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Homotopy-theoretic and categorical models of neural information networks

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In this paper we develop a novel mathematical formalism for the modeling of neural information networks endowed with additional structure in the form of assignments of resources, either computational or metabolic or informational. The starting point for this construction is the notion of summing functors and of Segal's Gamma-spaces in homotopy theory. This paper analyzes functorial assignments of different levels of structure (resources) to networks and their subsystems. Resources are described by categories, involving concurrent/distributed computing architectures, binary codes, and associated information structures and information cohomologies, including a cohomological version of integrated information. A categorical form of the Hopfield network dynamics is introduced, which recovers the usual Hopfield equations when applied to a suitable category of weighted codes, where the variables of the dynamics are these functorial assignments of resources to a network (summing functors).

Contents

1 Introduction and motivation . . . **3**
1.1 Background motivation . . . 4
1.1.1 Cognition and computation . . . 4
1.1.2 Homotopical representations of stimulus spaces . . . 5
1.1.3 Homology and stimulus processing . . . 6
1.1.4 Informational complexity and integrated information . . . 7
1.1.5 Perception, representation, computation . . . 7
1.2 Structure of the paper and main results . . . 8
1.2.1 Comparison with other approaches . . . 11

2 Summing functors on networks . . . **12**
2.1 The category of summing functors . . . 12
2.2 Networks and summing functors . . . 16
2.2.1 Conservation laws at vertices . . . 18
2.2.2 Vertex constraints by coequalizer . . . 18
2.3 Categories of summing functors on networks . . . 20
2.3.1 Graphs in terms of vertices and flags . . . 21
2.3.2 Constraints through grafting operations . . . 22
2.3.3 Inclusion-exclusion properties . . . 23

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3 Neural information networks and resources **24**
3.1 Networks with informational and metabolic constraints . . . 24
3.1.1 Types of neural codes . . . 25
3.1.2 Spikes, coding capacity, and firing rate . . . 25
3.1.3 Metabolic efficiency and information rate . . . 25
3.1.4 Connection weights and mutual information . . . 25
3.1.5 Resources and constraints . . . 26
3.2 The mathematical theory of resources . . . 26
3.2.1 Examples of resources . . . 26
3.2.2 Convertibility of resources . . . 27
3.2.3 Information loss . . . 27
3.3 Adjunction and optimality of resources . . . 27

4 Networks with computational structures **29**
4.1 Transition systems: a category of computational resources . . . 29
4.2 Computational architectures in neuronal networks . . . 30
4.3 Computational architectures and network summing functors . . . 31
4.3.1 Transition systems and network summing functor . . . 31
4.4 Larger-scale structures and distributed computing . . . 33
4.4.1 Distributed computing model of neuromodulation . . . 34
4.4.2 Network summing functors and automata with time delays . . . 35
4.4.3 Topological questions . . . 36

5 Codes, probabilities, and information **37**
5.1 Introducing neural codes . . . 37
5.1.1 Firing rates of codes . . . 37
5.1.2 Codes and finite probabilities . . . 38
5.1.3 Categories of codes . . . 39
5.1.4 Codes and associated probabilities . . . 41
5.2 Weighted codes and linear relations . . . 42
5.3 Information measures . . . 43
5.3.1 Category of simplices . . . 45
5.4 Summing functors and information measures . . . 45
5.4.1 Cohomological information theory . . . 45
5.4.2 Network summing functors and information . . . 47
5.4.3 Other functorial maps to information structures . . . 49
5.5 Codes and simplicial sets . . . 49
5.5.1 Nerves of coverings and functoriality . . . 50
5.6 Transition systems, codes, and information structures . . . 51
5.7 Clique complexes and information structures . . . 52

6 Categorical Hopfield dynamics **53**
6.1 Continuous and discrete Hopfield dynamics . . . 53
6.2 Categorical threshold non-linearity . . . 54
6.3 Discrete Hopfield dynamics in categories of summing functors . . . 55
6.4 Category of weighted codes and ordinary Hopfield dynamics . . . 57

7 Gamma-spaces and Gamma networks **59**
7.1 Gamma-spaces . . . 59
7.2 Gamma-spaces as endofunctors of simplicial sets . . . 60
7.3 Gamma-spaces and homotopy types . . . 61
7.4 Spectra and homotopy types . . . 62
7.4.1 Gamma-space representation of clique complexes . . . 62
7.4.2 The case of random graphs . . . 63
7.4.3 Feedforward networks . . . 64
7.5 Gamma networks . . . 64
7.5.1 Gamma networks, codes, and nerves of coverings . . . 66

8 Gamma networks and integrated information 67
8.1 Information geometry and integrated information . . . 67
8.2 Feedforward networks and integrated information . . . 69
8.3 Kullback–Leibler divergence and information cohomology . . . 69
8.4 Cohomological integrated information . . . 70
8.5 Categorical Hopfield dynamics and integrated information . . . 71
8.6 Integrated information and Gamma networks . . . 71
8.7 Homotopy types, spectra, information and cohomology . . . 74

A Probabilistic and persistent Gamma-spaces 75
A.1 Simplicial topology enriched with probabilities . . . 75
A.1.1 Probability distributions on finite sets . . . 76
A.2 Probabilistic Gamma-spaces . . . 76
A.3 Probabilistic transition systems . . . 76
A.4 Persistent Gamma-spaces and persistent spectra . . . 77
A.4.1 Thin categories and persistence diagrams . . . 77
A.4.2 Persistent Gamma-spaces . . . 77
A.5 Probabilistic persistent Gamma-spaces . . . 79
A.6 Modeling constraints and the role of persistence . . . 79
A.7 Persistence to model time and scale dependence . . . 80
A.8 Variants of the nerve construction . . . 81

1 Introduction and motivation

The main goal of this paper is the development of a new mathematical formalism for the modeling of networks endowed with several different levels of structure. The types of structures considered are formalized in terms of categories, whose objects represent various kinds of resources, such as computational architectures for concurrent/distributed computing, codes generated by spiking activity of neurons, probabilities and information structures, and possible other categories describing physical resources with metabolic and thermodynamical constraints. Morphisms in these categories represent ways in which resources can be converted and computational systems can be transformed into one another. All these different levels of structure are in turn related via several functorial mappings. We model a configuration space of consistent ways of assigning such resources to a network and all its subsystems, in the form of categories of "network summing functors" with invertible natural transformations as morphisms. These provide a categorical model of a moduli space of all possible assignments of resources to subnetworks (subject to various types of constraints), considered up to equivalence.

It is useful to consider an analogy with the usual description of physical systems, where one first introduces a suitable *configuration space*. This is the *kinematic* part of the model, which describes the underlying geometry (variables and constraints), in which the dynamics takes place. One then introduces the *dynamics* in the form of an equation of motion given by a dynamical system on the assigned configuration space. We are going to proceed along the same lines here. The categories of summing functors play the role of the physical configuration space, which determines the geometry and kinematics of the model. Namely, the basic variables of our model are the summing functors. Our physical configuration space is then given by a category of summing functors with invertible natural transformations, which describes these functorial assignments of resources up to equivalence. We then introduce dynamical systems on these categories of summing functors, describing the time evolution of the assignments of resources to the network with given constraints. The main advantage of adopting this categorical viewpoint lies in the fact that the entire system, with all its levels of structure, transforms simultaneously and consistently (for example, consistently over all possible subsystems as well as over all functorial relations between different layers of structure), under dynamical evolution, and in the course of interacting with and processing external stimuli. More precisely, we show that a discretized form of the Hopfield network dynamics can be formulated in this categorical setting, thus providing an evolution equation for the entire system of the network with all its resources and constraints, and we show that one recovers the usual Hopfield network dynamics when specializing this to a category of weighted codes.

The way we incorporate these different levels of structure is based on a notion from homotopy theory, the concept of Gamma-space introduced by Graeme Segal in the 1970s to realize homotopy-theoretic spectra in terms of symmetric monoidal categories. A Gamma-space functorially maps finite sets to simplicial sets, by assigning to a set the nerve of its category of summing functors with target a fixed category of resources. We extend this notion of Gamma-spaces to a similar notion of Gamma networks, which assign to a network a topological model (the nerve) of our configuration space given by the category of network summing functors. We view this as a functorial construction of a topological "configuration space" of all possible mappings of subsystems of a finite system to resources, in a way that is additive on independent subsystems. The categorical dynamics we introduce at the level of the category of summing functors induces a topological dynamical system on the nerves obtained via the associated Gamma network. Segal's Gamma-spaces extend to endofunctors of the category of simplicial sets. As such they can be used to construct Gamma networks that have as input certain simplicial sets that naturally arise in an activated network responding to a stimulus, such as clique complexes or nerves of coverings associated to receptor fields of neural codes, as well as simplicial sets associated to various forms of categorical information structures. The output is a new simplicial set that combines the topology of the input with additional topological structures coming from the category of resources, through the associated Gamma-space and spectrum. Thus, our configuration space also acts as an encoder that takes as input homotopy types coming from the activity of the network and produces as output a new collection of homotopy types, that also incorporate the topology of the configuration space itself. We show that these homotopy types have associated measurements of integrated information and that this encoding of homotopy types increases the integrated information by an amount described in terms of Shannon entropy of the Gamma-space. In an appendix we enrich the formalism of summing functors and Gamma-spaces with both probabilistic and persistent structures.

