the edges are contracted. The relation $\bowtie$ reflects morphisms of towers of the second type introduced in Section 3. □

Theorem 5.10. *The terminal* $\mathtt{ggGrc}$*-operad* $\mathbf{1}_{\mathtt{ggGrc}}$ *having* $\mathbf{1}_{\mathtt{ggGrc}}(\Gamma) := \Bbbk$ *for each* $\Gamma \in \mathtt{ggGrc}$ *and constant composition laws is binary quadratic.*

Proof. Let us define a collection $E \in \mathtt{Coll}_1^{\mathtt{V}}$ by

$$E[\Gamma] := \begin{cases} \Bbbk & \text{if } \Gamma \text{ has exactly one internal edge} \\ 0 & \text{otherwise} \end{cases} \tag{79}$$

with the constant $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaf structure. As we already noticed, the quadratic part $\mathbb{F}^2(E)$ of the free operad may be nontrivial only for graphs with precisely two internal edges, i.e. those analyzed in Examples 5.4 and 5.6. For X as in (70) and E in (79), formula (74) gives

$$\mathbb{F}^2(E)(X) \cong \{\Bbbk \otimes \Bbbk\} \oplus \{\Bbbk \otimes \Bbbk\}$$

which is a two-dimensional vector space with the basis

$$b_1^1 := \{1 \otimes 1\} \oplus \{0 \otimes 0\} \quad \text{and} \quad b_2^1 := \{0 \otimes 0\} \oplus \{1 \otimes 1\}. \tag{80}$$

For the situations portrayed in (76a)–(76b) and (77) we get similar spaces, with bases (b_1^t, b_2^t), $2 \le t \le 4$. Let us denote

$$r_1 := b_1^1 - b_2^1,\ r_2 := b_1^2 - b_2^2,\ r_3 := b_1^3 - b_2^3 \text{ and } r_4 := b_1^4 - b_2^4 \tag{81}$$

We define $R(X)$ to be the subspace of $\mathbb{F}^2(E)(X)$ spanned by
– r_1 for X with soul as in (75),
– r_2 for X with soul as in (76a),
– r_3 for X with soul as in (76b), and
– r_4 for X with soul as in (77),
while $R(X) := 0$ if X does not have exactly two internal edges. We are going to prove that

$$\mathbf{1}_{\mathtt{ggGrc}} \cong \mathbb{F}(E)/(R) \tag{82}$$

for the sub-presheaf $R = \{R(X)\}_{X \in \mathtt{ggGrc}}$ of $\mathbb{F}(E)$ defined above.

By Proposition 5.8 combined with formula (66a), the vector space $\mathbb{F}(E)(\Gamma)$ is spanned by the set of total orders on $\mathrm{edg}(\Gamma)$ modulo the relation $\bowtie$ that interchanges any two edges $x, y \in \mathrm{edg}(\Gamma)$ that *do not* share a common vertex in Γ.

All possible relative configurations of edges x, y that *do share* a common vertex are in (75)–(77). Relations in (81) guarantee that two orders that differ by the interchange $x \leftrightarrow y$ agree in the quotient (82). We conclude that all orders on $\mathrm{edg}(\Gamma)$ are mutually equivalent modulo (R), so $\mathbb{F}(E)/(R)(\Gamma) \cong \Bbbk$ as required. □

Proposition 5.11. *Algebras over the terminal* $\mathtt{ggGrc}$*-operad* $\mathbf{1}_{\mathtt{ggGrc}}$ *are modular operads.*

Proof. The key ingredients of the proof are presentation (82) together with Proposition 6.13 of [3] which describes $\mathbf{1}_{\mathtt{ggGrc}}$-algebras as morphisms to the endomorphism operad. We start by determining what the underlying collection

$$V = \{V_c \mid c \in \pi_0(\mathtt{O})\}$$

of the endomorphism operad is in this case.

We noticed in Example 5.1 that the local terminal objects of $\mathtt{ggGrc}$ are the n-corollas $c(\sigma)^g$ with the vertex of genus g and the local order given by a permutation $\sigma \in \Sigma_n$. The chosen local terminal objects are the n-corollas $c_n^g := c(\mathbb{1}_n)^g$, $n, g \in \mathbb{N}$. Therefore the set $\pi_0(\mathtt{ggGrc})$ is indexed by pairs $(n; g)$ of natural numbers consisting of an "arity" n and a "genus" g, i.e.

$$\pi_0(\mathtt{ggGrc}) = \big\{(n; g) \mid (n; g) \in \mathbb{N} \times \mathbb{N}\big\}.$$

The underlying collection of the endomorphism operad is thus a family

$$\mathcal{M} = \big\{\mathcal{M}(n; g) \in \mathtt{Vect} \mid (n; g) \in \mathbb{N} \times \mathbb{N}\big\}.$$

The actions $u : V_{\pi_0(s_1(u))} \to V_{\pi_0(u)}$ of the groupoid of local terminal objects in this particular case give rise to actions of the symmetric group Σ_n on each $\mathcal{M}(n; g)$. We recognize $\mathcal{M}$ as the skeletal version of a modular module recalled in Appendix A. Proposition 6.13 of [3] now identifies $\mathbb{1}_{\mathtt{ggGrc}}$-algebras with morphisms

$$a : \mathbb{F}(E)/(R) \to \mathcal{E}nd_{\mathcal{M}}, \tag{83}$$

where E is as in (79) and R is spanned by relations (81).

By Proposition 2.8, the unital operad $\mathcal{E}nd_{\mathcal{M}}$ determines a $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaf $\mathring{\mathcal{E}nd}_{\mathcal{M}}$. Although $\mathcal{E}nd_{\mathcal{M}}$ is not strictly unital, morphism (83) is still uniquely determined by a map $\tilde{a} : E \to \mathring{\mathcal{E}nd}_{\mathcal{M}}$ of $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaves given by a family

$$\tilde{a}_{[\Gamma]} : E[\Gamma] \to \mathring{\mathcal{E}nd}_{\mathcal{M}}([\Gamma]), \ [\Gamma] \in \mathtt{QO}_{\mathtt{vrt}}(e). \tag{84}$$

By definition, the generating collection E is supported on graphs with one internal edge portrayed in (69a) and (69b), whose souls are:

(85)

The operations $\tilde{a}_{[\Gamma]}$ may therefore be nontrivial only for graphs of this form.

Let us analyze the operation (84) induced by the virtual isomorphism class of the graph $\Gamma := \xi(\lambda_1, \ldots, \lambda_k | \lambda_{k+1}, \lambda_{k+2})$ in (69a). One clearly has $\pi_0(s_1(\Gamma)) = (k+2; g)$ and $\pi_0(\Gamma) = (k; g+1)$, therefore $\tilde{a}_{[\Gamma]}$ is by (34) a map

$$\tilde{a}_{[\Gamma]} : E[\Gamma] = \Bbbk \longrightarrow \operatorname*{colim}_{\tilde{\Gamma}} \mathcal{E}nd_{\mathcal{M}}(\tilde{\Gamma}) \cong \operatorname*{colim}_{\sigma \in \Sigma_k} \mathtt{Vect}\big(\mathcal{M}(k+2; g), \mathcal{M}(k; g+1)_\sigma\big),$$

where the first colimit is taken over all $\tilde{\Gamma}$'s virtually isomorphic to Γ, where $\sigma = (\sigma_1, \ldots, \sigma_k)$ and where $\mathcal{M}(k; g+1)_\sigma$ is the copy of $\mathcal{M}(k; g+1)$ corresponding to the graph

$$\xi(\sigma_1, \ldots, \sigma_k | \lambda_{k+1}, \lambda_{k+2}),$$

which is virtually isomorphic to $\xi(\lambda_1, \ldots, \lambda_k | \lambda_{k+1}, \lambda_{k+2})$. The map $\tilde{a}_{[\Gamma]}$ is clearly determined by

$$\tilde{a}_{[\Gamma]}(1) : \mathcal{M}(k+2; g) \to \operatorname*{colim}_{\sigma \in \Sigma_k} \mathcal{M}(k; g+1)_\sigma$$

which is the same as a collection of morphisms

$$\circ_{uv}^{\sigma} : \mathcal{M}(k+2; g) \longrightarrow \mathcal{M}(k; g+1), \ u := \lambda_{k+1}, \ v := \lambda_{k+2}, \ \sigma \in \Sigma_k,$$

satisfying

$$\circ_{uv}^{\sigma\delta}(x) = \sigma \circ_{uv}^{\delta}(x), \ x \in \mathcal{M}(k+2; g), \ \sigma, \delta \in \Sigma_k.$$

The operation $\circ_{uv} := \circ_{uv}^{\mathbb{1}_k}$ is the skeletal version of the contraction (116b). The identity $\circ_{uv} = \circ_{vu}$ follows from the Σ_2-symmetry of the graph Γ. In exactly the same manner, the graph in (69b) gives rise to the operations in (116a).

The map $\tilde{a}$ determines a morphism (83) if and only if it sends the generators (81) of R to 0. The vanishing $\tilde{a}(r_i) = 0$ for $1 \leq i \leq 4$ corresponds to the remaining axiom of modular operads:

Axiom (116h) corresponds to relation r_2, Axiom (116i) corresponds to relation r_3,

Axiom (116f) corresponds to relation r_4, Axiom (116g) corresponds to relation r_1.

This finishes the proof. □

Theorem 5.12. *The Koszul dual of the operad* $1_{\mathtt{ggGrc}}$*, denoted* $\mathfrak{K}_{\mathtt{ggGrc}}$*, is the operad whose algebras are odd modular operads.*

Proof. The Koszul dual $\mathfrak{K}_{\mathtt{ggGrc}} := 1^!_{\mathtt{ggGrc}}$ is, by definition, generated by the collection

$$\uparrow E^* := \begin{cases} \uparrow \Bbbk & \text{if } \Gamma \text{ has exactly one internal edge} \\ 0 & \text{otherwise.} \end{cases}$$

We get the similar type of generators d_1^i, d_2^i, $1 \leq i \leq 4$, for $\mathbb{F}^2(\uparrow E^*)$ as in the proof of Theorem 5.10 except that now they will be in degree 2. The pairing (67) in this particular case is given by

$$\langle\, b_k^i \mid d_l^j \,\rangle = \begin{cases} 1 & \text{if } i = j,\ k = l \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the annihilator $R^\perp$ of the relations (81) is spanned by

$$o_1 := d_1^1 + d_2^1,\ o_2 := d_1^2 + d_2^2,\ o_3 := d_1^3 + d_2^3 \text{ and } o_4 := d_1^4 + d_2^4.$$

Repeating the arguments in the proof of Theorem 5.10 we identify algebras over $\mathbb{F}(\uparrow E^*)/(R^\perp)$ with odd modular operads whose definition is recalled in Appendix A. □

Remark 5.13. As observed in [3, Example 4.19], the category $\mathtt{ggGrc}$ is similar to the category of graphs of [9, §2.15]. The difference is the presence of the local orders of graphs in $\mathtt{ggGrc}$ manifested e.g. by the fact that, while the category in [9, §2.15] has only one local terminal object for each arity n and genus g, the local terminal objects in $\mathtt{ggGrc}$ are indexed by n, g and by a permutation $\sigma \in \Sigma_n$, cf. Example 5.1. The category of operads over the operadic category $\mathtt{ggGrc}$ is however equivalent to the category of hyperoperads in the sense of [9, §4.1]. Moreover, there is a canonical isomorphism between the category of algebras for a $\mathtt{ggGrc}$-operad and the category of algebras for the corresponding hyperoperad.

This relation enables one to compare the operad $\mathfrak{K}_{\mathtt{ggGrc}}$ of Theorem 5.12 to a similar object considered in [9]. Recall that the *determinant* $\det(S)$ of a finite set S is the top-dimensional piece of the exterior (Grassmann) algebra generated by the elements of S placed in degree $+1$. In particular, $\det(S)$ is a one-dimensional vector space concentrated in degree k, with k the cardinality of S. Mimicking the arguments in the second half of the proof of Theorem 5.10 one can establish that $\mathfrak{K}_{\mathtt{ggGrc}}(\Gamma) \cong \det(\mathrm{edg}(\Gamma))$, the determinant of the set of internal edges of Γ. This relates $\mathfrak{K}_{\mathtt{ggGrc}}$ directly to the *dualizing cocycle* of [9, §4.8], cf. also Example II.5.52 of [21].

6 Other operad-like structures

In this section we analyze other operad-like structures whose pasting schemes are obtained from the basic operadic category $\mathtt{Gr}$ of graphs by means of the iterated Grothendieck construction. For all these categories the properties UFib and SGrad can be easily checked "manually." By the reasoning from the beginning of Section 5 they are rigid and fulfill *Assumptions 1.2*.

6.1 Cyclic operads

Cyclic operads introduced in [8] are, roughly speaking, modular operads without the genus grading and contractions (116b). Explicitly, a cyclic operad is a functor $\mathcal{C} : \mathtt{Fin} \to \mathtt{Vect}$ along with operations

$$ {}_a\circ_b : \mathcal{C}\big(S_1 \sqcup \{a\}\big) \otimes \mathcal{C}\big(S_2 \sqcup \{b\}\big) \longrightarrow \mathcal{C}(S_1 \sqcup S_2) \tag{86}$$

indexed by disjoint finite sets S_1, S_2 and symbols a, b. These operations satisfy axioms (116c), (116e) and (116f) of modular operads (without the genus grading). Let $\mathtt{Tr}$ be the full subcategory of $\mathtt{Gr}$ consisting of graphs of genus zero whose geometric realizations are contractible, i.e. which are trees. The local terminal objects of $\mathtt{Tr}$ are corollas $c(\sigma)$, $\sigma \in \Sigma_n$, as in (68) but without the genus labeling the vertex. The chosen local terminal objects are corollas $c_n := c(\mathbb{1}_n)$, $n \in \mathbb{N}$.

Theorem 6.1. *The terminal* $\mathtt{Tr}$*-operad* $1_{\mathtt{Tr}}$ *is binary quadratic. Its algebras are cyclic operads. Its Koszul dual* $\mathfrak{K}_{\mathtt{Tr}} := 1^!_{\mathtt{Tr}}$ *is the operad whose algebras are anticyclic operads.*

Anticyclic operads introduced in [8, §2.11] are "odd" versions of cyclic operads, see also [21, Definition II.5.20]. Due to the absence of the operadic units in our setup, the category of anticyclic operads is however isomorphic to the category of ordinary cyclic operads, via the isomorphism given by the suspension of the underlying collection.