1.1 Background motivation

In the rest of this introductory section we review some general background motivations behind the approach developed in this paper. The content of the paper and the main results are then summarized in §1.2.

1.1.1 Cognition and computation

A main motivation of this paper, as well as of many others, was briefly summarized in [71]: it is the heuristic value of comparative study of *"cognitive activity"* of human beings (and more generally, other biological systems) with *"computational processing"* by engineered objects, such as computing devices.

In [71] it was stressed, in particular, that such a comparison should be *not restricted,* but rather *widened*, by the existence of wide spectra of space and time scales relevant for understanding both of "cognition" and "computation".

In particular, in [71] it was argued that we must not *a priori* decide that brain should be compared to a computer, or neuron to a chip. We suggested, that there exist fruitful similarities between spatio-temporal activity patterns of a *one brain* and *the whole Web*; or between similar patterns on the levels of *history of civilizations*, and several functional neuronal circuits developing *in a brain of a single human being from birth to ageing.*

It was noticed long ago that various mathematical models of such processes have skeletons of common type: *oriented graphs* describing paths of transmission and/or transformation of information. Mathematical machinery of topological nature (geometric realization of graphs by simplicial complexes, and their topological invariants) must be connected, in such studies, with mathematical machinery of information theory (probability distributions, entropy, complexity ...): cf. [76] and [91].

The primary goal of this paper consists in the enrichment of the domain of useful tools: paths in oriented graphs can be considered as compositions of morphisms between objects of categories, and assignments of resources of different types (computational, metabolic, informational) to networks can be regarded as functors between suitable categories. Topological invariants of their geometric realizations might include *homotopical* rather than only (co)homological invariants. Respectively, we continue studying their possible interaction with information-theoretic properties started e.g. in [76] and [73].

As in classical theoretical mechanics, such invariants embody *configuration and phase spaces* of systems that we are studying, *equations of motion, conservation laws*, etc. In the setting we develop here, the main *configuration space* is the space of all consistent functorial mappings of a network and its subsystems to a monoidal category of resources (computational systems, codes, information structures). As in the case of classical mechanics, this kinematic setup describing the configuration space is then enriched with dynamics, in the form of categorical Hopfield networks.

One can view classical mechanics in categorical terms as well, by considering the assignment of configuration spaces to classical physical systems and their subsystems. This is a useful viewpoint, for instance, when considering the physics of open systems, and was developed in [8]. The assignments of configuration spaces to systems and subsystems form a category (of spans/cospans) of Riemannian manifolds and surjective Riemannian submersions in the Lagrangian formulation of classical mechanics, and of symplectic manifolds with surjective Poisson maps in the Hamiltonian formulation, with the Legendre transform relating the Lagrangian and Hamiltonian formalism realized functorially. The categorical setting we consider here is different, but it has some aspects in common with this categorical formulation of classical mechanics, in the sense of focusing on configuration spaces for systems and subsystems, realized by our categories of summing functors.

1.1.2 Homotopical representations of stimulus spaces

One of the main motivations behind the viewpoint developed in this paper comes from the idea that the neural code generates a representation of the stimulus space in the form of a homotopy type.

Indeed, it is known from [29], [32], [33], [73], [111] that the geometry of the stimulus space can be reconstructed *up to homotopy* from the binary structure of the neural code. The key observation behind this reconstruction result is a simple topological property: the binary code words in the neural code represent the overlaps between the place fields of the neurons, where the place field is the preferred region of the stimulus space that cause the neuron to respond with a high firing rate. The neural code determines in this way a simplicial complex, given by the simplicial nerve of the open covering of the stimulus space. Under the reasonable assumption that the place fields are convex open sets, the homotopy type of this simplicial complex is the same as the homotopy type of the stimulus space. Thus, the fact that the binary neural code captures the complete information on the intersections between the place fields of the individual neurons is sufficient to reconstruct the stimulus space, but only up to homotopy.

The homotopy equivalence relation in topology is weaker but also more flexible than the notion of homeomorphism. The most significant topological invariants, such as homotopy and homology groups, are homotopy invariants. Heuristically, homotopy describes the possibility of deforming a topological space in a one-parameter family. In particular, a *homotopy type* is an equivalence class of topological spaces up to (weak) homotopy equivalence, which roughly means that only the information about the space that is captured by its homotopy groups is retained. There is a direct connection between the formulation of topology at the level of homotopy types and "higher categorical structures". Homotopy theory and higher categorical structures have come to play an increasingly important role in contemporary mathematics, including important applications to theoretical physics and to computer science. We will argue here that it is reasonable to expect that they will also play a role in the mathematical modeling of neuroscience. This was in fact already suggested by Mikhail Gromov in [54].

This suggests that a good mathematical modeling of network architectures in the brain should also include a mechanism that generates homotopy types, through the information carried by the network via neural codes. One of the main goals in this paper is to show that, indeed, a mathematical framework that models networks with additional computational and information structure will also give rise to a mechanism that acts on homotopy types. The Gamma-spaces associated to our configuration spaces of assignments of resources to networks are functors that take as inputs homotopy types generated by the network activities (such as clique complexes of activated subnetworks, nerve complexes of response fields and neural codes) and encode these inputs into another class of homotopy type (which we call a representation). The new homotopy types obtained in this way combine the nontrivial input homotopy types that encode information about the stimulus space with topological information about the categories of resources, in a way that increases informational complexity (see §8.6 and especially Proposition 8.9).

1.1.3 Homology and stimulus processing

Another main motivation for the formalism developed in this paper is the detection, in neuroscience experiments and simulations, of a peak of non-trivial persistent homology. This arises in the clique complex of the network of neurons activated during the processing of external stimuli. A related motivation is given by increasing evidence of a functional role of these nontrivial topological structures.

The analysis of the simulations of neocortical microcircuitry in [91], as well as experiments on visual attention in rhesus monkeys [93], have shown the rapid formation of a peak of non-trivial homology generators in response to stimulus processing. These findings are very intriguing for two reasons: they link topological structures in the activated neural circuitry to phenomena like attention, and they suggest that a sufficient amount of topological complexity serves a functional computational purpose.

This suggests a possible mathematical setting for modeling neural information networks architectures in the brain. The work of [91] proposes the interpretation that these topological structures are necessary for the processing of stimuli in the brain cortex, but does not offer a theoretical explanation of why topology is needed for stimulus processing. However, there is a well-known context in the theory of computation where a similar situation occurs, which may provide the key for the correct interpretation, namely the theory of concurrent and distributed computing [17], [36], [57].

In the mathematical theory of distributed computing, one considers a collection of sequential computing entities (processes) that cooperate to solve a problem (task). The processes communicate by applying operations to objects in a shared memory, and they are asynchronous, in the sense that they run at arbitrary varying speeds. Distributed algorithms and protocols decide how and when each process communicates and shares with others. The main questions are how to design distributed algorithms that are efficient in the presence of noise, failures of communication, and delays, and how to understand when a distributed algorithm exists to solve a particular task.

Protocols for distributed computing can be modeled using simplicial sets. An initial or final state of a process is a vertex, any $d+1$ mutually compatible initial or final states are a d-dimensional simplex, and each vertex is labeled by a different process. The complete set of all possible initial and final states is then a simplicial set. A decision task consists of two simplicial sets of initial and final states and a simplicial map (or more generally correspondence) between them. The typical structure describing a distributed algorithm consists of an input complex, a protocol complex, and an output complex, with a certain number of topology changes along the execution of the protocol, [57].

There are very interesting topological obstruction results in the theory of distributed computing, [57], [58], which show that a sufficient amount of non-trivial homology in the protocol complex is necessary for a decision task problem to be solvable. Thus, the theory of distributed computing shows explicitly a setting where a sufficient amount of topological complexity (measured by non-trivial homology) is necessary for computation.

This suggests that the mathematical modeling of network architectures in the brain should be formulated in such a way as to incorporate additional structure keeping track of associated concurrent/distributed computational systems. This is indeed one of the main aspects of the formalism described in this paper: we will show how to associate functorially to a network and its subsystems a computational architecture in a category of transition systems, which is suitable for the modeling of concurrent and distributed computing. For additional discussion of topological and categorical models of concurrent and distributed computing see for instance [17], [19], [47], [48], [49], [53] [59].

1.1.4 Informational complexity and integrated information

In recent years there has been some serious discussion in the neuroscience community around the idea of possible computational models of consciousness based on some measure of informational complexity, in particular in the form of the proposal of Tononi's *integrated information* theory (also known as the Φ function) [104], see also [67], [81] for a general overview. This proposal for a quantitative correlate of consciousness roughly measures the least amount of effective information in a whole system that is not accounted for by the effective information of its separate parts. The main idea is therefore that integrated information is a measure of informational complexity and causal interconnectedness of a system.

This approach to a mathematical modeling of consciousness has been criticized on the ground that it is easy to construct simple mathematical models exhibiting high values of the Φ function. Generally, one can resort to the setting of coding theory to generate many examples of sufficiently good codes (for example the algebro-geometric Reed–Solomon error-correcting codes) that indeed exhibit precisely the typical form of high causal interconnectedness that leads to large values of integrated information. This indicates that integrated information alone does not suffice to imply consciousness. Thus, it seems that it would be preferable to interpret integrated information as a consequence of a more fundamental model of how networks in the brain process and represent stimuli, leading to high informational complexity and causal interdependence as a necessary but not in itself sufficient condition.

One of the goals of this paper is to show that integrated information can be incorporated as an aspect of the model of neural information network that we develop, and that many of its properties, such as the low values on feedforward architectures, are already built into the topological structures that we consider. One can then interpret the homotopy types generated by the topological model we consider as the "representations" of stimuli produced by the network through the neural codes, and the space of these homotopy types as a kind of "qualia space", [80]. While we will not pursue in the present paper the development of such a model of qualia, this motivation lies in the background of some of the results on integrated information that we obtain in this paper, in particular our result on the gain in integrated information caused by the encoding of homotopy types through Gamma-spaces.

1.1.5 Perception, representation, computation

We conclude this overview of motivational background by some broader and more general considerations. At these levels of generalization, additional challenges arise, both for researchers and students. Namely, even when we focus on some restricted set of observables, passage from one space/time scale to a larger or smaller one might require a drastic change of languages we use for description of these levels. The typical example is passage from classical to quantum physics. In fact, it is only one floor of the Babel Tower of imagery that humanity uses in order to keep, extend and transmit the vast body of knowledge, that makes us human beings: cf. a remarkable description of this in [60].

Studying neural information, we meet this challenge, for example, when we try to pass from one subgraph of the respective oriented graph to the next one by adding just one oriented arrow to each vertex. It might happen that each such step *implies a change of language*, but in fact such languages themselves cannot be reconstructed before the whole process is relatively well studied.

Actually, the drastic change of languages arises already in the passage between two wide communities of readers to which this paper is addressed: that of mathematicians and that of neuroscientists. Therefore, before moving to the main part of this paper, we wanted to make the mathematicians among our readers aware of this necessity of permanent change of languages.

A very useful example of successful approach to this problem is the book [99], in particular its Chapter 5, "Encoding Colour". Basically, this Chapter explains *mathematics* of color perception, by the retina in the human eye. But for understanding its *neural machinery*, the reader will have to return temporarily back each time, when it is necessary. Combination of both is a good lesson in neural information theory. Below we will give a brief sketch of that chapter.

Physics describes light on the macroscopic level as a superposition of electromagnetic waves of various lengths, with varying intensity. Light perception establishes bounds for these wavelengths, outside of which they stop to be perceived as light. Inside these bounds, certain bands may be perceived as light of various "pure" colors: long wavelengths (*red*), medium wavelengths (*green*), and short wavelengths (*blue*).

The description above refers to the "point" source of light. The picture perceived by photoreceptors in the eye and transmitted to neurons in the brain, is a space superposition of many such "point source" pictures, which then is decoded by the brain as a "landscape", or a "human face", or "several barely distinguishable objects in darkness", etc.

We will focus here upon the first stages of this encoding/decoding of an image in the human eye made by the retina. There are two types of photoreceptors in the retina: cones (responsible for color perception in daylight conditions) and rods (providing images under night-time conditions).

Each photoreceptor (as other types of neurons) receives information in the form of action potentials in its cell body, and then transmits it via its axon (a kind of "cable") to the next neuron in the respective neuronal network. Action potentials are physically represented by a flow of ions. Communication between two neurons is mediated by synapses (small gaps, collecting ions from several presynaptic neurons and transferring the resulting action potential into the cell body of the postsynaptic neuron).

Perception of visual information by the human eye starts with light absorption by (a part of) the retinal photoreceptors and subsequent exchange of arising action potentials in the respective part of the neural network. Then retinal ganglion cells, forming the optic nerves, transmit the information from the retina to the brain.

Encoding color bands into action potentials, and subsequently encoding relative amplitudes of respective potentials into their superpositions, furnish the first stage of "color vision". Mathematical modeling of this stage in [99] requires a full machinery of information theory and of chapters of statistical physics involving entropy and its role in efficient modeling of complex processes.

Our focus here is more abstract and general, as we deal with a formalism for describing networks endowed with different types of resources related by certain mutual constraints. The steps of encoding information coming from external stimuli can be regarded as a way of assigning codes to a network and probabilities and information measures to these resulting codes. Enrichment of all these models by topology, or vice versa, enrichment of topology by information formalisms plays an important role in our approach, as we will be discussing in the rest of the paper.

1.2 Structure of the paper and main results

In §2 we introduce the general problem of modeling networks with associated resources. We present our main configuration spaces, parameterizing assignments of resources to networks, given by categories of summing functors. In §2.1 we first present the case of categories of summing functors from the category of subsets of a finite set with inclusions to a category of resources, which is a category with sums and zero object, or more generally a symmetric monoidal category. We think here of the finite set as representing either the set of vertices (nodes) or of edges of a network. We give a simple characterization of these summing functors. In §2.2 we extend this notion by incorporating the network structure. Instead of considering finite sets, in §2.3.1 we consider directed finite graphs, seen as functors from a category with two objects V, E and two non-identity morphisms, source and target, $s, t : E \to V$. We introduce two preliminary examples of network summing functors, where the compatibility between vertices and edges of the directed graph is described via either an equalizer or a coequalizer construction. In §2.3 we introduce our more general definition of "network summing functors" and we show in §2.2.1 and §2.2.2 that the equalizer and coequalizer examples determine subcategories of the category of network summing functors. We then show that other subcategories of interest can be identified by specifying other forms of additional constraints at vertices and edges that the network summing functors should satisfy. In particular, in §2.3.2 we describe the case of network summing functors that are obtained through grafting operations, in cases where the category of resources has an additional compositional structure described by a properad. In §2.3.3 we describe another class of network summing functors, which satisfy inclusion-exclusion relations, in cases where the category of resources is either abelian or triangulated. These cases are presented to illustrate the fact that specific subcategories of our category of summing functors may be suitable for different types of models, depending on what kinds of resources on networks one is describing.

In §3 we analyze more closely the notion of category of resources. We recall in §3.1 various forms of resources that are associated to neuronal networks, in particular informational and metabolic constraints and computational resources. We then review in §3.2 the mathematical theory of resources and convertibility of resources developed in [27] and [40] using symmetric monoidal categories. We recall in §3.2.1 some simple examples of categories of resources, from [27] and [40]. We discuss briefly in §3.2.2 the notion of measuring semigroups associated to categories of resources, which was also introduced in [27] and [40] to keep track of resource convertibility. We will be using this notion of measuring semigroup to define the threshold-dynamics of Hopfield networks in our categorical setting in §6. In §3.2.3 we also recall the categorical characterization of information loss of [5]. In §3.3 we describe how adjunction of functors can be viewed in this setting as optimization of resources. This particular observation is not directly needed for our applications, but we have included it because it provides some further insight and intuition about the categorical formalism in discussing resources.

In §4 we look more specifically into how to model assignments of computational structures as resources attached to networks. We focus in §4.1 on one particular categorical model of computational resources for concurrent and distributed computing architectures, given by the category of transition systems of [110]. While there are many categorical models of concurrent and distributed computing, we have chosen this one as it is sufficiently flexible to accommodate various existing computational models of individual neurons, and at the same time it has a simple structure that makes it clear the category has the required properties of a category of resources in the sense recalled in §3.2. In §4.2 we mention briefly some of the existing approaches to computational models of individual neurons and how they can be made to fit in the category of transition systems, though a more detailed account for specific neuron models will be given elsewhere, [78]. In §4.3.1 we introduce a class of summing functors obtained via grafting operations in the category of transition systems, which provides a good configuration space in this setting. We finish this section with some subsections aimed at illustrating interesting possible directions of investigation related to this type of resources and summing functors: in §4.4 we outline the problem of including in this setting a good computational model of neuromodulation. In particular for this specific problem, we discuss in §4.4.2 how one can use a class of automata with time delays as transition systems. We finish in §4.4.3 with some questions on the possible role of the 3-dimensional topology of the network and of topological invariants that depend on the 3-dimensional embedding of graphs.