Proof of Theorem 6.1. The proof is a simplified version of calculations in Section 5. The soul of the only graph in $\mathtt{Tr}$ with one internal edge is the one on the left of (85) (without the genera, of course), the corresponding operation is (86). The souls of the only graphs in $\mathtt{Tr}$ with two internal edges are portrayed in (77). Let E be the restriction of the collection (79) to the virtual isomorphism classes of trees in $\mathtt{Tr}$. If R denotes the subspace of $\mathbb{F}^2(E)$ spanned by r_2 in (81), then $1_{\mathtt{Tr}} \cong \mathbb{F}(E)/(R)$. The arguments are the same as in the proof of Theorem 5.10. With the material of Section 5 at hand, the identification of $1_{\mathtt{Tr}}$-algebras with cyclic operads is immediate.

Algebras over $\mathfrak{K}_{\mathtt{Tr}}$ can be analyzed in the same way as $\mathfrak{K}_{\mathtt{ggGrc}}$-algebras in the proof of Theorem 5.12. $\mathfrak{K}_{\mathtt{Tr}}$-algebras posses degree $+1$ operations

$$ {}_a\bullet_b : \mathcal{C}\big(S_1 \sqcup \{a\}\big) \otimes \mathcal{C}\big(S_2 \sqcup \{b\}\big) \longrightarrow \mathcal{C}(S_1 \sqcup S_2) \tag{87} $$

satisfying non-genus graded variants of (117b), (117c) and (117d). The level-wise suspension $\uparrow\mathcal{C}$ with operations

$$ {}_a\circ_b : \uparrow\mathcal{C}\big(S_1 \sqcup \{a\}\big) \otimes \uparrow\mathcal{C}\big(S_2 \sqcup \{b\}\big) \longrightarrow \uparrow\mathcal{C}(S_1 \sqcup S_2) $$

defined as the composite

$$ \uparrow\mathcal{C}\big(S_1 \sqcup \{a\}\big) \otimes \uparrow\mathcal{C}\big(S_2 \sqcup \{b\}\big) \xrightarrow{\uparrow\otimes\uparrow} \mathcal{C}\big(S_1 \sqcup \{a\}\big) \otimes \mathcal{C}\big(S_2 \sqcup \{b\}\big) \xrightarrow{{}_a\bullet_b} \mathcal{C}(S_1 \sqcup S_2) \xrightarrow{\uparrow} \uparrow\mathcal{C}(S_1 \sqcup S_2) $$

can easily be shown to be an anticyclic operad [21, Definition II.5.20]. □

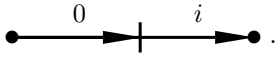
As in Remark 5.13, one may observe that $\mathfrak{K}_{\mathtt{Tr}}(T)$ equals the determinant of the set of internal edges of the tree T. Our description of anticyclic operads as $\mathfrak{K}_{\mathtt{Tr}}$-algebras is therefore parallel to the definition as $\mathbb{T}_{-}$-algebras given in [8, page 178].

6.2 Ordinary operads

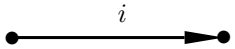
Let us consider a variant $\mathtt{RTr}$ of the operadic category $\mathtt{Tr}$ consisting of trees that are rooted in the sense explained in [3, Example 4.8]. By definition, the output half-edge of each vertex is the minimal element in the local order; we will denote this minimal element by 0 in the context of rooted trees. We use the same convention also for the smallest leg in the global order, i.e. for the root. Since $\mathtt{RTr}$ was obtained from the basic operadic category $\mathtt{Gr}$ by a Grothendieck construction, it is again an operadic category sharing all the nice properties of $\mathtt{Gr}$.

Theorem 6.2. *The terminal* $\mathtt{RTr}$*-operad* $1_{\mathtt{RTr}}$ *is binary quadratic. Its algebras are nonunital Markl operads recalled in Definition A.4 of Appendix A. The category of algebras over its Koszul dual* $\mathfrak{K}_{\mathtt{RTr}} := 1^!_{\mathtt{RTr}}$ *is isomorphic to the category of Markl operads, via the isomorphism given by the suspension of the underlying collection.*

Proof. The soul of graphs in $\mathtt{RTr}$ with one internal edge is the oriented interval consisting of two oriented half-edges as in

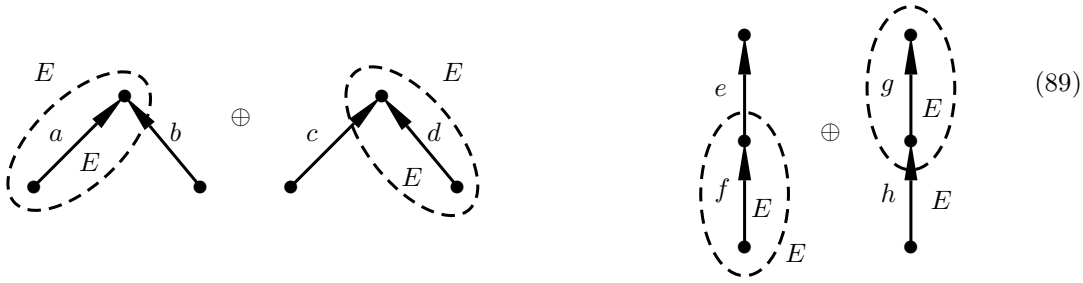


Since the label of the out-going half-edge is always the minimal one in the local order, we omit it from pictures and draw the internal edges as arrows acquiring the label of the in-going half-edge, see



(88)

Let E be an obvious modification of the constant collection (79) to the category $\mathtt{RTr}$. The display



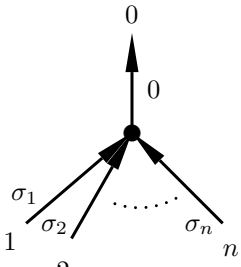
(89)

features souls of rooted trees with two internal edges. It shows that $\mathbb{F}^2(E)$ has two families of bases, (b_1^1, b_2^1) corresponding to the direct sum in the left part and (b_1^2, b_2^2) corresponding to the direct sum on the right of the display. Let R be the subspace of $\mathbb{F}^2(E)$ spanned by the relations

$$r_1 := b_1^1 - b_2^1 \quad \text{and} \quad r_2 := b_1^2 - b_2^2. \tag{90}$$

The isomorphism $\mathtt{1}_{\mathtt{RTr}} \cong \mathbb{F}(E)/(R)$ can be established as in the proof of Theorem 5.10.

To identify $\mathtt{1}_{\mathtt{RTr}}$-algebras with Markl operads we proceed as in the proof of Theorem 5.11. We start by realizing that the local terminal objects are rooted corollas $c^{\uparrow}(\sigma)$, $\sigma \in \Sigma_n$, shown in



while the chosen local terminal objects are $c_n^{\uparrow} := c^{\uparrow}(\mathbb{1}_n)$. The set $\pi_0(\mathtt{RTr})$ of connected components is therefore identified with the natural numbers $\mathbb{N}$. Analyzing the actions of local terminal objects in [3, display (60)] we conclude that the underlying collections for $\mathtt{1}_{\mathtt{RTr}}$-algebras are sequences $\mathcal{S}(n)$, $n \in \mathbb{N}$, of Σ_n-modules.

As in the proof of Theorem 5.11 we establish that the value of the generating collection E on graphs whose soul is the arrow in (88) produces partial compositions (118a), that the relation r_1 expresses the parallel associativity, i.e. the first and the last cases of the relation in (118b) of the Appendix, and r_2 the sequential associativity, i.e. the middle case of that relation.

We are sure that at this stage the reader will easily describe the annihilator $R^{\perp}$ of the space R of relations and identify algebras of the Koszul dual

$$\mathfrak{K}_{\mathtt{RTr}} := \mathtt{1}^!_{\mathtt{RTr}} = \mathbb{F}(\uparrow E^*)/(R^{\perp})$$

as structures with degree $+1$ operations

$$\bullet_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \to \mathcal{S}(m+n-1) \tag{91}$$

satisfying (118c) and the associativities (118b) with the minus sign. It can be verified directly that the level-wise suspension of such a structure is an ordinary Markl operad. However, a more conceptual approach based on coboundaries introduced in [3, Example 6.10] is available.

As in the cases of modular and cyclic operads we notice that, for a rooted tree $T \in \mathtt{RTr}$, we have $\mathfrak{K}_{\mathtt{RTr}}(T) \cong \det(\mathrm{edg}(T))$, the determinant of the set of internal edges of T. On the other hand, the correspondence that assigns to each vertex of T its out-going edge is an isomorphism

$$\mathrm{edg}(T) \cong \{\text{vertices of } T\} \setminus \{\text{the root}\} \tag{92}$$

which implies that $\det(\mathrm{edg}(T))$ is isomorphic to $\mathfrak{D}_{\mathfrak{l}}(T)$, where $\mathfrak{D}_{\mathfrak{l}}$ is the coboundary with $\mathfrak{l} : \pi_0(\mathtt{RTr}) \to \mathtt{Vect}$ the constant function with value the desuspension $\downarrow\Bbbk$ of the ground field. Therefore

$$\mathfrak{K}_{\mathtt{RTr}} = \mathtt{1}_{\mathtt{RTr}} \otimes \mathfrak{D}_{\mathfrak{l}}$$

and the identification of $\mathfrak{K}_{\mathtt{RTr}}$-algebras with Markl operads via the suspension of the underlying collection follows from [3, Proposition 6.11]. □

Similar statements can be proved also for the operadic categories $\mathtt{PTr}$ and $\mathtt{PRTr}$ of planar resp. planar rooted trees introduced in [3, Example 4.9]. The corresponding terminal operads $1_{\mathtt{PTr}}$ resp. $1_{\mathtt{PRTr}}$ will again be self-dual binary quadratic, with algebras nonsymmetric cyclic operads [21, page 257] resp. nonsymmetric Markl operads [21, Definition II.1.14]. We leave the details to the reader.

6.3 Pre-permutads.

Pre-permutads introduced in [13] form a link between nonsymmetric operads and permutads. They are structures satisfying all axioms of Markl operads as recalled in Definition A.4 except the parallel associativity, i.e. the first and the last case of (118b). Pre-permutads are algebras for a certain binary quadratic operad over the category $\mathtt{RTr}$ of rooted trees which is very far from being Koszul self-dual.

Definition 6.3. Let $\mathfrak{pp} := \mathbb{F}(E)/(R)$ be the $\mathtt{RTr}$-operad with the same collection E of generators as the $\mathtt{RTr}$-operad $1_{\mathtt{RTr}}$ for ordinary operads, cf. the proof of Theorem 6.2. The ideal of relations (R) is spanned by r_2 in (90) belonging to the direct sum on the right of (89).

Theorem 6.4. *Pre-permutads in the sense of [13, page 348] are algebras over* $\mathfrak{pp}$*. The category of algebras over the Koszul dual* $\mathfrak{pp}^!$ *is isomorphic to the category of structures satisfying all axioms of Markl operads, except the associativity (118b) which is replaced by*

$$(f \circ_j g) \circ_i h = \begin{cases} 0 & \text{for } 1 \le i < j \\ f \circ_j (g \circ_{i-j+1} h) & \text{for } j \le i \ < b+j \\ 0 & \text{for } j+b \le i \le a+b-1. \end{cases}$$

Proof. The first part of the theorem is an immediate consequence of the definition of $\mathfrak{pp}$. Let d_1^1, d_2^1 resp. d_1^2, d_2^2 be the bases of $\mathbb{F}^2(\uparrow E^*)$ dual to b_1^1, b_2^1 resp. b_1^2, b_2^2. Then the annihilator $R^\perp$ is clearly spanned by

$$o := d_1^2 + d_2^2,\ d_1^1 \text{ and } d_2^1.$$

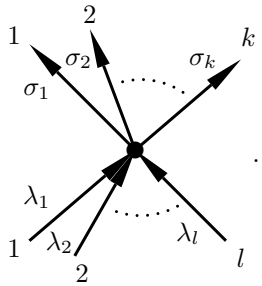
As before, we identify algebras over $\mathfrak{pp}^! = \mathbb{F}(\uparrow^* E)/R^\perp$ with structures equipped with degree $+1$ operations (91) satisfying

$$(f \bullet_j g) \bullet_i h = \begin{cases} 0 & \text{for } 1 \le i < j \\ -f \bullet_j (g \bullet_{i-j+1} h) & \text{for } j \le i < b+j \\ 0 & \text{for } j+b \le i \le a+b-1, \end{cases}$$

whose first case corresponds to d_1^1, the middle to o, and the last one to d_2^1. The level-wise suspension of this object is the structure described in Theorem 6.4. □

7 PROP-like structures and permutads

In this section we treat some important variants of PROPs whose associated operadic categories are sundry modifications of the category $\mathtt{Whe}$ of connected ordered oriented graphs introduced in [3, Example 4.20]. The orientation divides the set of half-edges adjacent to each vertex of the graphs involved into two subsets – inputs and outputs of that vertex. The local terminal objects in these categories will thus be the ordered corollas $c\binom{\sigma}{\lambda}$, $\sigma \in \Sigma_k$, $\lambda \in \Sigma_l$, as in



The chosen local terminal objects are the ordered corollas $c_l^k := c\binom{\mathbb{1}_k}{\mathbb{1}_l}$, $k,l \in \mathbb{N}$. The underlying collections of the corresponding algebras will be families

$$D(m,n),\ m,n \in \mathbb{N}, \tag{93}$$

of $\Sigma_m \times \Sigma_n$-modules. We will see that the orientation of the underlying graphs implies that the corresponding terminal operads are self-dual.

7.1 Wheeled properads

These structures were introduced in [20] as an extension of Vallette's properads [26] that allowed "back-in-time" edges in order to capture traces and therefore also master equations in mathematical physics. Surprisingly, this extended theory is better behaved than the theory of properads in that their composition laws are iterated composites of elementary ones, that is, in terms of pasting schemes, of those given by contraction of a single edge.

The operadic category relevant for wheeled properads is the category $\mathtt{Whe}$ of connected oriented ordered graphs. Since $\mathtt{Whe}$ was constructed in [3, Example 4.20] from the basic operadic category $\mathtt{Gr}$ by iterating the Grothendieck construction, and since it clearly satisfies the conditions UFib and SGrad, we conclude as in the previous sections that our theory of Koszul duality applies to it.

Theorem 7.1. *The terminal* $\mathtt{Whe}$*-operad* $1_{\mathtt{Whe}}$ *is binary quadratic. Its algebras are wheeled properads introduced in [20, Definition 2.2.1]. The operad* $1_{\mathtt{Whe}}$ *is self-dual in the sense of Definition 4.4.*

Proof. The proof goes along the same lines as the proofs of similar statements, namely Theorems 5.10, 5.12, 6.1, 6.2, 6.4 and Proposition 5.11, so we will be telegraphic. As before, for a wheeled graph Γ we put

$$\mathfrak{K}_{\mathtt{Whe}}(\Gamma) := 1^!_{\mathtt{Whe}}(\Gamma) \cong \det(\mathrm{edg}(\Gamma)),$$

the determinant of the set of internal edges of Γ. On the other hand, the correspondence that assigns to each vertex v of Γ the set $\mathrm{out}(v)$ of its out-going edges defines an isomorphism

$$\mathrm{edg}(\Gamma) \cong \bigcup_{v\in \mathrm{Ver}(\Gamma)} \mathrm{out}(v) \setminus \mathrm{out}(\Gamma) \tag{94}$$

which implies that $\det(\mathrm{edg}(\Gamma))$ is isomorphic to $\mathfrak{D}_{\mathfrak{l}}(\Gamma)$, where $\mathfrak{D}_{\mathfrak{l}}$ is the coboundary with $\mathfrak{l} : \pi_0(\mathtt{RTr}) \to \mathtt{Vect}$ the function defined by

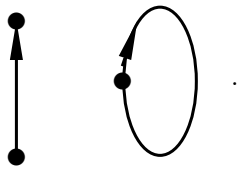
$$\mathfrak{l}(c_l^k) :=\downarrow^k \Bbbk,$$

the desuspension of the ground field iterated k times. Therefore

$$\mathfrak{K}_{\mathtt{Whe}} \cong 1_{\mathtt{Whe}} \otimes \mathfrak{D}_{\mathfrak{l}}$$

which, by [3, Proposition 6.11], implies the self-duality of $1_{\mathtt{Whe}}$.

It follows from the description of the local terminal objects in $\mathtt{Whe}$ that the underlying structure of a $1_{\mathtt{Whe}}$-algebra is a collection of bimodules as in (93). The composition laws are given by wheeled graphs with one internal edge, whose souls are depicted in



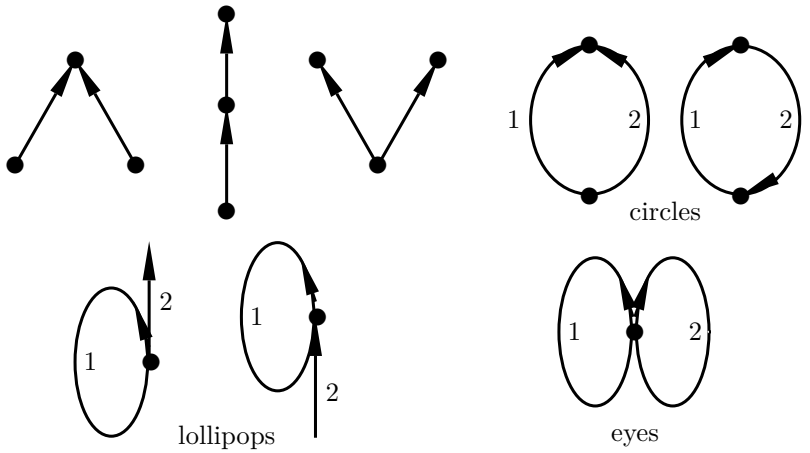
We recognize them as the operations

$$\circ_j^i : D(m,n) \otimes D(k,l) \longrightarrow D(m+k-1, n+l-1),\ 1 \le i \le n,\ 1 \le j \le k, \text{ and} \tag{95a}$$

$$\xi_j^i : D(m,n) \longrightarrow D(m-1, n-1),\ 1 \le i \le m,\ 1 \le j \le n \tag{95b}$$

in formulas (16) and (17) of [20].

As in the previous cases, the axioms that these operations satisfy are determined by graphs with two internal edges whose souls are depicted in the following display.



(96)

The graphs with three vertices induce the parallel and sequential associativity of the ξ-operations, similar to that for Markl operads (118b). They were explicitly given in the dioperadic context as axioms (a) and (b) in [7, page 111].

The circles in (96) represent the rules of the type $\circ_1\xi_2 = \circ_2\xi_1$, where ξ_1 resp. ξ_2 is the operation corresponding to the shrinking of the edge labeled 1 resp. 2, and similarly for $\circ_1$ and $\circ_2$. The lollipops in (96) force the interchange rule $\xi_2\circ_1 = \circ_1\xi_2$, and the eyes the rule $\circ_1\circ_2 = \circ_2\circ_1$. To expand these remaining axioms into explicit forms similar to that on [7, page 111] would not be very helpful; we thus leave it as an exercise for a determined reader. □

7.2 Dioperads

They were introduced in [7] as PROP-like structures whose algebras are objects such as Lie or infinitesimal bialgebras (called mock bialgebras in [15]). A short definition is that a dioperad is a wheeled properad without the ξ^i_j-operations (95b). The underlying operadic category is the category `Dio` of ordered simply connected oriented graphs introduced in [3, Example 4.21]. As before, one may check that `Dio` meets all requirements of our theory. One has the expected:

Theorem 7.2. *Dioperads are algebras over the terminal* `Dio`*-operad* $\mathtt{1}_{\mathtt{Dio}}$*, which is binary quadratic and self-dual.*

Proof. The proof is a simplified version of the wheeled case. The self-duality of $\mathtt{1}_{\mathtt{Dio}}$ is established in precisely the same way as the self-duality of the terminal `Whe`-operad $\mathtt{1}_{\mathtt{Whe}}$; the existence of the relevant coboundary is given by isomorphism (94) which clearly holds in `Dio` as well. The soul of graphs in `Dio` with one internal edge is the oriented interval, with the corresponding operation as in (95a). The souls of graphs in `Dio` with two internal edges are the three upper left graphs in (96). The resulting axioms are the parallel and sequential associativities which are the same as for ξ^i_j-operations of wheeled properads, see [7, §1.1]. □

7.3 $\frac{1}{2}$PROPs

These structures were introduced, following a suggestion of Kontsevich, in [22] as a link between dioperads and PROPs. A $\frac{1}{2}$PROP is a collection of bimodules (93) which is *stable* in that it fulfills

$$D(m,n) = 0 \quad \text{if} \quad m+n < 3,$$

together with partial vertical compositions

$$\circ_i : D(m_1,n_1) \otimes D(1,l) \to D(m_1, n_1+l-1),\ 1 \le i \le n_1, \text{ and}$$
$$_j\circ : D(k,1) \otimes D(m_2,n_2) \to D(m_2+k-1, n_2),\ 1 \le j \le m_2,$$

that satisfy the axioms of vertical compositions in PROPs. The corresponding operadic category $\frac{1}{2}\mathtt{Gr}$ is introduced in [3, Example 4.22]. We have the expected statement whose proof is left to the reader.

Theorem 7.3. *$\frac{1}{2}$PROPs are algebras over the terminal $\frac{1}{2}\mathtt{Gr}$-operad $1_{\frac{1}{2}\mathtt{Gr}}$. This operad is binary quadratic and self-dual.*

The operadic categories considered so far in this section were based on graphs. Let us give one example where this is not the case.

7.4 Permutads

They are structures introduced by Loday and Ronco in [13] to handle the combinatorial structure of objects like the permutahedra. We will describe an operadic category $\mathtt{Per}$ such that permutads are algebras over the terminal operad for this category.

Let $\underline{n}$ denote the finite ordered set $(1,\dots,n)$, $n\geq 1$. Objects of $\mathtt{Per}$ are surjections $\alpha:\underline{n}\twoheadrightarrow\underline{k}$, $n\geq 1$, and the morphisms are diagrams

$$\begin{array}{ccc} \underline{n} & = & \underline{n} \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha''} \\ \underline{k'} & \xrightarrow{\gamma} & \underline{k''} \end{array} \tag{97}$$

in which γ is order preserving (and necessarily a surjection).

The cardinality functor is defined by $|\alpha:\underline{n}\twoheadrightarrow\underline{k}|:=k$. The i-th fiber of the morphism in (97) is the surjection $(\gamma\alpha')^{-1}(i)\twoheadrightarrow\gamma^{-1}(i)$, $i\in\underline{k}$. The only local terminal objects are $\underline{n}\to\underline{1}$, $n\geq 1$, which are also the chosen ones. The category $\mathtt{Per}$ is graded by $e(\underline{n}\twoheadrightarrow\underline{k}):=k-1$. All quasibijections, and isomorphisms in general, are identities. The first sentence of the following theorem is the content of [19, Proposition 26].

Theorem 7.4. *Algebras over the terminal $\mathtt{Per}$-operad $1_{\mathtt{Per}}$ are the permutads of [13]. The operad $1_{\mathtt{Per}}$ is binary quadratic. It is self-dual in the sense that the category of algebras over $1^!_{\mathtt{Per}}$ is isomorphic to the category of permutads via the functor induced by the suspension of the underlying collection.*

Proof. Let us give a quadratic presentation of the terminal operad $1_{\mathtt{Per}}$. As noticed in Example 2.3, the category $\mathtt{QO}_{\mathtt{vrt}}(e)$ of virtual isomorphisms related to $\mathtt{Per}$ is isomorphic to the category $\mathtt{O}_{\mathtt{iso}}$ of isomorphisms in $\mathtt{Per}$. Since all isomorphisms in $\mathtt{Per}$ are identities, we infer that in fact $\mathtt{QO}_{\mathtt{vrt}}(e)\cong\mathtt{Per}_{\mathrm{disc}}$, the discrete category with the same objects as $\mathtt{Per}$. Therefore a $\mathtt{QO}_{\mathtt{vrt}}(e)$-presheaf is just a rule that assigns to each $\alpha\in\mathtt{Per}$ a vector space $E(\alpha)\in\mathtt{Vect}$. Let us define a 1-connected $\mathtt{Per}$-collection, in the sense of Definition 3.5, by

$$E(\alpha):=\begin{cases}\Bbbk & \text{if } |\alpha|=2\\ 0 & \text{otherwise,}\end{cases} \tag{98}$$

and describe the free operad $\mathbb{F}(E)$ generated by E.

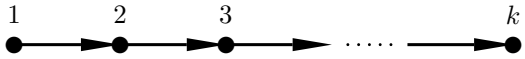
The first step is to understand the labeled towers in $lTw(\alpha)$. As all isomorphisms in $\mathtt{Per}$ are identities, the labeling is the identity map, so these towers are of the form

$$\alpha:=\alpha\xrightarrow{\tau_1}\alpha_1\xrightarrow{\tau_2}\alpha_2\xrightarrow{\tau_3}\cdots\xrightarrow{\tau_{s-1}}\alpha_{s-1}.$$

Since the generating collection E is such that $E(\alpha)\neq 0$ only if $|\alpha|=2$, we may consider only towers in which each τ_i, $1\leq i\leq s-1$, decreases the cardinality by one. For $\alpha:\underline{n}\twoheadrightarrow\underline{k}$, such a tower is a diagram

$$\begin{array}{ccccccccc} \underline{n} & = & \underline{n} & = & \underline{n} & = & \cdots & = & \underline{n} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & & & {\scriptstyle\alpha_{k-1}}\downarrow \\ \underline{k} & \xrightarrow{\nu_1} & \underline{k-1} & \xrightarrow{\nu_2} & \underline{k-2} & \xrightarrow{\nu_3} & \cdots & \xrightarrow{\nu_{k-2}} & \underline{2} \end{array} \tag{99}$$

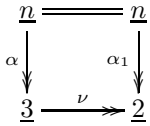
with $\nu_1,\dots,\nu_{k-2}$ order-preserving surjections. Notice that all vertical maps are determined by α and $\nu_1,\dots,\nu_{k-2}$. It will be convenient to represent $\underline{k}$ by a linear graph with k vertices:



and denote by $\mathrm{edg}(\underline{k})$ or $\mathrm{edg}(\alpha)$ the set of $k-1$ edges of this graph. In this graphical presentation, each $\nu_1, \dots, \nu_{k-2}$ contracts one of the edges of our linear graph; thus $\nu_1, \dots, \nu_{k-2}$ and therefore also the tower (99) is determined by the linear order on $\mathrm{edg}(\underline{k})$ in which the edges are contracted. We readily get the following analog of Proposition 5.8, which we formulate as a separate claim so we can refer to it later in the proof.

Claim 7.5. *The isomorphism classes of labeled towers (99) are in one-to-one correspondence with the linear orders on* $\mathrm{edg}(\underline{k})$ *modulo the relation* $\bowtie$ *that interchanges two edges adjacent in this linear order that do not share a common vertex.*

Let us continue the proof of Theorem 7.4. By the above claim, $\mathbb{F}(E)(\alpha)$ equals the span of the set of linear orders on $\mathrm{edg}(\underline{k})$ modulo the equivalence $\bowtie$. Let us inspect in detail its component $\mathbb{F}^2(E)(\alpha)$. It might be nonzero only for $\alpha : \underline{n} \to \underline{k} \in \mathtt{Per}$ with $k = 3$, for which (99) takes the form



and the relation $\bowtie$ is vacuous.

There are two possibilities for the map ν and therefore also for α_1. The map ν may either equal $\nu_{\{1,2\}} : \underline{3} \to \underline{2}$ defined by

$$\nu_{\{1,2\}}(1) = \nu_{\{1,2\}}(2) := 1, \ \nu_{\{1,2\}}(3) := 2$$

which corresponds to the linear order

on $\mathrm{edg}(3)$, or equal $\nu_{\{2,3\}} : \underline{3} \to \underline{2}$ defined by

$$\nu_{\{2,3\}}(1) := 1, \ \nu_{\{2,3\}}(2) = \nu_{\{2,3\}}(3) := 2,$$

corresponding to the order

.