In §5 we consider neural codes generated by networks of neurons and associated probabilities and information measures. We introduce neural codes in §5.1.2 and we recall their main structure and properties. In §5.1.3 we construct a simple category of codes and we show that one can think of the neural codes as determining summing functors to this category of codes. In §5.1.4 we then show that the probabilities associated to neural codes by the firing frequencies of the neurons fit into a functor from this category of codes to a category of probability measures. However, we show that this construction is not fully satisfactory because it does not in general translate to a functorial assignment of information measures (see §5.3). In §5.2 we show that our setting, with a category of weighted codes, recovers a simple model of the linear neurons. (We discuss threshold-nonlinearities in §6.) The problem with functorial assignments of information measures is solved in §5.4, using the more sophisticated formalism of cohomological information theory introduced by Baudot and Bennequin [10] and developed by Vigneaux [106]. In §5.4.1 we give a very quick review of the cohomological information setting of [106], with finite information structures, probability functors, and the Hochschild cohomology interpretation of information functionals like Shannon and Tsallis entropy. We start in §5.4.2, by considering the subcategory of network summing functors given by the equalizer condition, discussed in §2.2.1. We construct a functor from the category of codes to the category of finite information structures, and from there to an abelian category of modules as in [106]. We obtain an associated category of summing functors by composition. These describe assignments of informational resources to the network. We show that these satisfy inclusion-exclusion properties as discussed in §2.3.3. A variant of this construction is described in §5.4.3, with a functor from codes to information structures and then to the category of chain complexes, and a resulting category of network summing functors. In §5.5 we show that the formalism of finite information structures and probability functors of [106] incorporates as a particular case the assignment to a neural code of the simplicial set given by the nerve of the covering associated to the receptor fields of the neurons. In §5.6 we further refine these functorial relations between the different categories of resources introduced in the previous sections by constructing a functor from the category of transition systems to the category of codes, describing the codes generated by the automata. We also construct a functor from transition systems to information structures, and we show that it agrees with the composition of the functor to codes with the functors from codes to information structures described in §5.4.2. We show in §5.7 that we can also fit into the formalism of finite information structures and probability functors the clique complexes of networks, by exhibiting a specific choice of finite information structures and probability functor for which the output simplicial set is the clique complex. These various cases are meant to show the functorial consistency between the various categories of resources of interest to us (neural codes, computational systems, information structures) and how significant examples of topological structures associated to neuronal networks, such as nerves of coverings of receptor fields and clique complexes of activated subnetworks, fit inside the same broader formalism.

In all the sections of the paper up to this point we have only dealt with a static model, in the sense that we have focused on constructing the configuration space parameterizing the assignment of resources to a network and the relations between these configuration spaces determined by the relations between different types of resources. In §6 we make the setting dynamical, in the sense that we introduce equations of motion on our kinematic space. This is done by introducing a suitable form of the Hopfield equations of networks which is categorical in the sense that the variables of the equation are now summing functors. We start by recalling in §6.1 the classical Hopfield equations of networks, in both the continuous and the discretized form. The equations are non-linear due to the presence of a threshold non-linearity that accounts for the non-linear properties of neurons. In §6.2 we discuss how one can formulate threshold non-linearity in a categorical setting using the measuring semigroups on categories of resources, that we recalled in §3.2.2. We then formulate in §6.3 the categorical form of the Hopfield equations with variables that are summing functors and the dynamics determined by an endofunctor and by the threshold non-linearity. We show that the resulting dynamics in the category of endofunctors induces a topological dynamical system on the associated nerve, which can be used to study the dynamics through traditional topological dynamical systems methods. While in the present paper we do not present a detailed study of the properties of these equations, which is left to future work, we do discuss in §6.4 a basic consistency check, by showing that, in a very special case with the category of resources given by our category of weighted codes of §5.2, one recovers the classical Hopfield equations of networks. This in particular shows how to extend the result of §5.2 from the over-simplistic linear neuron to a more realistic non-linear case.

In §7 we introduce another level of structure, focused more on simplicial sets and homotopy types. We have already seen the role of the nerve of the category of summing functors in discussing the Hopfield equations in §6.3, as an associated topological dynamical system. We focus here more generally on functorial assignments of simplicial sets to networks. We present these through the classical Segal construction of Gamma-spaces, which are functorial assignments of simplicial sets to finite sets, through the construction of the nerve of a category of summing functors. We think of these nerves as the geometric realizations of our categorical configuration spaces. In §7.1 we review Segal's notion of Gamma-spaces and the construction of Gamma-spaces associated to categories of resources. We then recall in §7.2 how Gamma-spaces extend to endofunctors of the category of simplicial sets through a coend construction. In §7.3 we observe how, correspondingly, a Gamma-space generates a collection of homotopy types from input simplicial sets. In §7.4 we recall the relation between Gamma-spaces and homotopy-theoretic spectra. In §7.4.1, §7.4.2, and §7.4.3 we discuss certain special cases that are useful in preparation for the more general discussion in §7.5. In particular, in §7.4.1 we analyze the topological properties of the output simplicial sets when the input of the Gamma-space endofunctor is a clique complex of a network; in §7.4.2 we present a similar discussion for the case where the input is the (un-oriented) clique complex of an Erdős–Rényi random graph; while in §7.4.3 we discuss briefly the case of feedforward networks. In §7.5 we then introduce our notion of Gamma networks, which generalizes Gamma-spaces, as functors from directed graphs to simplicial sets, and we focus on two main classes of Gamma networks: those obtained by composing a functorial assignments of simplicial sets to graphs (such as clique complexes or assignments coming from probability functors) with a classical Gamma space, and those obtained by taking the nerve of a category of network summing functors. Combinations (via smash product of Gamma-spaces) of these two types cover most of the needs for our model. The special cases discussed in §7.4.1, §7.4.2, and §7.4.3 are all examples of the first kind.

In §8 we enrich our setting with a notion of integrated information. This is a notion of informational complexity of a system, such as a network with resources in our setting, which is designed to capture the amount of information carried by the system that cannot be accounted for in terms of any partition into independent subsystems. In that sense it is a measure of both information and causal interrelatedness between subsystems. Integrated information has been introduced in neuroscience (see [9], [67], [81], [104]) as a possible quantitative correlate of consciousness. We are interested here in how two aspects of our model affect integrated information: our categorical Hopfield dynamics, and the mapping of simplicial sets via Gamma networks. In §8.1 we recall the mathematical formulation of integrated information, using the construction of [88], based on information geometry. As an example of the type of structure that integrated information detects, we recall in §8.2 the reason why it is trivial on feedforward network architectures. In §8.3 and §8.4 we present a way of formulating integrated information in the setting of cohomological information theory of [106], by first recalling in §8.3 how the Kullback–Leibler divergence is formulated in that formalism, and then presenting in §8.4 our cohomological construction of integrated information. In §8.5 we show that we can assign a measurement of integrated information to the summing functors that are solutions of our categorical Hopfield equation, in such a way as to keep track of the changes in integrated information along the dynamics. In §8.6 we consider Gamma networks that are obtained as composition of a probability functor from a category of random graphs with a classical Gamma-space, where the Gamma-space accounts for the type of resources associated to the network. We show that there is an associated cohomological integrated information and that this form of integrated information increases under composition with the Gamma-space, by an amount described in terms of Shannon entropy associated to the Gamma-space. This shows that the encoding of homotopy types affected by a Gamma-space increases the amount of integrated information they carry. We conclude this section by formulating in §8.7 some questions about the possible role of generalized cohomologies associated to the spectra defined by Gamma-spaces in combination with the cohomological formulation of information functionals.

The Appendix discusses two generalizations of the summing functors and Gamma-spaces formalism, one that incorporates probabilities and one that incorporates persistent structures. In §A.1 we present a general setting for the categorical formulation of probabilities and its specialization to the simpler case of probabilities over finite sets. In §A.2 we show how to use this category of probabilities to construct a probabilistic version of Gamma-spaces, following the setting of [76]. As an example we describe the case of probabilistic transition systems in §A.3. In §A.4 we describe how to include a notion of persistence for Gamma-spaces and corresponding persistent spectra. In §A.5 we show that these two generalizations can be combined to obtain Gamma-spaces that are both probabilistic and persistent. We discuss in §A.6 and §A.7 how these generalizations can be useful to incorporate descriptions of constraints and of time and scale dependence. Finally, in §A.8 we also discuss briefly the possible role of generalizations of the nerve construction.

1.2.1 Comparison with other approaches

The idea of considering assignments of various types of data to networks, as well as the use of topological methods, have also been considered in other forms, for example along the lines of constructions involving bundle/sheaf-theoretic methods. These include, for instance, the approach of [94], [95], based on vector bundles, with a notion of approximate and discrete Euclidean vector bundle and a dimensionality-reduction method for large data sets based on embeddability of such bundles. Such a construction can be organized in a categorical form, and it encodes topological information about the data sets. Another viewpoint that pursues similar ideas is the cellular sheaves method of [55], that extends spectral graph theory to a spectral theory (with a Hodge Laplacian) on cellular sheaves of vector spaces over cell complexes. When considered over graphs, this allows for assignments of data to networks, encoded by vectors, with applications to distributed algorithms, such as consensus problems, or distributed optimization. This sheaf-theoretic context also has a natural categorical formulation.