The fiber sequence associated to $\nu_{\{1,2\}}$ is $\alpha|_{\alpha^{-1}\{1,2\}}, \nu_{\{1,2\}}\alpha$, and the one associated to $\nu_{\{2,3\}}$ is $\alpha|_{\alpha^{-1}\{2,3\}}, \nu_{\{2,3\}}\alpha$; therefore

$$\mathbb{F}^2(E)(\alpha) \cong \{E(\alpha|_{\alpha^{-1}\{1,2\}}) \otimes E(\nu_{\{1,2\}}\alpha)\} \oplus \{E(\alpha|_{\alpha^{-1}\{2,3\}}) \otimes E(\nu_{\{2,3\}}\alpha)\}.$$

Since $E(\alpha|_{\alpha^{-1}\{1,2\}}) = E(\alpha|_{\alpha^{-1}\{2,3\}}) = E(\nu_{\{1,2\}}\alpha) = E(\nu_{\{2,3\}}\alpha) = \Bbbk$ by definition, $\mathbb{F}^2(E)(\alpha)$ has a basis formed by

$$b_1 := [1 \otimes 1] \oplus [0 \otimes 0] \quad \text{and} \quad b_2 := [0 \otimes 0] \oplus [1 \otimes 1].$$

Let R be the subspace of $\mathbb{F}^2(E)$ spanned by $b_2 - b_1$. Quotienting by the ideal (R) generated by R extends the relation $\bowtie$ of Claim 7.5 by allowing edges that do share a common vertex, thus $\mathbb{F}(E)/(R)(\alpha) \cong \Bbbk$ for any α, in other words,

$$\mathbb{1}_{\mathtt{Per}} \cong \mathbb{F}(E)/(R).$$

Now we describe $\mathbb{1}_{\mathtt{Per}}$-algebras. Since $\pi_0(\mathtt{Per}) = \{1, 2, \dots\}$, their underlying collections are sequences of vector spaces $P(n)$, $n \geq 1$. As we saw several times before, the structure operations of $\mathbb{1}_{\mathtt{Per}}$-algebras are parametrized by the generating collection E, therefore, by (98), by surjections $r : \underline{n} \twoheadrightarrow \underline{2} \in \mathtt{Per}$. If $n_i := |r^{-1}(i)|$, $i = 1, 2$, the operation corresponding to r is of the form

$$\circ_r : P(n_1) \otimes P(n_2) \to P(n_1 + n_2) \tag{100}$$

by [3, display (57)]. It is easy to verify that the vanishing of the induced map $\mathbb{F}(E) \to \mathcal{E}nd_P$ on the generator $b_2 - b_1$ of the ideal of relations (R) is equivalent to the associativity

$$\circ_t(\circ_s \otimes \mathbb{1}) = \circ_u(\mathbb{1} \otimes \circ_v) \tag{101}$$

with $s := \alpha|_{\alpha^{-1}\{1,2\}}$, $t := \nu_{\{1,2\}}\alpha$, $u := \alpha|_{\alpha^{-1}\{2,3\}}$ and $v := \nu_{\{2,3\}}\alpha$. We recognize it as the associativity of [13, Lemma 2.2] featured in the biased definition of permutads.

It can easily be seen that $\mathfrak{K}_{\mathtt{Per}}(\alpha) := 1^!_{\mathtt{Per}}(\alpha) \cong \det(\mathrm{edg}(\alpha))$. As in §6.2 we identify $\mathfrak{K}_{\mathtt{Per}}$-algebras as structures with degree $+1$ operations

$$\bullet_r : P(n_1) \otimes P(n_2) \to P(n_1 + n_2)$$

with r as in (100) satisfying an odd version

$$\bullet_t(\bullet_s \otimes \mathbb{1}) + \bullet_u(\mathbb{1} \otimes \bullet_v) = 0$$

of (101). It is elementary to show that the structure induced on the component-wise suspension of the underlying collection is that of a permutad. □

In [19] we prove the following theorem:

Theorem 7.6. *The terminal* $\mathtt{P}$*-operad* $1_{\mathtt{Per}}$ *is Koszul.*

Its meaning is that the canonical map $\Omega(1^!_{\mathtt{Per}}) \to 1_{\mathtt{Per}}$ from a suitably defined bar construction of $1^!_{\mathtt{Per}}$ to $1_{\mathtt{Per}}$ is a component-wise homology equivalence. In other words, the dg-$\mathtt{Per}$ operad $\Omega(1^!_{\mathtt{Per}})$ is the minimal model of $1_{\mathtt{Per}}$; therefore, according to the philosophy of [17, Section 4], $\Omega(1^!_{\mathtt{Per}})$-algebras are *strongly homotopy* permutads. An explicit description of these objects is given in [19] as well.

8 Derivations and the cobar construction

Derivations of traditional operads defined in terms of partial compositions (118a) (i.e. "traditional" Markl operads) were introduced in [17, Definition 1.5], and the cobar construction in that context then implicitly in [17, Theorem 1.9]. The aim of this section is to generalize these notions to Markl (co)operads over operadic categories.

We require that the base operadic category $\mathtt{O}$ fulfills *Assumptions 1.2*. Then $e(X) = 0$ if and only if $X \in \mathtt{O}$ is local terminal by [3, Lemma 3.25]. All Markl operads are tacitly assumed to be strictly unital and 1-connected. The base monoidal category $\mathtt{V}$ will be the category $\mathtt{Vect}$ of graded vector spaces over a field $\Bbbk$ of characteristic 0.

8.1 Derivations, cooperads

To simplify the notation, we will use the same symbol both for a (co)operad and for its underlying collection when the meaning is clear from the context.

Definition 8.1. A degree s *derivation* of a Markl operad $\mathcal{M}$ is defined as a degree s endomorphism $\varpi : \mathcal{M} \to \mathcal{M}$ of the underlying collection such that, for each elementary morphism $F \triangleright T \xrightarrow{\phi} S$ and the related composition law $\circ_\phi : \mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T)$, one has the equality

$$\varpi_T \circ_\phi = \circ_\phi(\mathbb{1}_S \otimes \varpi_F) + \circ_\phi(\varpi_S \otimes \mathbb{1}_T) \tag{102}$$

of maps $\mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T)$. A *dg Markl operad* is a pair $(\mathcal{M}, \partial)$ consisting of a Markl operad $\mathcal{M}$ and a degree -1 derivation ∂ such that $\partial\partial = 0$.

In the following proposition, $\mathbb{F}(E)$ is the free Markl operad generated by a 1-connected collection E, which is considered as a subcollection of $\mathbb{F}(E)$ via the inclusion $\iota : E \hookrightarrow \mathbb{F}(E)$ in (63).

Proposition 8.2. *Each degree* s *map of collections* $\zeta : E \to \mathbb{F}(E)$ *uniquely extends to a degree* s *derivation* ϖ *of the free operad* $\mathbb{F}(E)$ *satisfying* $\varpi|_E = \zeta$.

Proof. Our proof follows the scheme of the proof of an analogous statement for operad algebras [14, Proposition 12.3.11]. To avoid cumbersome but conceptually insignificant sign issues, we assume that $s = 0$. The modification for a general s is indicated at the end of the proof. Given a Markl operad $\mathcal{M}$, we give the component-wise direct sum $\mathcal{M}\oplus\mathcal{M}$ of the underlying collections the structure of an operad with composition laws $\circ_\phi^\oplus$ given by

$$\circ_\phi^\oplus(a'\oplus b', a''\oplus b'') := \circ_\phi(a', a'') \oplus \big(\circ_\phi (a', b'') + (-1)^{|a''|\cdot|b'|} \circ_\phi (a'', b')\big),$$

for $a', a'' \in \mathcal{M}(S)$, $b', b'' \in \mathcal{M}(F)$, where $\circ_\phi$ is the composition law of $\mathcal{M}$ associated to an elementary morphism $F \triangleright T \xrightarrow{\phi} S$. Then clearly $\varpi : \mathcal{M} \to \mathcal{M}$ is a degree 0 derivation if and only if the map $(\mathbb{1}, \varpi) : \mathcal{M} \to \mathcal{M}\oplus\mathcal{M}$ is a morphism of Markl operads.

Let $\iota : E \to \mathbb{F}(E)$ be the inclusion (63). The map $(\iota, \zeta) : E \to \mathbb{F}(E)\oplus\mathbb{F}(E)$ extends to a unique operad morphism $\mathbb{F}(E) \to \mathbb{F}(E)\oplus\mathbb{F}(E)$ by the freeness of $\mathbb{F}(E)$. This extension is of the form $(\mathbb{1}, \varpi)$, where ϖ is the required derivation. If $s \neq 0$, we replace the direct sum $\mathcal{M}\oplus\mathcal{M}$ by $\mathcal{M}\oplus\uparrow^s\mathcal{M}$ and introduce the canonical Koszul signs. □

Definition 8.3. A *Markl* $\mathtt{O}$*-cooperad* is a functor $\mathcal{W} : \mathtt{O}_{\mathtt{iso}} \to \mathtt{V}$ equipped, for each elementary morphism $F \triangleright T \xrightarrow{\phi} S$ as in Definition 1.3, with the "partial coproduct"

$$\delta_\phi : \mathcal{W}(T) \to \mathcal{W}(S) \otimes \mathcal{W}(F). \tag{103}$$

These operations must fulfill the axioms obtained by reversing the arrows in the diagrams in Definition 1.8 of Markl operads.

Example 8.4. Assume that $\mathcal{M}$ is a Markl operad whose components $\mathcal{M}(T)$, $T \in \mathtt{O}$, are either finite-dimensional, or non-negatively, or non-positively graded dg vector spaces of finite type. Then its component-wise linear dual $\mathcal{M}^* := \{\mathcal{M}(T)^*\}_{T\in\mathtt{O}}$ is a Markl cooperad. The partial coproducts (103) are given by dualizing the operations (11), i.e.

$$\delta_\psi := \circ_\psi^* : \mathcal{M}(T)^* \to (\mathcal{M}(S)\otimes\mathcal{M}(F))^* \cong \mathcal{M}(S)^*\otimes\mathcal{M}(F)^*.$$

The finiteness assumption guarantees that the inclusion $(\mathcal{M}(S)\otimes\mathcal{M}(F))^* \hookleftarrow \mathcal{M}(S)^*\otimes\mathcal{M}(F)^*$ is an isomorphism. Since $\mathcal{M}$ is a $\mathtt{V}$-presheaf on $\mathtt{O}_{\mathtt{iso}}$, its dual is a functor $\mathcal{M}^* : \mathtt{O}_{\mathtt{iso}} \to \mathtt{V}$ as required in our definition of a cooperad.

Markl cooperad $\mathcal{W}$ is *counital* if there is given, for each trivial U, a "counit" $\epsilon_U : \mathcal{W}(U) \to \Bbbk$ such that the diagram

$$\begin{array}{ccc} \mathcal{W}(T) & \xrightarrow{\delta_!} & \mathcal{W}(U)\otimes\mathcal{W}(T) \\ \| & & \downarrow{\scriptstyle \epsilon_U\otimes\mathbb{1}} \\ \mathcal{W}(T) & \xrightarrow{\cong} & \Bbbk\otimes\mathcal{W}(T) \end{array}$$

commutes whenever T is such that $e(T) \geq 1$ and $T \triangleright T \xrightarrow{!} U$ the unique map. By reversing the arrows of (19) we obtain a map $\vartheta(T, u) : \mathcal{W}(T) \to \mathcal{W}(F)$ for each T with $e(T) \geq 1$ and $F \triangleright T \xrightarrow{!} u$, with u a local terminal object.

Definition 8.5. A counital Markl cooperad $\mathcal{W}$ is *strictly counital* if all the maps $\vartheta(T, u)$ are identities. It is 1-*connected* if $\epsilon_U : \mathcal{W}(U) \to \Bbbk$ is an isomorphism for each trivial $U \in \mathtt{O}$.

From this moment on, all Markl cooperads will be tacitly assumed to be *strictly counital* and 1-*connected*. The main source of examples will be component-wise linear duals of strictly unital 1-connected Markl operads that satisfy the finiteness assumption of Example 8.4.

8.2 The cobar construction

The underlying collection of a Markl cooperad $\mathcal{W}$ is a covariant functor $\mathtt{O}_{\mathtt{iso}} \to \mathtt{V}$. Since $\mathtt{O}_{\mathtt{iso}}$ is a groupoid, we may consider $\mathcal{W}$ also as a $\mathtt{V}$-presheaf with the contravariant action of $\omega \in \mathtt{O}_{\mathtt{iso}}$ given by $\omega^* := (\omega^{-1})_*$. With this convention in mind, the family $\overline{\mathcal{W}} = \{\mathcal{W}(T)\}_{T\in\mathtt{O}}$ defined by

$$\overline{\mathcal{W}}(T) := \begin{cases} \mathcal{W}(T) & \text{if } e(T) \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and also its component-wise desuspension $\downarrow\overline{\mathcal{W}}$, becomes a 1-connected $\mathtt{O}$-collection in the sense of Definition 3.5, so it make sense to form the free operad $\mathbb{F}(\downarrow\overline{\mathcal{W}})$ which it generates. We denote the restrictions of the partial coproducts δ_ϕ of (103) to $\overline{\mathcal{W}}$ by

$$\overline{\delta}_\phi : \overline{\mathcal{W}}(T) \to \overline{\mathcal{W}}(S) \otimes \overline{\mathcal{W}}(F), \text{ for } F \triangleright T \xrightarrow{\phi} S.$$

Note that $\overline{\mathcal{W}}$ with the above operations is an analog of the coaugmentation coideal of an coaugmented coalgebra featured in the classical cobar construction. We finally define, for every elementary map $F \triangleright T \xrightarrow{\phi} S$, degree -1 operations

$$\overline{\delta}^{\downarrow}_\phi := (\uparrow \otimes \uparrow)\, \overline{\delta}_\phi \downarrow : \downarrow\overline{\mathcal{W}}(T) \to \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(F).$$

A prominent rôle in the calculations below will be played by labeled towers of height 2. Recall that such a tower $\boldsymbol{\tau} = (\ell, \mathfrak{T}) \in \mathtt{lTw}^2(X)$ consists of an elementary morphism $\tau : T \to S$ and an isomorphism $\ell : X \xrightarrow{\cong} T$ which can always be replaced by a quasibijection. Its associated fiber sequence is the pair (F, S), with F the unique nontrivial fiber of τ. We will denote such a tower by

$$\begin{array}{ccc} & & F \\ & & \triangledown \\ X & \xrightarrow{\ell} & T\,. \\ & \boldsymbol{\tau} & \downarrow{\scriptstyle \tau} \\ & & S \end{array} \tag{104}$$

In the rest of this section we also assume that the groupoid $\mathtt{lTw}^2(X)$ has, for each $X \in \mathtt{O}$, only *finitely many* connected components.

Let us denote by $\Omega(\mathcal{W})$ the free operad $\mathbb{F}(\downarrow\overline{\mathcal{W}})$ generated by the 1-connected collection $\downarrow\overline{\mathcal{W}}$, with the natural grading $\Omega(\mathcal{W}) = \bigoplus_{n\geq 0} \Omega^n(\mathcal{W})$ inherited from $\mathbb{F}(\downarrow\overline{\mathcal{W}})$ by $\Omega^n(\mathcal{W}) := \mathbb{F}^n(\downarrow\overline{\mathcal{W}})$. We are going to introduce a degree -1 derivation $\partial_\Omega : \Omega(\mathcal{W}) \to \Omega(\mathcal{W})$ that squares to zero, thus making $\Omega(\mathcal{W})$ a differential graded Markl operad. Referring to Proposition 8.2, ∂_Ω will be defined as the unique extension of its restriction to the generators of $\Omega(\mathcal{W})$. For $x \in \downarrow\overline{\mathcal{W}}(X) \cong \Omega^1(\mathcal{W})(X)$ and $X \in \mathtt{O}$, this restriction is the finite sum

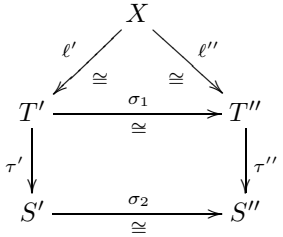
$$\partial_\Omega(x) = \sum_{[\boldsymbol{\tau}] \in \pi_0(\mathtt{lTw}^2(X))} \partial_{\boldsymbol{\tau}}(x), \tag{105}$$

with $\boldsymbol{\tau}$ running over representatives of the connected components of the groupoid $\mathtt{lTw}^2(X)$. If $\boldsymbol{\tau} = (\ell, \mathfrak{T})$ as in (104) is such a representative, we put

$$\partial_{\boldsymbol{\tau}}(x) := \ell^* \circ_\tau (\overline{\delta}^{\downarrow}_\tau(\ell_* x)) \in \Omega^2(\mathcal{W})(X), \tag{106}$$

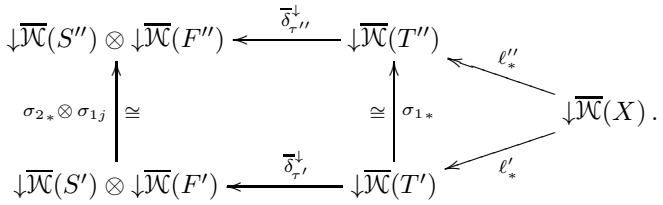
where $\circ_\tau : \Omega(\mathcal{W})(S) \otimes \Omega(\mathcal{W})(F) \to \Omega(\mathcal{W})(T)$ is the partial composition in the operad $\Omega(\mathcal{W})$ associated to the elementary map τ. In detail, $\ell_* x \in \downarrow\overline{\mathcal{W}}(T)$, thus $\overline{\delta}^{\downarrow}_\tau(\ell_* x) \in \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(F)$ which is canonically a subspace of $\Omega(\mathcal{W})(S) \otimes \Omega(\mathcal{W})(F)$, so the application of $\circ_\tau$ makes sense.

We must show that $\partial_{\boldsymbol{\tau}}(x)$ does not depend on the choices of the representatives of the connected components of $\mathtt{lTw}^2(X)$. Suppose thus that $\boldsymbol{\tau}' = (\ell', \mathfrak{T}')$ and $\boldsymbol{\tau}'' = (\ell'', \mathfrak{T}'')$ are two isomorphic labeled towers as in



with associated fiber sequences (F', S') resp. (F'', S''). Denote by $F = \phi^{-1}(j)$ the unique nontrivial fiber of the auxiliary morphism $\phi := \sigma_2 \tau' : T' \to S''$. By the strong counitality assumption, we

have $\mathcal{W}(F) = \mathcal{W}(F'')$, so the dual of commutative diagram (22) leads to



We conclude from this diagram that $\overline{\delta}^{\downarrow}_{\tau'}(\ell'_* x) \in \downarrow\overline{\mathcal{W}}(\mathfrak{T}')$ and $\overline{\delta}^{\downarrow}_{\tau''}(\ell''_* x) \in \downarrow\overline{\mathcal{W}}(\mathfrak{T}'')$ are related by the isomorphism $\boldsymbol{\sigma}^*$ associated to (σ_2, σ_1) as in (58), thus $\partial_{\boldsymbol{\tau}'}(x) = \ell'^* \circ_{\tau'} (\overline{\delta}^{\downarrow}_{\tau'}(\ell'_* x))$ and $\partial_{\boldsymbol{\tau}''}(x) = \ell''^* \circ_{\tau''} (\overline{\delta}^{\downarrow}_{\tau''}(\ell''_* x))$ are the same.

Proposition 8.6. *The derivation $\partial_{\Omega} : \Omega(\mathcal{W}) \to \Omega(\mathcal{W})$ introduced above squares to zero.*

Proof. It is simple to verify, using the defining equation (102), that the square of an odd-degree derivation is a derivation again. In particular, ∂^2_{Ω} is a derivation, thus it suffices, by Proposition 8.2, to verify that $\partial^2_{\Omega}(x) = 0$ for $x \in \downarrow\overline{\mathcal{W}}(X)$, $X \in \mathsf{0}$, i.e. that

$$\sum_{\boldsymbol{\tau}} \partial_{\Omega} \partial_{\boldsymbol{\tau}}(x) = \sum_{\boldsymbol{\tau}} \partial_{\Omega} \ell^* \circ_{\tau} (\overline{\delta}^{\downarrow}_{\tau}(\ell_* x)) = \sum_{\boldsymbol{\tau}} \ell^* \partial_{\Omega} \circ_{\tau} (\overline{\delta}^{\downarrow}_{\tau}(\ell_* x)) = 0.$$

In the above display, as well as in the following ones, we will not specify the summation range where it is clear from the context. By the derivation rule (102) we have

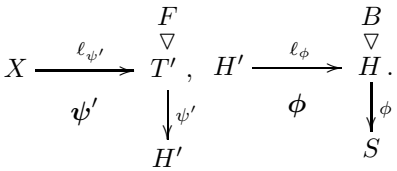
$$\sum_{\boldsymbol{\tau}} \ell^* \partial_{\Omega} \circ_{\tau} (\overline{\delta}^{\downarrow}_{\tau}(\ell_* x)) = \sum_{\boldsymbol{\tau}} \ell^* \circ_{\tau} (\partial_{\Omega} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\tau}(\ell_* x)) + \sum_{\boldsymbol{\tau}} \ell^* \circ_{\tau} (\mathbb{1} \otimes \partial_{\Omega})(\overline{\delta}^{\downarrow}_{\tau}(\ell_* x)),$$

thus $\partial^2_{\Omega}(x)$ decomposes as $\partial^2_{\Omega}(x) = \partial^2_L(x) + \partial^2_R(x)$, where $\partial^2_L(x)$, resp. $\partial^2_R(x)$, is the left, resp. right, term at the left-hand side of the above equation.

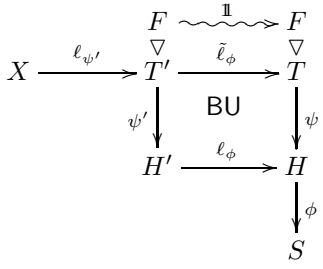
Analyzing $\partial^2_L(x)$. Let us focus on $\partial^2_L(x)$ first. Expanding further using definitions, we get

$$\begin{aligned}
\partial^2_L(x) &= \sum_{\boldsymbol{\psi}'} \ell^*_{\psi'} \circ_{\psi'} (\partial_{\Omega} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\psi'}(\ell^*_{\psi'} x)) = \sum_{\boldsymbol{\phi}, \boldsymbol{\psi}'} \ell^*_{\psi'} \circ_{\psi'} (\partial_{\boldsymbol{\phi}} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\psi'}(\ell_{\psi'*} x)) \\
&= \sum_{\boldsymbol{\phi}, \boldsymbol{\psi}'} \ell^*_{\psi'} \circ_{\psi'} (\ell^*_{\phi} \circ_{\phi} \overline{\delta}^{\downarrow}_{\phi} \ell_{\phi *} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\psi'}(\ell_{\psi'*} x)) \\
&= \sum_{\boldsymbol{\phi}, \boldsymbol{\psi}'} \ell^*_{\psi'} \circ_{\psi'} (\ell^*_{\phi} \otimes \mathbb{1})(\circ_{\phi} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\phi} \otimes \mathbb{1})(\ell_{\phi *} \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\psi'}(\ell_{\psi'*} x)),
\end{aligned}$$

where $\boldsymbol{\psi}'$ and $\boldsymbol{\phi}$ run over the representatives of the isomorphism classes of labeled towers of the form



Consider now the diagram



whose square was obtained using the blow-up axiom with the condition that the prescribed maps of fibers are identities, which is symbolized by the wavy arrow decorated by $\mathbb{1}$. From (22) we conclude that $\circ_{\psi'}(\ell_\phi^* \otimes \mathbb{1}) = \tilde{\ell}_\phi^* \circ_\psi$ and, dually, $(\ell_{\phi_*} \otimes \mathbb{1})\delta_{\psi'} = \delta_\psi \tilde{\ell}_{\phi_*}$, thus also $(\ell_{\phi_*} \otimes \mathbb{1})\overline{\delta}^{\downarrow}_{\psi'} = \overline{\delta}^{\downarrow}_{\psi} \tilde{\ell}_{\phi_*}$, and therefore

$$\partial_L^2(x) = \sum_{\boldsymbol{\phi},\boldsymbol{\psi}'} (\tilde{\ell}_\phi \ell_{\psi'})^* \circ_{\psi'} (\circ_\phi \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\phi} \otimes \mathbb{1})\overline{\delta}^{\downarrow}_{\psi} (\tilde{\ell}_\phi \ell_{\psi'})_*(x).$$

This can clearly be rewritten as

$$\partial_L^2(x) = \sum_{\boldsymbol{\phi},\boldsymbol{\psi}} \ell_* \circ_\psi (\circ_\phi \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_{\phi} \otimes \mathbb{1})\, \overline{\delta}^{\downarrow}_{\psi} \ell_*(x), \tag{107}$$

where $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ runs over the representatives of the isomorphism classes of labeled towers as in

$$\begin{array}{ccccc} X & \xrightarrow{\ell} & \begin{array}{c} F \\ \triangledown \\ T \\ \downarrow{\scriptstyle \psi} \\ H \end{array} & , \quad H & \xrightarrow{\mathbb{1}} & \begin{array}{c} B \\ \triangledown \\ H \\ \downarrow{\scriptstyle \phi} \\ S \end{array} \\ & \boldsymbol{\psi} & & & \boldsymbol{\phi} & \end{array} . \tag{108}$$

Let us further decompose

$$\partial_L^2(x) = \partial_L^2(x)' + \partial_L^2(x)'',$$

where $\partial_L^2(x)'$, resp. $\partial_L^2(x)''$, is the part of the sum in (107) running over ϕ and ψ with joint, resp. disjoint, fibers in the sense of Definition 1.4. We are going to show that $\partial_L^2(x)'' = 0$.

To start, let $k, i \in |S|$, $k \neq i$. Recall that (ϕ, ψ) is a (k,i)-pair if there is $j \in |\phi|^{-1}(k)$ such that $\psi^{-1}(j)$ is the only nontrivial fiber of ψ, and $\phi^{-1}(i)$ the only nontrivial fiber of ϕ. This results in the decomposition

$$\partial_L^2(x)'' = \sum_{k \neq i} \partial_L^2(x)''_{(k,i)}$$

where $\partial_L^2(x)''_{(k,i)}$ is the part of the sum in (107) taken over (k,i)-pairs. Out next aim will be to prove that

$$\partial_L^2(x)''_{(k,i)} + \partial_L^2(x)''_{(i,k)} = 0, \tag{109}$$

for each $k \neq i \in |S|$, using Lemma 3.9. The commutative diagram

$$\begin{array}{ccccc} & & P' & & \\ & {\scriptstyle (\psi', j)} \nearrow & & \searrow {\scriptstyle (\phi', i)} & \\ T & & & & S \\ & {\scriptstyle (\psi'', l)} \searrow & & \nearrow {\scriptstyle (\phi'', k)} & \\ & & P'' & & \end{array}$$

leads, with the help of the dual of (14), to

$$\begin{array}{ccc} \mathcal{W}(P') \otimes \mathcal{W}(F) & \xrightarrow{\delta_{\phi'} \otimes \mathbb{1}} & \mathcal{W}(S) \otimes \mathcal{W}(G) \otimes \mathcal{W}(F) \\ {\scriptstyle \delta_{\psi'}} \uparrow & & \\ \mathcal{W}(T) & & \downarrow {\scriptstyle \mathbb{1} \otimes \text{transposition}} \\ {\scriptstyle \delta_{\psi''}} \downarrow & & \\ \mathcal{W}(P'') \otimes \mathcal{W}(G) & \xrightarrow{\delta_{\phi''} \otimes \mathbb{1}} & \mathcal{W}(S) \otimes \mathcal{W}(F) \otimes \mathcal{W}(G)\,. \end{array}$$

As a simple application of the Koszul sign rule we obtain the diagram

$$\begin{array}{ccc} \downarrow\overline{\mathcal{W}}(P') \otimes \downarrow\overline{\mathcal{W}}(F) & \xrightarrow{\overline{\delta}^{\downarrow}_{\phi'} \otimes \mathbb{1}} & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(G) \otimes \downarrow\overline{\mathcal{W}}(F) \\ {\scriptstyle \overline{\delta}^{\downarrow}_{\psi'}} \uparrow & & \\ \downarrow\overline{\mathcal{W}}(T) & \text{commutes up to } -1 & \downarrow {\scriptstyle \mathbb{1} \otimes \text{transposition}} \\ {\scriptstyle \overline{\delta}^{\downarrow}_{\psi''}} \downarrow & & \\ \downarrow\overline{\mathcal{W}}(P'') \otimes \downarrow\overline{\mathcal{W}}(G) & \xrightarrow{\overline{\delta}^{\downarrow}_{\phi''} \otimes \mathbb{1}} & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(F) \otimes \downarrow\overline{\mathcal{W}}(G) \end{array}$$

commuting up to multiplication by -1. Combining it with diagram (14) applied to the composition laws of the free Markl operad $\Omega(\mathcal{W})$, we get

$$\begin{array}{ccccc}
\downarrow\overline{\mathcal{W}}(P')\otimes\downarrow\overline{\mathcal{W}}(F) & \xrightarrow{\overline{\delta}^{\downarrow}_{\phi'}\otimes\mathbb{1}} & \downarrow\overline{\mathcal{W}}(S)\otimes\downarrow\overline{\mathcal{W}}(G)\otimes\downarrow\overline{\mathcal{W}}(F) & \xrightarrow{\circ_{\phi'}\otimes\mathbb{1}} & \downarrow\overline{\mathcal{W}}(P')\otimes\downarrow\overline{\mathcal{W}}(F) \\
\uparrow{\scriptstyle \overline{\delta}^{\downarrow}_{\psi'}} & & & & \downarrow{\scriptstyle \circ_{\psi'}} \\
\downarrow\overline{\mathcal{W}}(T) & & \text{commutes up to } -1 & & \downarrow\overline{\mathcal{W}}(T) \\
\downarrow{\scriptstyle \overline{\delta}^{\downarrow}_{\psi''}} & & & & \uparrow{\scriptstyle \circ_{\psi''}} \\
\downarrow\overline{\mathcal{W}}(P'')\otimes\downarrow\overline{\mathcal{W}}(G) & \xrightarrow{\overline{\delta}^{\downarrow}_{\phi''}\otimes\mathbb{1}} & \downarrow\overline{\mathcal{W}}(S)\otimes\downarrow\overline{\mathcal{W}}(F)\otimes\downarrow\overline{\mathcal{W}}(G) & \xrightarrow{\circ_{\phi''}\otimes\mathbb{1}} & \downarrow\overline{\mathcal{W}}(P'')\otimes\downarrow\overline{\mathcal{W}}(G)\,,
\end{array}$$

expressing the equation

$$\circ_{\psi'}(\circ_{\phi'}\otimes\mathbb{1})(\overline{\delta}^{\downarrow}_{\phi'}\otimes\mathbb{1})\overline{\delta}^{\downarrow}_{\psi'}+\circ_{\psi''}(\circ_{\phi''}\otimes\mathbb{1})(\overline{\delta}^{\downarrow}_{\phi''}\otimes\mathbb{1})\overline{\delta}^{\downarrow}_{\psi''}=0.$$

Invoking (107), we conclude that each summand of $\partial^2_L(x)''_{(k,i)}$ has its counterterm in $\partial^2_L(x)''_{(i,k)}$. This establishes (109), and therefore $\partial^2_L(x)''=0$, so we may assume that the sum in (107) runs over isomorphism classes of pairs (108) with ϕ and ψ having *joint fibers*.