Some of the motivations for adopting the type of construction described in this present paper, with summing functors and Gamma-spaces, rather than a simple elaboration on one of the pre-existing approaches mentioned above, are summarized by the following observations.

1. Not all optimization problems are reducible to real-valued (or vector-valued) functions: there

are more general settings where one deals with objects in more abstract categories. A general discussion of such categorical notions of optimization is given in [79].

2. A discussed briefly in [77], our process of building homotopy types from network Gamma spaces provides a unifying context where several different occurrences of simplicial sets and homotopy arising in a neuroscience-related setting are simultaneously accounted for. For example, three different roles of topology in neuroscience are clique complexes of subnetworks that activate in response to stimuli, nerve complexes of neural codes that encode homotopy types of external stimuli, and simplicial sets of probabilities in information structures. We will see that these are all accounted for simultaneously in the same formalism, through the construction of simplicial sets and homotopy types through Gamma-spaces.

3. The functoriality of the construction (through categories of summing functors) allows for the possibility of describing simultaneously several different types of assignments to networks, such as computational architectures (automata), neural codes, information structures, along with (functorial) relations between them, in such a way that the dynamics simultaneously involves all these levels of structure, compatibly with their relations.

4. In addition to direct applications to models of neuronal networks, the formalism considered here makes it possible also to study dynamical systems with threshold non-linearity in other categories, of independent interest in other mathematical setting. An example related to rational points on arithmetic algebraic varieties and “invisible varieties”, inspired by our previous work [75], will be discussed separately, in a forthcoming paper.

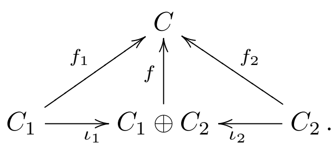
While this paper is mostly dedicated to presenting the general construction and its properties, specific examples of the resulting categorical Hopfield dynamics are described in detail in [78], where a very simple example of threshold non-linear dynamics is presented with resources given by a category of deep neural networks (DNN). It is shown that the simplest possible case of Hopfield dynamics with that category of resources reproduces, in a functorial form, the backpropagation mechanism for the weights of the DNN based on gradient descent. Other explicit examples of categorical Hopfield equations with different categories of resources will be presented separately.

2 Summing functors on networks

In this section we introduce the main formalism we will be using in the modeling of networks with associated resources and their dynamical behavior. Namely we construct certain “moduli spaces” (described by categories) parameterizing all possible assignments of resources of a given type (also described by categories) to a network and its subsystems. These categories of summing functors provide our configuration space attached to a network. The focus of most of this paper will be on understanding relations between these configuration spaces for various specific choices of categories of resources, representing neuronal computational architectures, neural codes, and information structures, and in introducing equations on these configuration spaces describing the dynamical evolution of the network and its resources.

2.1 The category of summing functors

Let $\mathcal{C}$ be a category with a categorical sum (coproduct) and a zero object. A zero object is an object $0 \in \mathrm{Obj}(\mathcal{C})$ that is both initial and terminal, namely for any object $C \in \mathrm{Obj}(\mathcal{C})$ there is a unique morphism $0 \to C$ and a unique morphism $C \to 0$. The categorical sum $C_1 \oplus C_2$ is characterized by the following universal property. There are morphisms $\iota_i : C_i \to C_1 \oplus C_2$ such that, for any object $C \in \mathrm{Obj}(\mathcal{C})$ and any pair of morphisms $f_i : C_i \to C$, there exists a unique morphism $f : C_1 \oplus C_2 \to C$ such that the following diagram commutes



More generally, one can consider categories $\mathcal{C}$ that are unital symmetric monoidal categories. This is especially relevant in view of interpreting $\mathcal{C}$ as a category of resources, in the sense we will discuss in §3. The main point in the paper where we will need to work with this more general setting of unital symmetric monoidal categories, instead of restricting to the case of categories with zero object and sum, is when we introduce the categorical Hopfield dynamics in §6. In the setting of [101], which we will refer to in §7, morphisms in the category of small symmetric monoidal categories are taken to be lax symmetric monoidal functors, that is, functors $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $f : F(A) \oplus F(B) \to F(A \oplus B)$ with commutativity of the diagrams determining the compositions $F(\alpha) \circ f \circ (1 \oplus f) = f \circ (f \oplus 1) \circ \alpha$, with $\alpha : F(A) \oplus (F(B) \oplus F(C)) \to (F(A) \oplus F(B)) \oplus F(C)$ the associativity natural isomorphism, and $F(\gamma) \circ f = f \circ \gamma$, with $\gamma : F(A) \oplus F(B) \to F(B) \oplus F(A)$ the commutativity natural isomorphism. In our setting it is preferable to work with strict symmetric monoidal functors, where the natural transformation f is the identity.

In the following, we will refer to symmetric monoidal categories, without making explicit the unital condition, except where it is explicitly used, as in the setting of categories of resources mentioned above.

Most of the cases we will be discussing in the following sections fit into the stronger case of a category $\mathcal{C}$ with sums and zero object. These include the category of computational systems as in §4.1, a category of neural codes as in §5.1.3, or categories of information structures as discussed in §5.4. Thus, we will assume throughout our discussion that $\mathcal{C}$ has sum and zero object, except where we need to adopt the more general setting of unital symmetric monoidal categories, as in §6.

Let X be a finite pointed set, with $*$ denoting the base point. For most of this section we do not need to work with pointed sets, but the presence of base points will become relevant for the homotopy-theoretic constructions used in §7 and §7.4. Adding a base point should simply be regarded as a computational artifact (introduced for the purpose of homotopy theory), while the "relevant set" is just the complement $X \smallsetminus \{*\}$.

The notion of summing functors was introduced in [96] (see also [20]) in the construction of Gamma-spaces, which we will discuss in §7.

Definition 2.1 *Let $P(X)$ denote the category whose objects are pointed subsets $A \subset X$ with morphisms given by inclusions. A summing functor $\Phi_X : P(X) \to \mathcal{C}$ is a functor with the property that the object $\{*\}$ of $P(X)$ has image $\Phi_X(\{*\}) = 0$, the zero object of $\mathcal{C}$, and for any $A, A' \in \mathrm{Obj}(P(X))$ with $A \cap A' = \{*\}$ one has*

$$\Phi_X(A \cup A') = \Phi_X(A) \oplus \Phi_X(A') . \tag{2.1}$$

In the following, we will interpret the complement $X \smallsetminus \{*\}$ as describing a certain system of neurons, with $A \subset X$ ranging over all possible choices of subsystems $A \smallsetminus \{*\}$. A summing functor $\Phi_X : P(X) \to \mathcal{C}$ describes a way of assigning to every subsystem A a corresponding object $\Phi_X(A)$ in the category $\mathcal{C}$. The target category $\mathcal{C}$ represents a certain type of resources, either computational architectures, describing resources of concurrent or distributed computing in the form of the category of transition systems described in §4.1, or other forms of resources associated to the neurons. The summing-functor property (2.1), that a union of two disjoint sets (which after adding a basepoint means $A \cap A' = \{*\}$) is mapped to the coproduct $\Phi_X(A) \oplus \Phi_X(A')$, describes the requirement that this assignment of resources is *additive on independent subsystems.*

Summing functors are themselves organized into a category, which is a subcategory of the category of functors $\mathrm{Func}(P(X), \mathcal{C})$.

Definition 2.2 *Let $\mathcal{C}$ be a category with sums and zero object. The category $\Sigma_{\mathcal{C}}(X)$ of summing functors has objects the summing functors $\Phi_X : P(X) \to \mathcal{C}$ as in Definition 2.1 and morphisms given by the* invertible *natural transformations.*

Note that if we allow all natural transformations as morphisms rather than restricting to only the invertible ones, the resulting category would not be interesting in a topological sense, since the nerve would be contractible, given that the category $\mathcal{C}$ has a zero object so the category of summing functors has an initial object. Restricting to only invertible natural transformations as morphisms precisely avoids having an initial or terminal object in the category of summing functors, hence allowing for non-contractible topologies: with this restriction to invertible natural transformations, the nerve of the category $\Sigma_{\mathcal{C}}(X)$ of summing functors becomes topologically very non-trivial, as we will recall more in detail in §7 and §7.4, Indeed, it was shown in [101] that, for $\mathcal{C}$ ranging over symmetric monoidal categories, the nerves of the corresponding categories of summing functors generate (in a sense we will make more precise in §7) all connective spectra. In our perspective it is a feature of the model to be able to generate a large supply of sufficiently complex homotopy types (this will be further discussed in §7.4 and in following work, see also [77], [80]).