Analyzing $\partial^2_R(x)$. Let us perform a similar analysis of $\partial^2_R(x)$. We have

$$\begin{aligned}
\partial^2_R(x)&=\sum_{\boldsymbol{\alpha}'}\ell^*_{\alpha'}\circ_{\alpha'}(\mathbb{1}\otimes\partial_{\Omega})\overline{\delta}^{\downarrow}_{\alpha'}\ell_{\alpha'*}(x)=\sum_{\boldsymbol{\beta},\boldsymbol{\alpha}'}\ell^*_{\alpha'}\circ_{\alpha'}(\mathbb{1}\otimes\partial_{\boldsymbol{\beta}})\overline{\delta}^{\downarrow}_{\alpha'}\ell_{\alpha'*}(x)\\
&=\sum_{\boldsymbol{\beta},\boldsymbol{\alpha}'}\ell^*_{\alpha'}\circ_{\alpha'}(\mathbb{1}\otimes\ell^*_{\beta}\circ_{\beta}\overline{\delta}^{\downarrow}_{\beta}\ell_{\beta*})\overline{\delta}^{\downarrow}_{\alpha'}\ell_{\alpha'*}(x)\\
&=\sum_{\boldsymbol{\beta},\boldsymbol{\alpha}'}\ell^*_{\alpha'}\circ_{\alpha'}(\mathbb{1}\otimes\ell^*_{\beta})(\mathbb{1}\otimes\circ_{\beta})(\mathbb{1}\otimes\overline{\delta}^{\downarrow}_{\beta})(\mathbb{1}\otimes\ell_{\beta*})\overline{\delta}^{\downarrow}_{\alpha'}\ell_{\alpha'*}(x)\,,
\end{aligned}$$

where $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$ run over representatives of the isomorphism classes of labeled towers as in

$$\begin{array}{ccccc}
X & \xrightarrow{\ell_{\alpha'}} & A' & \qquad & F \\
 & & \triangledown & & \triangledown \\
 & \boldsymbol{\alpha}' & T' & , \quad A' \xrightarrow[\boldsymbol{\beta}]{\ell_{\beta}} & A\,. \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta} \\
 & & S & & B
\end{array}$$

Let us construct, out of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$, the commutative diagram

$$\begin{array}{ccccc}
 & & A' & \overset{\ell_{\beta}}{\rightsquigarrow} & A \\
 & & \triangledown & & \triangledown \\
X & \xrightarrow{\ell_{\alpha'}} & T' & \xrightarrow{\underline{\ell}_{\beta}} & T \\
 & & & \searrow\quad\swarrow & \\
 & & {\scriptstyle \alpha'} & \text{WBU} & {\scriptstyle \alpha} \\
 & & & S &
\end{array}$$

whose triangle is given by the weak blow-up axiom, under the condition that the prescribed map between the only nontrivial fibers is ℓ_β and the remaining maps of the fibers are identities. Using (22) and its dual as before, we rewrite $\partial^2_R(x)$ as

$$\partial^2_R(x)=\sum_{\boldsymbol{\beta},\boldsymbol{\alpha}'}(\underline{\ell}_{\beta}\ell_{\alpha'})^*\circ_{\alpha}(\mathbb{1}\otimes\circ_{\beta})(\mathbb{1}\otimes\overline{\delta}^{\downarrow}_{\beta})\overline{\delta}^{\downarrow}_{\alpha}(\underline{\ell}_{\beta}\ell_{\alpha'})_*(x)$$

which is the same as

$$\partial^2_R(x)=\sum_{\boldsymbol{\beta},\boldsymbol{\alpha}}\ell^*\circ_{\alpha}(\mathbb{1}\otimes\circ_{\beta})(\mathbb{1}\otimes\overline{\delta}^{\downarrow}_{\beta})\,\overline{\delta}^{\downarrow}_{\alpha}\ell_*(x),\tag{110}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ run over representatives of the isomorphism classes of towers as in

$$\begin{array}{ccc} X & \xrightarrow{\ell} & \begin{array}{c} A \\ \triangledown \\ T \\ \downarrow{\scriptstyle \alpha} \\ S \end{array} \\ & \boldsymbol{\alpha} & \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{\mathbb{1}} & \begin{array}{c} F \\ \triangledown \\ A \\ \downarrow{\scriptstyle \beta} \\ B \end{array} \\ & \boldsymbol{\beta} & \end{array}. \tag{111}$$

The weak blow-up axiom produces a unique factorization $\alpha = \phi\psi$ in the diagram

$$\begin{array}{ccc} A & \overset{\beta}{\rightsquigarrow} & B \\ \triangledown & & \triangledown \\ S & \xrightarrow{\psi} & H \\ & {\scriptstyle\alpha}\searrow \quad \text{WBU} \quad \swarrow{\scriptstyle\phi} & \\ & T & \end{array} \tag{112}$$

in which α is elementary and the map between the only nontrivial fibers of α resp. ϕ is $\beta : A \to B$, as symbolized by the wavy arrow. Notice that the pair (ϕ, ψ) has joint fibers. The dual of (12) gives

$$\begin{array}{ccccc} & & \mathcal{W}(S) \otimes \mathcal{W}(A) & & \\ & {\scriptstyle\delta_\alpha}\nearrow & & \searrow{\scriptstyle\mathbb{1}\otimes\delta_\beta} & \\ \mathcal{W}(T) & & & & \mathcal{W}(S) \otimes \mathcal{W}(B) \otimes \mathcal{W}(F) \\ & {\scriptstyle\delta_\psi}\searrow & & \nearrow{\scriptstyle\delta_\phi\otimes\mathbb{1}} & \\ & & \mathcal{W}(H) \otimes \mathcal{W}(F) & & \end{array}$$

which, combined with the Koszul sign rule, leads to

$$\begin{array}{ccccc} & & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(A) & & \\ & {\scriptstyle\overline{\delta}^{\downarrow}_\alpha}\nearrow & & \searrow{\scriptstyle\mathbb{1}\otimes\overline{\delta}^{\downarrow}_\beta} & \\ \downarrow\overline{\mathcal{W}}(T) & & \text{commutes up to } -1 & & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(B) \otimes \downarrow\overline{\mathcal{W}}(F) \\ & {\scriptstyle\overline{\delta}^{\downarrow}_\psi}\searrow & & \nearrow{\scriptstyle\overline{\delta}^{\downarrow}_\phi\otimes\mathbb{1}} & \\ & & \downarrow\overline{\mathcal{W}}(H) \otimes \downarrow\overline{\mathcal{W}}(F) & & \end{array}$$

which commutes up to multiplication by -1. Combining it with diagram (12) for the composition laws of the free operad $\Omega(\mathcal{W})$, we obtain the diagram

$$\begin{array}{ccccccc} & & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(A) & & & & \\ & {\scriptstyle\overline{\delta}^{\downarrow}_\alpha}\nearrow & & \searrow{\scriptstyle\mathbb{1}\otimes\overline{\delta}^{\downarrow}_\beta} & & & \\ \downarrow\overline{\mathcal{W}}(T) & & \text{commutes up to } -1 & & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(B) \otimes \downarrow\overline{\mathcal{W}}(F) & & \\ & {\scriptstyle\overline{\delta}^{\downarrow}_\psi}\searrow & & \nearrow{\scriptstyle\overline{\delta}^{\downarrow}_\phi\otimes\mathbb{1}} & & & \\ & & \downarrow\overline{\mathcal{W}}(H) \otimes \downarrow\overline{\mathcal{W}}(F) & & & & \end{array} \quad \begin{array}{ccc} & \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(A) & \\ {\scriptstyle\mathbb{1}\otimes\circ_\beta}\nearrow & & \searrow{\scriptstyle\circ_\alpha} \\ \downarrow\overline{\mathcal{W}}(S) \otimes \downarrow\overline{\mathcal{W}}(B) \otimes \downarrow\overline{\mathcal{W}}(F) & & \downarrow\overline{\mathcal{W}}(T) \\ {\scriptstyle\circ_\phi\otimes\mathbb{1}}\searrow & & \nearrow{\scriptstyle\circ_\psi} \\ & \downarrow\overline{\mathcal{W}}(H) \otimes \downarrow\overline{\mathcal{W}}(F) & \end{array}$$

which translates into the equation

$$\circ_\psi(\circ_\phi \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_\phi \otimes \mathbb{1})\overline{\delta}^{\downarrow}_\psi + \circ_\alpha(\mathbb{1} \otimes \circ_\beta)(\mathbb{1} \otimes \overline{\delta}^{\downarrow}_\beta)\overline{\delta}^{\downarrow}_\beta = 0. \tag{113}$$

Notice that the assignment $(\alpha, \beta) \mapsto (\phi, \psi)$ described by diagram (112) induces a bijection between the set of isomorphism classes of towers (111) and (108) with (ϕ, ψ) having joint fibers. Its inverse $(\phi, \psi) \mapsto (\alpha, \beta)$ produces α as the composite $\phi\psi$ while β is the induced map between the unique nontrivial fibers as in

$$\begin{array}{ccc} A & \overset{\beta}{\rightsquigarrow} & B \\ \triangledown & & \triangledown \\ S & \xrightarrow{\psi} & H \\ & {\scriptstyle\phi\psi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & T\,. & \end{array}$$

Therefore, by (113), each summand $\ell^* \circ_\psi (\circ_\phi \otimes \mathbb{1})(\overline{\delta}^{\downarrow}_\phi \otimes \mathbb{1})\overline{\delta}^{\downarrow}_\psi \ell_*(x)$ of (107) possesses a unique counterterm $\ell^* \circ_\alpha (\mathbb{1} \otimes \circ_\beta)(\mathbb{1} \otimes \overline{\delta}^{\downarrow}_\beta)\overline{\delta}^{\downarrow}_\beta \ell_*(x)$ in the sum (110), which proves that

$$\partial^2_{\mathfrak{Q}}(x) = \partial^2_L(x) + \partial^2_R(x) = 0$$

as claimed. □

Definition 8.7. We call the dg operad $\mathfrak{Q}(\mathcal{W}) = (\mathfrak{Q}(\mathcal{W}), \partial_{\mathfrak{Q}})$ the *cobar construction* of a strictly counital 1-connected Markl cooperad $\mathcal{W}$.

9 The dual dg operad and Koszulity

The dual dg operad of a traditional operad was introduced in Section 3 of [10], and the Koszulity for quadratic operads in Section 4 of the same article. In this section we generalize these notions to Markl operads over operadic categories and prove that operads whose algebras are the most common structures are Koszul.

Remark 9.1. The duality for the (colored) operad whose algebras are non-unital nonsymmetric operads was studied in 2002 by van der Laan [27]. Nineteen years later, Dehling and Vallette proved in [5] that the colored operad whose algebras are the classical (symmetric) operad is curved Koszul. Koszulity of the groupoid-colored operad whose algebras are modular operads was proved by Ward [25]. A general approach to graph-based operadic structures in the language of Feynman categories was suggested by Kaufmann and Ward in [12].

The base operadic category $\mathtt{O}$ is required to fulfill *Assumptions 1.2*. Let $\mathcal{M}$ be a strictly unital 1-connected Markl operad satisfying the assumptions of Example 8.4, so that the component-wise linear dual $\mathcal{M}^* := \{\mathcal{M}(T)^*\}_{T \in \mathtt{O}}$ is a strictly counital 1-connected Markl cooperad.

Definition 9.2. The *dual dg operad* of a Markl operad $\mathcal{M}$ as above is the dg operad

$$\mathbb{D}(\mathcal{M}) = (\mathbb{D}(\mathcal{M}), \partial_{\mathbb{D}}) := (\mathfrak{Q}(\mathcal{M}^*), \partial_{\mathfrak{Q}}),$$

the cobar construction of the component-wise linear dual of $\mathcal{M}$.

Assume that $\mathcal{M} = \mathbb{F}(E)/(R)$ is a quadratic Markl operad as in Definition 4.2 and $\mathcal{M}^!$ its Koszul dual, cf. Definition 4.3. To introduce the Koszulity, we start from the injection $\uparrow E \hookrightarrow \mathcal{M}^!$ of collections defined as the composite

$$\uparrow E \hookrightarrow \mathbb{F}(\uparrow E) \twoheadrightarrow \mathbb{F}(\uparrow E)/(R^\perp) = \mathcal{M}^!.$$

Its linear dual $\mathcal{M}^{!*} \twoheadrightarrow \uparrow E$ desuspends to a map $\pi : \downarrow \mathcal{M}^{!*} \twoheadrightarrow E$. The related twisting morphism $\downarrow \mathcal{M}^{!*} \to \mathcal{M}$, defined as the composite

$$\downarrow \mathcal{M}^{!*} \overset{\pi}{\twoheadrightarrow} E \hookrightarrow \mathbb{F}(E) \twoheadrightarrow \mathbb{F}(E)/(R) = \mathcal{M},$$

extends, by the freeness of $\mathbb{F}(\downarrow \mathcal{M}^{!*})$, to a morphism $\rho_{\mathcal{M}} : \mathbb{F}(\downarrow \mathcal{M}^{!*}) \to \mathcal{M}$ of Markl $\mathtt{O}$-operads. One verifies by direct calculation:

Proposition 9.3. *The morphism $\rho_{\mathcal{M}}$ induces the* canonical map

$$can_{\mathcal{M}} : \mathbb{D}(\mathcal{M}^!) = (\mathbb{F}(\downarrow \mathcal{M}^{!*}), \partial_{\mathbb{D}}) \longrightarrow (\mathcal{M}, 0) \tag{114}$$

of Markl dg $\mathtt{O}$-operads.