Another reason why it is desirable, in our setting, to restrict morphisms between summing functors to be invertible natural transformations is that we want to interpret summing functors as consistent assignments of resources to a system. Invertible natural transformations identify which of such assignments should be regarded as *equivalent* to each other. So we can interpret $\Sigma_{\mathcal{C}}(X)$ as a categorical "moduli space" of all possible such assignments up to equivalence. Note, however, that classical geometric intuition here may be misleading, as one does *not* take a quotient by equivalence: one simply maintains all the equivalences explicitly as morphisms in the category. A better intuition is provided by the notion of "action groupoid": given a space Ω with a group action by a group G, instead of considering the quotient Ω/G where points in the same orbit are identified, one considers the action groupoid (sometimes denoted by $\Omega//G$), which is a category with objects the points $\omega \in \Omega$ and morphisms the elements $(g,\omega) \in G \times \Omega$ with source $s(g,\omega) = \omega$ and target $t(g,\omega) = g \cdot \omega$. This construction "resolves" the quotient Ω/G in the sense that the identifications of points in Ω/G are replaced by (invertible) morphisms in the category $\Omega//G$. It is well known that the action groupoid $\Omega//G$ is a better behaved notion of quotient than Ω/G in the case of non-free actions [16]. Thus, one should view here the category $\Sigma_{\mathcal{C}}(X)$ of summing functors as playing a similar role as the action groupoids, in describing assignments of resources to subsystems of X and keeping track of their equivalence structure.

Note that the summing condition (2.1) gives an equivalent and very simple description of summing functors, stated as follows.

Lemma 2.3 *Let $\mathcal{C}$ denote a category with sums and zero object.*

1. *A summing functor $\Phi_X : P(X) \to \mathcal{C}$ as in Definition 2.1 is completely determined by its values $\Phi_X(x) := \Phi_X(A_x)$ on the sets $A_x = \{x, *\}$ for $x \in X \smallsetminus \{*\}$.*

2. *Let $\hat{\mathcal{C}}$ denote the category with the same objects as $\mathcal{C}$ and with morphisms the invertible morphisms of $\mathcal{C}$. For X a finite pointed set with $\#X = n+1$, the category $\Sigma_{\mathcal{C}}(X)$ of summing functors is equivalent to $\hat{\mathcal{C}}^n$, the n-fold product with objects the n-tuples of objects in $\hat{\mathcal{C}}$ and morphisms the n-tuples of arrows (invertible morphisms) with component-wise composition.*

Proof. The first statement is obtained directly from Definition 2.1. For the second statement, an invertible natural transformation $\eta : \Phi \to \Psi$ of summing functors $\Phi, \Psi \in \Sigma_{\mathcal{C}}(X)$ consists of a family of isomorphisms $\eta_A : \Phi(A) \to \Psi(A)$ in the category $\mathcal{C}$ that are compatible with the morphisms of $P(X)$, given by the inclusions of pointed subsets $j : A \hookrightarrow A'$. This compatibility with inclusions shows that, in fact, the isomorphisms η_A must be of the form $\eta_A = \oplus_{x \in A \smallsetminus \{*\}} \eta_x$, with the isomorphisms $\eta_x : \Phi(x) \to \Psi(x)$, as can be seen inductively on the number of elements of A. □

Note that in the proof of Lemma 2.3 we are explicitly using the fact that $\oplus$ is a coproduct, with the pointed inclusions $j : A \hookrightarrow A'$ in $P(X)$ inducing the canonical morphisms $\Phi_X(A) \to \Phi_X(A) \oplus \Phi_X(A' \smallsetminus A \cup \{*\}) = \Phi_X(A')$ defined by the universal property of the coproduct.

In the following we will also consider cases where the category $\mathcal{C}$ is, more generally, a unital symmetric monoidal category. For this case we write here the monoidal product and the unit as $(\oplus, 0)$ rather than in the more usual form $(\otimes, \mathbb{I})$, for consistency of notation with Lemma 2.3. Note, however, that here $\oplus$ is not a coproduct and 0 is not a zero object.

In this more general setting Lemma 2.3 no longer holds as stated. Indeed, first observe that using the same definition of summing functor implies the existence of morphisms $0 \to \Phi_X(A)$ for all $A \in P(X)$, coming from the inclusions $j : \{*\} \hookrightarrow A$. Since 0 is no longer required to be an initial object of $\mathcal{C}$, morphisms $0 \to C$ for $C \in \mathrm{Obj}(\mathcal{C})$ need not always exist. This then imposes a constraint on the possible range of the summing functors, namely summing functors $\Phi_X : P(X) \to \mathcal{C}$ have range in the subcategory of $\mathcal{C}$ of the "no-cost resources", namely the subcategory of $\mathcal{C}$ with objects those $C \in \mathrm{Obj}(\mathcal{C})$ with $\mathrm{Mor}_{\mathcal{C}}(0, C) \neq \emptyset$ (see §3.2 for an explanation of the "no-cost" terminology).

The summing-functor property $\Phi_X(A \cup A') = \Phi_X(A) \oplus \Phi_X(A')$ for $A, A' \in P(X)$ with $A \cap A' = \{*\}$ gives an identification

$$\Phi_X(A) \simeq \bigoplus_{x \in A, x \neq *} \Phi_X(x)\,, \tag{2.2}$$

up to the associators and braiding isomorphisms of the symmetric monoidal structure, that relate the different bracketing and reordering of terms in the right-hand side of (2.2). Indeed, the coherence theorem for unital symmetric monoidal categories ensures that all these different choices of bracketing and reordering differ by a canonical isomorphism. We still have, as in Lemma 2.3, that the values (up to isomorphism) of a summing functor $\Phi_X : P(X) \to \mathcal{C}$ on objects $A \in P(X)$ are completely determined by the collection of objects $\{\Phi_X(x)\}_{x \in A}$.

In the special case where the unital symmetric monoidal category $\mathcal{C}$ is a commutative monoidal category, (2.2) is an identification, and ordering and bracketing of the right-hand side is irrelevant. Indeed, a commutative monoidal category is a permutative category (strictly associative and unital) that is also strictly commutative, so that the natural transformations that give the associators, braiding, and unitors of the monoidal category are all identities. Examples of commutative monoidal categories include Petri nets and categories of line bundles and invertible sheaves [7].

Inclusions $j : A \hookrightarrow A'$ correspond to some morphisms $\Phi_X(j) : \Phi_X(A) \to \Phi_X(A') = \Phi_X(A) \oplus \Phi_X(A' \smallsetminus A \cup \{*\})$ that are no longer canonically determined by the universal property of a coproduct. Thus, invertible natural transformation $\eta : \Phi_X \to \Psi_X$ between summing functors are now determined by the invertible morphisms $\eta_x : \Phi_X(x) \to \Psi_X(x)$, together with this additional datum of the morphisms $\Phi_X(j)$ and $\Psi_X(j)$ for inclusions $j : \{x, *\} \hookrightarrow A$, with $\Psi_X(j) \circ \eta_x = \eta_A \circ \Phi_X(j)$. In the special case of a commutative monoidal category, composition of morphisms and the monoidal product $\oplus$ satisfy the interchange relation $(\varphi \circ \psi) \oplus (\varphi' \circ \psi') = (\varphi \oplus \varphi') \circ (\psi \oplus \psi')$. This implies that the morphisms $\Phi_X(j)$ and $\Psi_X(j)$ are completely determined by the morphisms $\varphi_{x,y} := \Phi_X(j : \{x, *\} \hookrightarrow \{*, x, y\})$ with $\varphi_{x,y} : \Phi_X(x) \to \Phi_X(x) \oplus \Phi_X(y)$ (and similarly for the $\Psi_X(j)$), and these in turn are determined by the morphisms $\varphi_x : 0 \to \Phi_X(x)$ determined by the inclusions $\{*\} \hookrightarrow \{*, x\}$. (As observed above, summing functors necessarily have range in the subcategory of "no-cost" resources so that these morphisms can exist.)

Thus, in the case of commutative monoidal categories, we have obtained the following simple modification of Lemma 2.3.

Corollary 2.4 *Let $(\mathcal{C}, \oplus, 0)$ be a commutative monoidal category. Let $\mathcal{C}^{\text{no-cost}}$ be the full subcategory with objects those $C \in \mathrm{Obj}(\mathcal{C})$ with $\mathrm{Mor}_{\mathcal{C}}(0, C) \neq \emptyset$.*

- *A summing functor $\Phi_X : P(X) \to \mathcal{C}$, defined as in Definition 2.1, takes values in the subcategory $\mathcal{C}^{\text{no-cost}}$.*
- *Such a summing functor is completely determined by a collection of objects*

  $$\{\Phi_X(x)\}_{x \in X \smallsetminus *} \in \mathrm{Obj}(\mathcal{C}^{\text{no-cost}})$$

  and a collection of morphisms

  $$\{\varphi_x : 0 \to \Phi_X(x)\}_{x \in X \smallsetminus *}.$$

- *Invertible natural transformations $\eta : \Phi_X \to \Psi_X$ of two summing functors are completely determined by the isomorphisms $\{\eta_x : \Phi_X(x) \to \Psi_X(x)\}$.*

It is desirable in general to work with arbitrary unital symmetric monoidal categories, not just with the more restrictive class of commutative monoidal categories. In [101], Thomason extended Segal's construction of [96] to the case where $\mathcal{C}$ is an arbitrary unital symmetric monoidal category, see also [102], [103].

In this general setting of arbitrary unital symmetric monoidal categories (see the Appendix of [103]), one proceeds by modifying the notion of summing functor of Definition 2.1, and replacing its characterization in terms of the collection of objects $\{\Phi_X(x)\}_{x \in X \smallsetminus *}$ of Lemma 2.3 into a definition. For our purposes, we take the definition of the category of summing functors for unital symmetric monoidal categories to be of the following form.

Definition 2.5 *Let* $(\mathcal{C}, \oplus, 0)$ *be a symmetric monoidal category. For a finite pointed set* X*, the category* $\Sigma_{\mathcal{C}}(X)$ *has objects*

$$\Phi_X := \{\Phi_X(x)\}_{x \in X \smallsetminus *}$$

given by objects in the Cartesian product $\hat{\mathcal{C}}^n$*, with* $\#X = n+1$*, and morphisms given by morphisms in* $\hat{\mathcal{C}}^n$*.*

Note that here, because the category $\mathcal{C}$ does not have, in general, an initial or a terminal object, one does not have to restrict to invertible natural transformations of summing functors in order to ensure a non-trivial topology of the resulting category of summing functors. Thus, in [103] one just considers the category $\mathcal{C}^n$ instead of $\hat{\mathcal{C}}^n$. However, in our setting we are interested in maintaining this constraint, as we want to interpret the category of summing functors as assignments of resources *up to equivalence*, hence we modified the setting of [103], [101], to include the requirement that summing functors take values in $\hat{\mathcal{C}}$. One may worry here whether this restriction might affect the main result of [101], that Gamma-spaces obtained from this construction realize *all* connective spectra. However, this is still the case. Indeed, our setting includes in particular the case where $\mathcal{C}$ is a unital symmetric monoidal *groupoid*, in which case $\mathcal{C} = \hat{\mathcal{C}}$, and it is known by Theorem 5.3 of [45] that Gamma-spaces associated to unital symmetric monoidal groupoids already suffice to realize all connective spectra. Thus, this restriction does not affect the main homotopy-theoretic properties we will be discussing in §7. This leaves an ambiguity of two slightly different possible definitions of summing functors in the case of commutative monoidal categories, so one will need to specify, in those cases, which notion of $\Sigma_{\mathcal{C}}(X)$ is used. In the following, we will mostly discuss summing functors based on Definition 2.1, without specifying explicitly how to incorporate the case of Definition 2.5, except where directly needed, as the latter case usually follows by a simple modification.

The reason why it is useful to consider both of these slightly different notions of category of summing functors will be discussed more in detail in §6.3 below, when we introduce categorical Hopfield equations with threshold non-linearities. We will see that, while the case of categories of resources with a zero-object reduces to a linear dynamics, the more general case of symmetric monoidal categories gives rise to genuinely non-linear models. (As shown above, summing functors necessarily take values in "no-cost resources", that is, in the subcategory of objects C with a morphism $0 \to C$, while as we will see in §6.3 the threshold dynamics is designed to detect the opposite convertibility $C \to 0$.) The case of the symmetric monoidal category of deep neural networks introduced in [78] provides such an example with non-linear dynamics.

2.2 Networks and summing functors

Our goal is to assign resources to networks of neurons. So far, we have only described a notion of consistent assignments of $\mathcal{C}$-type resources to subsets of a given finite set. We next describe how to introduce the network structure. The setting we described in the previous subsections can be modified by regarding the data of neurons and connections as a directed graph and incorporating it in the construction.

A categorical description of networks and flows on networks was introduced in [56]. In that generality, one considers networks to be directed graphs, where a priori no restriction on edges is imposed (that is, one allows multiple edges and looping edges). In more specific cases (for example when considering cliques), one only allows graphs without these types of edges. The standard description of directed graphs in categorical terms is as follows (see e.g. [56]).

Definition 2.6 *Let* $\mathbf{2}$ *denote the category with two objects* E, V *and as only non-identity morphisms two parallel morphisms* $s, t : E \to V$ *(called source and target morphisms). A directed finite graph is a functor* $G : \mathbf{2} \to \mathcal{F}$ *where* $\mathcal{F}$ *is the category of finite sets.*

In the following we will refer to a functor $G \in \text{Func}(\mathbf{2}, \mathcal{F})$ as a directed graph, or equivalently as a (directed) network, to the set $V_G = G(V)$ as either the set of *vertices* or, equivalently, as the set of nodes of G, and to $E_G = G(E)$ as the set of edges.

Note that some variants of this categorical notion of directed graphs are possible, and useful to consider in some cases. For example, with the notion given in Definition 2.6, morphisms of directed graphs do not include contraction of edges (mapping an edge to a vertex). If one wants to work with directed graphs where it is important to also consider such transformations, then a simple modification of the category **2** achieves this purpose. We refer the reader to §2.1.1 of [78] where different categorical formulations of directed graphs are compared. A specific example where contractions of edges are need is also discussed in [78].

Because of the need to work with pointed sets for homotopy-theory purposes, we can alter slightly this standard definition with the addition of base-point data. Again, these base-point data should be regarded only as an artificial computational device introduced here for later use (see §7 and §7.4). For the purpose of what we discuss here, the reader can easily ignore this extension to the pointed case and just rephrase everything in terms of the original Definition 2.6.

Definition 2.7 *A pointed directed finite graph is a functor $G : \mathbf{2} \to \mathcal{F}_*$ to the category of pointed finite sets.*

Note that this definition differs from other notions of flow graphs, since we do not require the distinguished root vertex to be a source or a sink, nor do we require the existence of directed paths from the root to all other vertices. Moreover, since the source and target maps are mapped by the functor G to morphisms of pointed sets, these graphs have a distinguished looping edge with both source and target equal to the root vertex. This root vertex and its looping edge do not play a direct role in our model and are only an artificial device to introduce base points for homotopy-theoretic purposes.

For all the practical aspects of the model, we can assume that we work with directed graphs $G : \mathbf{2} \to \mathcal{F}$ in the usual sense. Indeed the pointed directed graphs we will be considering are obtained from an ordinary directed graph in the following way.

Lemma 2.8 *Given a functor $G : \mathbf{2} \to \mathcal{F}$, we associate to it a pointed directed graph $G^* : \mathbf{2} \to \mathcal{F}_*$ defined by $E_{G^*} = E_G \sqcup \{e_*\}$ and $V_{G^*} = V_G \sqcup \{v_*\}$ with $s, t : E_{G^*} \to V_{G^*}$ given by the source and target maps $s, t : E_G \to V_G$ for all edges $e \in E_G$ and as $s, t : e_* \mapsto v_*$.*

Thus the pointed graphs G^* we will consider here are just ordinary directed graphs G together with a disjoint base-point vertex with a single looping edge attached to it. In the following, in cases where we consider the case without looping edges, we mean that the underlying G has no looping edges.

Lemma 2.9 *The source and target maps $s, t : E \to V$ determine functors between the categories of summing functors (still denoted s, t),*

$$s, t : \Sigma_{\mathcal{C}}(E_{G^*}) \to \Sigma_{\mathcal{C}}(V_{G^*})\,.$$

Proof. The source and target maps $s, t : E \to V$ transform summing functors $\Phi_{E_{G^*}} : P(E_{G^*}) \to \mathcal{C}$ into functors $\Phi^{s,t}_{V_{G^*}} : P(V_{G^*}) \to \mathcal{C}$ given by

$$\Phi^s_{V_{G^*}}(A) := \Phi_{E_{G^*}}(s^{-1}(A)) = \oplus_{e \in E_{G^*} : s(e) \in A} \Phi_{E_{G^*}}(e),$$

for $A \in P(V_{G^*})$ where $\Phi_{E_{G^*}}(e)$ means the functor $\Phi_{E_{G^*}}$ evaluated on the pointed set $\{e, *\} \in P(E_{G^*})$. Because of the way the pointed directed graph G^* is constructed from the directed graph G, we see that the functor $\Phi^s_{V_{G^*}}$ obtained in this way is by construction still a summing functor. Indeed, for $A \cap A' = \{v_*\}$ in $P(V_{G^*})$, we have

$$\Phi^s_{V_{G^*}}(A \cup A') = \bigoplus_{e \in E_{G^*} : s(e) \in A \smallsetminus \{v_*\}} \Phi_{E_{G^*}}(e) \oplus \bigoplus_{e \in E_{G^*} : s(e) \in A' \smallsetminus \{v_*\}} \Phi_{E_{G^*}}(e) \oplus \Phi_{E_{G^*}}(e_*),$$

where $\Phi_{E_{G^*}}(e_*)$ is the zero object in $\mathcal{C}$, so the above equals $\Phi^s_{V_{G^*}}(A) \oplus \Phi^s_{V_{G^*}}(A')$. The case of $\Phi^t_{V_{G^*}}$ is similar. □