Definition 9.4. A quadratic Markl $\mathtt{O}$-operad $\mathcal{M}$ is *Koszul* if the canonical map (114) is a component-wise homology isomorphism.

Remark 9.5. In the "classical" operad theory one proves that a quadratic operad is Koszul if and only if its Koszul dual is Koszul [10, Proposition 4.1.4]. We believe the same is true also in our setup, but postpone the proof for future work.

In the rest of this section we establish the Koszulity of some of the binary quadratic operads introduced in Sections 5–7. Namely, we prove the Koszulity of the operad $\mathbf{1}_{\mathtt{ggGrc}}$ whose algebras are modular operads, of the operad $1_{\mathtt{Tr}}$ whose algebras are cyclic operads, of the operad $1_{\mathtt{RTr}}$ whose algebras are ordinary Markl operads and of the operad $1_{\mathtt{Whe}}$ whose algebras are wheeled properads. The Koszulity of the operad $1_{\mathtt{Per}}$ whose algebras are permutads was already established in [19, Corollary 49].

Theorem 9.6. *The binary quadratic operads* $\mathbf{1}_{\mathtt{ggGrc}}$, $1_{\mathtt{Tr}}$, $1_{\mathtt{RTr}}$ *and* $1_{\mathtt{Whe}}$ *are Koszul.*

Proof. In the proof we drop the subscripts of ρ and *can* since they will always be clear from the context. We start with the most complex case of the terminal $\mathtt{ggGrc}$-operad $\mathbf{1}_{\mathtt{ggGrc}}$. Our strategy will be to show that its minimal model $\mathfrak{M}_{\mathtt{ggGrc}}$ constructed in [4, Section 3] is isomorphic to the dual dg operad $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}})$ of its Koszul dual $\mathfrak{K}_{\mathtt{ggGrc}} = \mathbf{1}^!_{\mathtt{ggGrc}}$, via an isomorphism compatible with the resolving maps $\mathfrak{M}_{\mathtt{ggGrc}} \xrightarrow{\rho} \mathbf{1}_{\mathtt{ggGrc}}$ resp. $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) \xrightarrow{can} \mathbf{1}_{\mathtt{ggGrc}}$.

It is not difficult to determine, using the presentation in the proof of Theorem 5.12, the operad structure of $\mathfrak{K}_{\mathtt{ggGrc}}$. As we noticed in Remark 5.13, given a genus-graded connected graph $\Gamma \in \mathtt{ggGrc}$, the corresponding piece $\mathfrak{K}_{\mathtt{ggGrc}}(\Gamma)$ can be identified with the one-dimensional vector space $\det(\Gamma) := \det(\mathrm{edg}(\Gamma))$, the determinant of the set of internal edges of Γ, placed in degree $e(\Gamma)+1$, where by definition $e(\Gamma)$, the grade of Γ is the number of internal edges. The unit map associated to a corolla $c \in \mathtt{ggGrc}$ is the canonical isomorphism

$$\Bbbk = \det(\emptyset) \cong \mathfrak{K}_{\mathtt{ggGrc}}(c).$$

Given two finite sets $S_1 = \{e^1_1, \dots, e^1_a\}$ and $S_2 = \{e^2_1, \dots, e^2_b\}$, we define the canonical isomorphism

$$\omega_{S_1,S_2} : \det(S_1 \sqcup S_2) \to \det(S_1) \otimes \det(S_2)$$

by

$$\omega_{S_1,S_2}(e^1_1 \wedge \cdots \wedge e^1_a \wedge e^2_1 \wedge \cdots \wedge e^2_b) := (e^1_1 \wedge \cdots \wedge e^1_a) \otimes (e^2_1 \wedge \cdots \wedge e^2_b).$$

Notice that if $\Gamma \triangleright \Gamma' \xrightarrow{\tau} \Gamma''$ is an elementary morphism, there is a natural isomorphism

$$\mathrm{edg}(\Gamma') \cong \mathrm{edg}(\Gamma'') \sqcup \mathrm{edg}(\Gamma)$$

of the sets of internal edges. The partial composition related to τ then equals

$$\circ_\tau : \mathfrak{K}_{\mathtt{ggGrc}}(\Gamma'') \otimes \mathfrak{K}_{\mathtt{ggGrc}}(\Gamma) \cong \det(\Gamma'') \otimes \det(\Gamma) \xrightarrow{\omega^{-1}_{\mathrm{edg}(\Gamma''),\mathrm{edg}(\Gamma)}} \det(\Gamma') \cong \mathfrak{K}_{\mathtt{ggGrc}}(\Gamma').$$

Our next step will be to describe the dual dg operad $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}})$ which, by definition, equals the cobar construction $\Omega(\mathfrak{K}^*_{\mathtt{ggGrc}})$ of the cooperad $\mathfrak{K}^*_{\mathtt{ggGrc}}$. Paying attention to the Koszul sign rule, we determine the cooperad structure operation of $\mathfrak{K}^*_{\mathtt{ggGrc}}$ related to τ as

$$\delta_\tau := (-1)^{(e(\Gamma'')+1)(e(\Gamma)+1)} \cdot \circ^*_\tau : \mathfrak{K}^*_{\mathtt{ggGrc}}(\Gamma') \to \mathfrak{K}^*_{\mathtt{ggGrc}}(\Gamma'') \otimes \mathfrak{K}^*_{\mathtt{ggGrc}}(\Gamma).$$

The underlying collection of the reduced cooperad $\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}$ is given by

$$\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma) := \begin{cases} \mathfrak{K}^*_{\mathtt{ggGrc}}(\Gamma) & \text{if } \Gamma \text{ has at least one internal edge} \\ 0 & \text{otherwise.} \end{cases}$$

The desuspended operation

$$\overline{\delta}^{\downarrow}_\tau := (\downarrow \otimes \downarrow)\overline{\delta}_\tau \uparrow : \downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma') \to \downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma'') \otimes \overline{\downarrow}\mathfrak{K}^*_{\mathtt{ggGrc}}(\Gamma)$$

is then, with the help of the identification $\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma) \cong \det(\Gamma)$ for Γ with at least one internal edge, described as

$$\overline{\delta}^{\downarrow}_\tau = (-1)^{(e(\Gamma'')+1)+(e(\Gamma'')+1)(e(\Gamma)+1)} \cdot \omega_{\mathrm{edg}(\Gamma''),\mathrm{edg}(\Gamma)} = (-1)^{e(\Gamma)(e(\Gamma'')+1)} \cdot \omega_{\mathrm{edg}(\Gamma''),\mathrm{edg}(\Gamma)}.$$

The new sign $(-1)^{(e(\Gamma'')+1)}$ is the contribution of the commutation of the first tensor factor of the image of $\overline{\delta}_\tau\uparrow$ over the desuspension $\downarrow$.

Let $\boldsymbol{\tau} = (\ell, \mathfrak{T}) \in \mathtt{1Tw}^2(X)$ be the labeled tower

$$\begin{array}{ccc} X & \xrightarrow{\ell} & \begin{array}{c}\Gamma\\ \triangledown\\ \Gamma'\\ \downarrow\tau\\ \Gamma''\end{array} \\ & \boldsymbol{\tau} & \end{array}.$$

We may assume that ℓ is a quasibijection by Proposition 3.13. Since quasibijections obviously act trivially on $\mathfrak{K}_{\mathtt{ggGrc}}$ and thus also on $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}$, formula (106) for the component $\partial_{\boldsymbol{\tau}}$ of the differential (105) associated to $\boldsymbol{\tau}$ reads

$$\partial_{\boldsymbol{\tau}}(x) = (-1)^{e(\Gamma)(e(\Gamma'')+1)} \cdot \circ_\tau \omega_{\mathrm{edg}(\Gamma),\mathrm{edg}(\Gamma'')}(x) \in \mathbb{F}^2(\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}})(X), \tag{115a}$$

for $x \in \downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(X) = \downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma)$. Thus $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) = (\mathbb{F}(\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}), \partial_{\mathbb{D}})$, with $\partial_{\mathbb{D}}$ given by the right-hand side of (106) with $\partial_{\boldsymbol{\tau}}$ as in (115a).

To describe the map $can : \mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) \to \mathbf{1}_{\mathtt{ggGrc}}$ notice that, for $\Gamma \in \mathtt{ggGrc}$ with precisely one internal edge, the graded vector space $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma)$ is *canonically* isomorphic to $\Bbbk$ placed in degree 0, since it is the desuspension of the dual of the determinant of a one-point set. The operad morphism *can* is the unique extension of the map $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}} \to \mathbf{1}_{\mathtt{ggGrc}}$ of collections whose component $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma) \to \mathbf{1}_{\mathtt{ggGrc}}(\Gamma)$ is trivial if Γ has at least two internal edges, and which is the canonical isomorphism $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma) \cong \Bbbk \cong \mathbf{1}_{\mathtt{ggGrc}}(\Gamma)$ if Γ has exactly one internal edge. This finishes the description of the dual dg operad $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}})$ and the associated canonical map.

Recall that the underlying non-dg operad of the minimal model $\mathfrak{M}_{\mathtt{ggGrc}}$ of $\mathbf{1}_{\mathtt{ggGrc}}$ in [4] is the free operad $\mathbb{F}(D)$ generated by the collection D, which in fact coincides with $\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}$, so $\mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) = \mathfrak{M}_{\mathtt{ggGrc}}$ as non-dg operads. With this identification, the resolving map $\rho : \mathfrak{M}_{\mathtt{ggGrc}} \to \mathbf{1}_{\mathtt{ggGrc}}$ equals the morphism $can : \mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) \to \mathbf{1}_{\mathtt{ggGrc}}$ described above.

The differential ∂ of the minimal model translated to the formalism used in this article is given by the sum in the right-hand side of (105), but the component $\partial_{\boldsymbol{\tau}}$ is now

$$\partial_{\boldsymbol{\tau}}(x) = (-1)^{e(\Gamma)} \cdot \circ_\tau \omega_{\mathrm{edg}(\Gamma),\mathrm{edg}(\Gamma'')}(x) \in \mathbb{F}^2(D)(X), \tag{115b}$$

cf. formula (18b) of [4]. We see that the expressions in (115a) and (115b) agree up to the sign factor $(-1)^{e(\Gamma'')e(\Gamma)}$. To compensate this discrepancy, we define a map $\chi : \overline{\mathfrak{K}^*_{\mathtt{ggGrc}}} \to D$ of collections by

$$\chi(x) := (-1)^{\frac{e(\Gamma)(e(\Gamma)-1)}{2}} \cdot x \text{ for } x \in \downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}(\Gamma)$$

and denote by the same symbol also its unique extension $\chi : \mathbb{F}(\downarrow\overline{\mathfrak{K}^*_{\mathtt{ggGrc}}}) \to \mathbb{F}(D)$ to an operad morphism. It is simple to verify that χ commutes with the differentials and that the diagram

$$\begin{array}{ccc} \mathbb{D}(\mathfrak{K}_{\mathtt{ggGrc}}) & \xrightarrow{can} & \\ \chi \downarrow \cong & & \mathbf{1}_{\mathtt{ggGrc}} \\ \mathfrak{M}_{\mathtt{ggGrc}} & \xrightarrow{\rho} & \end{array}$$

of dg operads commutes. Since ρ is a component-wise homology isomorphism by [4, Theorem 31], so is *can*. This establishes the Koszulity of $\mathbf{1}_{\mathtt{ggGrc}}$.

The proofs of the remaining cases are similar. One describes the dual dg operad by an obvious modification of the method above, and compares it with the corresponding minimal models described in Theorem 32 and Sections 3.4 and 3.5 of [4]. □

Appendix

A (Odd) modular operads, and classical Markl operads.

In this appendix we recall three structures referred to in this work. All definitions given here are standard today, see e.g. [9, 21], so the purpose is merely to fix the notation and terminology.

Let $\mathtt{fSet}$ denote the category of finite sets and $\mathtt{Vect}$, the category of graded vector spaces. For $S_1, S_2 \in \mathtt{fSet}$, we write $S_1 \sqcup S_2 \in \mathtt{fSet}$ to denote the disjoint union, the notation implying that S_1 and S_2 are disjoint. When we write e.g. "elements a, b, c" we tacitly assume that a, b and c are mutually distinct.

Recall that a *modular module* is a functor $\mathtt{fSet} \times \mathbb{N} \to \mathtt{Vect}$, with $\mathbb{N}$ interpreted as a discrete category with objects called *genera* in this context.

Definition A.1. A *modular operad* is a modular module

$$\mathcal{M} = \big\{\mathcal{M}(S;g) \in \mathtt{Vect} \mid (S;g) \in \mathtt{fSet} \times \mathbb{N}\big\}$$

together with degree 0 morphisms (composition laws)

$$ {}_a\circ_b : \mathcal{M}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{M}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{M}(S_1 \sqcup S_2; g_1 + g_2) \tag{116a}$$

given for arbitrary finite sets S_1, S_2, elements a, b, and genera $g_1, g_2 \in \mathbb{N}$. There are, moreover, degree 0 contractions

$$\circ_{uv} = \circ_{vu} : \mathcal{M}\big(S \sqcup \{u\} \sqcup \{v\}; g\big) \to \mathcal{M}(S; g+1) \tag{116b}$$

given for any finite set S, genus $g \in \mathbb{N}$, and elements u, v. These data are required to satisfy the following axioms.

(i) For arbitrary isomorphisms $\rho : S_1 \sqcup \{a\} \to T_1$ and $\sigma : S_2 \sqcup \{b\} \to T_2$ of finite sets and genera g_1, $g_2 \in \mathbb{N}$, one has the equality

$$\mathcal{M}\big(\rho|_{S_1} \sqcup \sigma|_{S_2}\big)\, {}_a\circ_b = {}_{\rho(a)}\circ_{\sigma(b)}\ \big(\mathcal{M}(\rho) \otimes \mathcal{M}(\sigma)\big) \tag{116c}$$

of maps

$$\mathcal{M}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{M}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{M}\big(T_1 \sqcup T_2 \setminus \{\rho(a), \sigma(b)\}; g_1 + g_2\big).$$

(ii) For an isomorphism $\rho : S \sqcup \{u\} \sqcup \{v\} \to T$ of finite sets and a genus $g \in \mathbb{N}$, one has the equality

$$\mathcal{M}\big(\rho|_S\big)\ \circ_{uv} = \circ_{\rho(u)\rho(v)} \mathcal{M}(\rho) \tag{116d}$$

of maps $\mathcal{M}\big(S \sqcup \{u\} \sqcup \{v\}; g\big) \to \mathcal{M}\big(T \setminus \{\rho(u), \rho(v)\}; g+1\big)$.

(iii) For S_1, S_2, a, b and g_1, g_2 as in (116a), one has the equality

$$ {}_a\circ_b = {}_b\circ_a \tau \tag{116e}$$

of maps $\mathcal{M}(S_1 \sqcup \{a\}; g_1) \otimes \mathcal{M}(S_2 \sqcup \{b\}; g_2) \to \mathcal{M}\big(S_1 \sqcup S_2; g_1 + g_2\big)$; here τ denotes the commutativity constraint in $\mathtt{Vect}$.

(iv) For finite sets S_1, S_2, S_3, elements a, b, c, d and genera $g_1, g_2, g_3 \in \mathbb{N}$, one has the equality

$$ {}_a\circ_b(\mathbb{1} \otimes {}_c\circ_d) = {}_c\circ_d({}_a\circ_b \otimes \mathbb{1}) \tag{116f}$$

of maps

$$\mathcal{M}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{M}\big(S_2 \sqcup \{b\} \sqcup \{c\}; g_2\big) \otimes \mathcal{M}\big(S_3 \sqcup \{d\}; g_3\big) \longrightarrow \mathcal{M}\big(S_1 \sqcup S_2 \sqcup S_3; g_1 + g_2 + g_3\big).$$

(v) For a finite set S, elements a, b, c, d and a genus $g \in \mathbb{N}$ one has the equality

$$\circ_{ab}\ \circ_{cd} = \circ_{cd}\ \circ_{ab} \tag{116g}$$

of maps $\mathcal{M}\big(S \sqcup \{a\} \sqcup \{b\} \sqcup \{c\} \sqcup \{d\}; g\big) \to \mathcal{M}(S; g+2)$.

(vi) For finite sets S_1, S_2, elements a, b, c, d and genera $g_1, g_2 \in \mathbb{N}$, one has the equality

$$\circ_{ab}\ \ {}_c\circ_d = \circ_{cd}\ \ {}_a\circ_b \tag{116h}$$

of maps $\mathcal{M}\big(S_1 \sqcup \{a\} \sqcup \{c\}; g_1\big) \otimes \mathcal{M}\big(S_2 \sqcup \{b\} \sqcup \{d\}; g_2\big) \to \mathcal{M}(S_1 \sqcup S_2; g_1 + g_2 + 1)$.

(vii) For finite sets S_1, S_2, elements a, b, u, v, and genera $g_1, g_2 \in \mathbb{N}$, one has the equality

$$ {}_a\circ_b \ (\circ_{uv} \otimes \mathbb{1}) = \circ_{uv} \ {}_a\circ_b \tag{116i}$$

of maps $\mathcal{M}\big(S_1 \sqcup \{a\} \sqcup \{u\} \sqcup \{v\}; g_1\big) \otimes \mathcal{M}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{M}(S_1 \sqcup S_2; g_1 + g_2 + 1)$.

Definition A.2. An *odd modular operad* is a modular module

$$\mathcal{O} = \big\{\mathcal{O}(S; g) \in \texttt{Vect} \mid (S; g) \in \texttt{fSet} \times \mathbb{N}\big\}$$

together with degree $+1$ morphisms (${}_a\bullet_b$-operations)

$${}_a\bullet_b : \mathcal{O}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{O}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{O}(S_1 \sqcup S_2; g_1 + g_2) \tag{117a}$$

given for arbitrary finite sets S_1, S_2, elements a, b, and arbitrary $g_1, g_2 \in \mathbb{N}$. There are, moreover, degree 1 morphisms (the contractions)

$$\bullet_{uv} = \bullet_{vu} : \mathcal{O}\big(S \sqcup \{u\} \sqcup \{v\}; g\big) \to \mathcal{O}(S; g + 1)$$

given for any finite set S, $g \in \mathbb{N}$, and elements u, v; we are using the notation for composition laws of odd modular operads introduced in [11]. These data are required to satisfy the following axioms.

(i) For arbitrary isomorphisms $\rho : S_1 \sqcup \{a\} \to T_1$ and $\sigma : S_2 \sqcup \{b\} \to T_2$ of finite sets and g_1, $g_2 \in \mathbb{N}$, one has the equality

$$\mathcal{O}\big(\rho|_{S_1} \sqcup \sigma|_{S_2}\big) \ {}_a\bullet_b = \ {}_{\rho(a)}\bullet_{\sigma(b)} \ \big(\mathcal{O}(\rho) \otimes \mathcal{O}(\sigma)\big) \tag{117b}$$

of maps

$$\mathcal{O}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{O}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{O}\big(T_1 \sqcup T_2 \setminus \{\rho(a), \sigma(b)\}; g_1 + g_2\big).$$

(ii) For an isomorphism $\rho : S \sqcup \{u\} \sqcup \{v\} \to T$ of finite sets and $g \in \mathbb{N}$, one has the equality

$$\mathcal{O}\big(\rho|_S\big) \ \bullet_{uv} = \bullet_{\rho(u)\rho(v)} \mathcal{O}(\rho)$$

of maps $\mathcal{O}\big(S \sqcup \{u\} \sqcup \{v\}; g\big) \to \mathcal{O}\big(T \setminus \{\rho(u), \rho(v)\}; g + 1\big)$.

(iii) For S_1, S_2, a, b and g_1, g_2 as in (117a), one has the equality

$${}_a\bullet_b = \ {}_b\bullet_a \, \tau \tag{117c}$$

of maps $\mathcal{O}(S_1 \sqcup \{a\}; g_1) \otimes \mathcal{O}(S_2 \sqcup \{b\}; g_2) \to \mathcal{O}\big(S_1 \sqcup S_2; g_1 + g_2\big)$.

(iv) For finite sets S_1, S_2, S_3, elements a, b, c, d and $g_1, g_2, g_3 \in \mathbb{N}$, one has the equality

$${}_a\bullet_b(\mathbb{1} \otimes \ {}_c\bullet_d) = -\ {}_c\bullet_d({}_a\bullet_b \otimes \mathbb{1}) \tag{117d}$$

of maps

$$\mathcal{O}\big(S_1 \sqcup \{a\}; g_1\big) \otimes \mathcal{O}\big(S_2 \sqcup \{b\} \sqcup \{c\}; g_2\big) \otimes \mathcal{O}\big(S_3 \sqcup \{d\}; g_3\big) \longrightarrow \mathcal{O}\big(S_1 \sqcup S_2 \sqcup S_3; g_1 + g_2 + g_3\big).$$

(v) For a finite set S, elements a, b, c, d and $g \in \mathbb{N}$ one has the equality

$$\bullet_{ab} \ \bullet_{cd} = -\ \bullet_{cd} \ \ \bullet_{ab}$$

of maps $\mathcal{O}\big(S \sqcup \{a\} \sqcup \{b\} \sqcup \{c\} \sqcup \{d\}; g\big) \to \mathcal{O}(S; g + 2)$.

(vi) For finite sets S_1, S_2, elements a, b, c, d and $g_1, g_2 \in \mathbb{N}$, one has the equality

$$\bullet_{ab} \ \ {}_c\bullet_d = -\ \bullet_{cd} \ \ {}_a\bullet_b$$

of maps $\mathcal{O}\big(S_1 \sqcup \{a\} \sqcup \{c\}; g_1\big) \otimes \mathcal{O}\big(S_2 \sqcup \{b\} \sqcup \{d\}; g_2\big) \to \mathcal{O}(S_1 \sqcup S_2; g_1 + g_2 + 1)$.

(vii) For finite sets S_1, S_2, elements a, b, u, v, and $g_1, g_2 \in \mathbb{N}$, one has the equality

$$ {}_a\bullet_b\ (\bullet_{uv} \otimes \mathbb{1}) = -\bullet_{uv}\ \ {}_a\bullet_b $$

of maps $\mathcal{O}\big(S_1 \sqcup \{a\} \sqcup \{u\} \sqcup \{v\}; g_1\big) \otimes \mathcal{O}\big(S_2 \sqcup \{b\}; g_2\big) \to \mathcal{O}(S_1 \sqcup S_2; g_1 + g_2 + 1)$.

Remark A.3. Odd modular operads appeared in [9, Section 4] as *modular $\mathfrak{K}$-operads* for the dualizing cocycle $\mathfrak{K}$. The terminology we use was suggested by Ralph Kaufmann. A discussion of odd modular operads and similar structures can be found e.g. in [18].

Definition A.4. A (classical) *Markl operad* is a collection $\mathcal{S} = \{\mathcal{S}(n)\}_{n\geq 0}$ of right $\Bbbk[\Sigma_n]$-modules, together with $\Bbbk$-linear maps ($\circ_i$-compositions)

$$ \circ_i : \mathcal{S}(m) \otimes \mathcal{S}(n) \to \mathcal{S}(m+n-1), \tag{118a} $$

for $1 \leq i \leq m$ and $n \geq 0$. These data fulfill the following axioms.

(i) For each $1 \leq j \leq a$, $b, c \geq 0$, $f \in \mathcal{S}(a)$, $g \in \mathcal{S}(b)$ and $h \in \mathcal{S}(c)$,

$$ (f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g & \text{for } 1 \leq i < j \\ f \circ_j (g \circ_{i-j+1} h) & \text{for } j \leq i\ < b+j \\ (f \circ_{i-b+1} h) \circ_j g & \text{for } j+b \leq i \leq a+b-1. \end{cases} \tag{118b} $$

(ii) For each $1 \leq i \leq m$, $n \geq 0$, $\tau \in \Sigma_m$ and $\sigma \in \Sigma_n$, let $\tau \circ_i \sigma \in \Sigma_{m+n-1}$ be given by inserting the permutation σ at the ith place in τ. Let $f \in \mathcal{S}(m)$ and $g \in \mathcal{S}(n)$. Then

$$ (f\tau) \circ_i (g\sigma) = (f \circ_{\tau(i)} g)(\tau \circ_i \sigma). \tag{118c} $$

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References

[1] D.W. Anderson. Fibrations and geometric realizations. *Bull. Amer. Math. Soc.*, 84:765–786, 1978.

[2] M.A. Batanin and M. Markl. Operadic categories and duoidal Deligne's conjecture. *Adv. Math.*, 285:1630–1687, 2015. https://doi.org/10.1016/j.aim.2015.07.008.

[3] M.A. Batanin and M. Markl. Operadic categories as a natural environment for Koszul duality. *Compositionality*, 5(3), 2023. https://doi.org/10.32408/compositionality-5-3.

[4] M.A. Batanin, M. Markl, and J. Obradović. Minimal models for graph-related (hyper)operads. *J. Pure Appl. Algebra*, 227(7):107329, 2023. https://doi.org/10.1016/j.jpaa.2023.107329.

[5] M. Dehling and B. Vallette. Symmetric homotopy theory of operads. *Algebr. Geom. Topol.*, 21(4):1595–1660, 2021. https://doi.org/10.2140/agt.2021.21.1595.

[6] M. Doubek, B. Jurčo, M. Markl, and I. Sachs. *Algebraic Structure of String Field Theory.* Volume 973 of *Lecture Notes in Physics.* Springer Verlag 2020.

[7] W.L. Gan. Koszul duality for dioperads. *Math. Res. Lett.*, 10(1):109–124, 2003. https://doi.org/10.4310/MRL.2003.v10.n1.a11.

[8] E. Getzler and M.M. Kapranov. Cyclic operads and cyclic homology. In S.-T. Yau, editor, *Geometry, Topology and Physics for Raoul Bott*, volume 4 of *Conf. Proc. Lect. Notes. Geom. Topol.*, pages 167–201. International Press, 1995.

[9] E. Getzler and M.M. Kapranov. Modular operads. *Compos. Math.*, 110(1):65–126, 1998. https://doi.org/10.1023/A:1000245600345.

[10] V. Ginzburg and M.M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994. https://doi.org/10.1215/S0012-7094-94-07608-4.

[11] R.M. Kaufmann, B.C. Ward, and J.J. Zúñiga. The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators and the master equations. *J. Math. Phys.*, 56:103504, 2015. https://doi.org/10.1063/1.4932962.

[12] R.M. Kaufmann and B.C. Ward. Koszul Feynman categories. Preprint, arXiv:2108.09251, 2021.

[13] J.-L. Loday and M.O. Ronco. Permutads. *J. Combin. Theory Ser. A*, 120(2):340–365, 2013. https://doi.org/10.1016/j.jcta.2012.08.005.

[14] J.-L. Loday and B. Vallette. *Algebraic operads.* Volume 346 of *Grundlehren der Mathematischen Wissenschaften.* Springer, Heidelberg, 2012.

[15] M. Markl. Distributive laws and Koszulness. *Ann. Inst. Fourier (Grenoble)*, 46(4):307–323, 1996. http://www.numdam.org/item?id=AIF_1996__46_2_307_0

[16] M. Markl. Intrinsic brackets and the L_∞-deformation theory of bialgebras. *J. Homotopy Relat. Struct.*, 5(1):177–212, 2010. https://tcms.org.ge/Journals/JHRS/xvolumes/2010/n1a11/v5n1a11.pdf.

[17] M. Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996. https://doi.org/10.1080/00927879608825647.

[18] M. Markl. Odd structures are odd. *Adv. Appl. Clifford Algebr.*, 27(2):567–1580, 2017. https://doi.org/10.1007/s00006-016-0720-8.

[19] M. Markl. Permutads via operadic categories, and the hidden associahedron. *J. Combin. Theory Ser. A*, 175:105277, 2020. https://doi.org/10.1016/j.jcta.2020.105277.

[20] M. Markl, S.A. Merkulov, and S. Shadrin. Wheeled PROPs, graph complexes and the master equation. *J. Pure Appl. Algebra*, 213(4):496–535, 2009. https://doi.org/10.1016/j.jpaa.2008.08.007.

[21] M. Markl, S. Shnider, and J.D. Stasheff. *Operads in algebra, topology and physics.* Volume 96 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002.

[22] M. Markl and A.A. Voronov. PROPped-up graph cohomology. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of Progress in Mathematics, pages 249–281. Birkhäuser, Boston, MA, 2009. https://doi.org/10.1007/978-0-8176-4747-6_8.

[23] S.B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152(1):39–60, 1970. https://doi.org/10.2307/1995637.

[24] R. Rosebrugh and R.J. Wood. Distributive laws and factorization. *J. Pure Appl. Algebra*, 175(1-3):327–353, 2002. https://core.ac.uk/download/pdf/82048781.pdf.

[25] B.C. Ward. Massey products for graph homology. *Int. Math. Res. Not.*, 11:8086–8161, 2022. https://doi.org/10.1093/imrn/rnaa346.

[26] B. Vallette. A Koszul duality for props. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007. https://doi.org/10.1090/S0002-9947-07-04182-7.

[27] P. Van der Laan. Operads up to homotopy and deformations of operad maps. Preprint, arXiv:math.QA/0208041, 2002.