We can interpret this explicitly in terms of our model in the following way. The directed graph G represents a network of neurons (the nodes V_G) and connections between them (the directed edges E_G). The introduction of the artificial base vertex v_* with its single looping edge e_* is merely a computational artifact that does not affect the structure of the network. The category $\Sigma_{\mathcal{C}}(V_{G^*})$

parameterizes all the possible consistent assignments of resources of type $\mathcal{C}$ over subsets of nodes in V_G (in fact at the individual nodes of G, by Lemma 2.3). In a similar way $\Sigma_{\mathcal{C}}(E_{G^*})$ describes assignments of resources of type $\mathcal{C}$ to the edges of the network. The induced source and target maps can be used to express possible compatibility requirements between the assignments at nodes and at edges. The images of the source and target maps describe assignments of $\mathcal{C}$-resources at sets A of nodes of the network that come from an assignment at either the outgoing or the incoming edges at those nodes. We will describe in §2.2.1 and §2.2.2 below some specific examples of possible ways of imposing constraints relating assignments of resources at vertices and edges.

2.2.1 Conservation laws at vertices

The first and simplest example of compatibility condition between assignments of resources to vertices and edges consists of imposing the standard physical conservation law at vertices. This is a typical feature, for example, of electrical networks with flows of electric currents, where conservation at vertices holds because of Kirchhoff's current law. For biological neuronal networks, this very simple conservation law is not always adequate, but we present it here as the first case because of its very simple description. In categorical terms, this kind of conservation law is literally expressed by the equalizer construction.

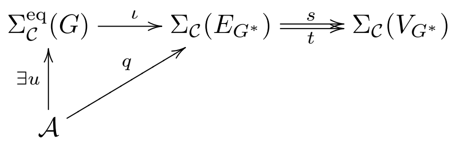
Proposition 2.10 *The equalizer $\Sigma^{\rm eq}_{\mathcal{C}}(G)$ of the two functors*

$$s, t : \Sigma_{\mathcal{C}}(E_{G^*}) \rightrightarrows \Sigma_{\mathcal{C}}(V_{G^*})$$

is a category consisting of the summing functors $\Phi_E \in \Sigma_{\mathcal{C}}(E_{G^*})$ that satisfy the Kirchhoff conservation law at vertices

$$\bigoplus_{e\,:\,s(e)=v} \Phi_E(e) = \bigoplus_{e\,:\,t(e)=v} \Phi_E(e)\,. \tag{2.3}$$

Proof. Consider the two functors $s, t : \Sigma_{\mathcal{C}}(E_{G^*}) \rightrightarrows \Sigma_{\mathcal{C}}(V_{G^*})$ as above, between the small categories of summing functors, induced by the source and target morphisms of the directed graph $G : \mathbf{2} \to \mathcal{F}$. The equalizer of this diagram is the small category $\Sigma^{\rm eq}_{\mathcal{C}}(G)$ with functor $\iota : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \Sigma_{\mathcal{C}}(E_{G^*})$ such that $s \circ \iota = t \circ \iota$ satisfying the universal property expressed for any $\mathcal{A}$ and q with $s \circ q = t \circ q$ by the commutative diagram



This can be realized as summing functors $\Phi_E : P(E_{G^*}) \to \mathcal{C}$ in $\Sigma_{\mathcal{C}}(E_{G^*})$ such that, for all $A \in P(V_{G^*})$

$$\Phi_E(s^{-1}(A)) = \Phi_E(t^{-1}(A)). \tag{2.4}$$

The relation (2.4) is exactly expressing the Kirchhoff conservation law at vertices since by Lemma 2.3 it can be reduced to the case where A has a single (non base-point) vertex where it reduces to (2.3). □

2.2.2 Vertex constraints by coequalizer

Another way of imposing a Kirchhoff-type conservation at vertices is provided by the dual coequalizer construction. While the equalizer construction selects those summing functors on edges, with the given target category $\mathcal{C}$, that satisfy conservation at vertices, the coequalizer construction modifies the target category to a suitable quotient where the conservation laws hold.

The coequalizer construction is more subtle for various reasons: the nerve functor (that we will be using in §7) only preserves directed colimits, and in general coequalizers in the category of small categories are more subtle to construct. However, we can still consider the following construction at the level of summing functors.

Following [12], coequalizers in the category of small categories can be described in terms of a quotient construction based on the notion of generalized congruences. For a small category $\mathcal{C}$, let

$\mathrm{Mor}^+(\mathcal{C})$ denote the set of n-tuples of (not necessarily composable) morphisms of $\mathcal{C}$ for arbitrary n. For $\phi \in \mathrm{Mor}^+(\mathcal{C})$ one denotes by $\mathrm{dom}(\phi)$ and $\mathrm{codom}(\phi)$, respectively, the objects of $\mathcal{C}$ given by the domain of the first morphism in the tuple and the codomain of the last morphism in the tuple.

Definition 2.11 [12] *A generalized congruence on $\mathcal{C}$ consists of an equivalence relation on the set of objects $\mathrm{Obj}(\mathcal{C})$ and a partial equivalence relation on the tuples of morphisms of $\mathcal{C}$ with the properties*

1. *if $A \sim B$ for $A, B \in \mathrm{Obj}(\mathcal{C})$ then $\mathrm{id}_A \sim \mathrm{id}_B$;*
2. *if $\phi \sim \psi$ for $\phi, \psi \in \mathrm{Mor}^+(\mathcal{C})$ then $\mathrm{dom}(\phi) \sim \mathrm{dom}(\psi)$ and $\mathrm{codom}(\phi) \sim \mathrm{codom}(\psi)$;*
3. *if $\phi_1\phi_2 \sim \psi$ with $\phi_i, \psi \in \mathrm{Mor}^+(\mathcal{C})$ then $\mathrm{dom}(\phi_2) \sim \mathrm{codom}(\phi_1)$;*
4. *if $\phi \sim \psi$ and $\chi \sim \xi$ for $\phi, \psi, \chi, \xi \in \mathrm{Mor}^+(\mathcal{C})$ with $\mathrm{codom}(\phi) \sim \mathrm{dom}(\chi)$ then $\phi\chi \sim \psi\xi$;*
5. *if $\mathrm{codom}(\phi) = \mathrm{dom}(\psi)$ for single morphisms $\phi, \psi \in \mathrm{Mor}(\mathcal{C})$ then the chain $\phi\psi$ is composable and $\phi\psi \sim \psi \circ \phi$ in $\mathrm{Mor}^+(\mathcal{C})$.*

The quotient $\mathcal{C}/_\sim$ of $\mathcal{C}$ by a generalized congruence is a small category with objects the equivalence classes of objects $\mathrm{Obj}(\mathcal{C}/_\sim) = \mathrm{Obj}(\mathcal{C})/_\sim$ and morphisms given by equivalence classes of tuples $\phi_1 \cdots \phi_n$ in $\mathrm{Mor}^+(\mathcal{C})$ with $\mathrm{codom}(\phi_i) \sim \mathrm{dom}(\phi_{i+1})$ (that is, chains that become composable in the quotient), with the composition determined by concatenation of tuples of paths. There is a quotient functor $Q : \mathcal{C} \to \mathcal{C}/_\sim$. A generalized congruence is principal if it is generated by a relation on single morphisms.

It is shown in [12] that, given two functors $F, G : \mathcal{A} \to \mathcal{C}$ in the category of small categories Cat, the coequalizer $\mathrm{coeq}(F, G)$ with functor $Q : \mathcal{C} \to \mathrm{coeq}(F, G)$ is the quotient category $\mathcal{C}/_\sim$ with quotient functor $Q : \mathcal{C} \to \mathcal{C}/_\sim$ with respect to the principal generalized congruence generated by $F(A) \sim G(A)$ in $\mathrm{Obj}(\mathcal{C})$ and $F(\phi) \sim G(\phi)$ for $\phi \in \mathrm{Mor}(\mathcal{C})$.

For a fixed network specified by a directed graph $G \in \mathrm{Func}(\mathbf{2}, \mathcal{F})$, let G^* be the pointed directed graph obtained as above. As above, given a summing functor $\Phi_E : P(E_{G^*}) \to \mathcal{C}$, we consider the two functors Φ_V^s and Φ_V^t from $P(V_{G^*})$ to $\mathcal{C}$ given by $\Phi_V^s(A) = \Phi_E(s^{-1}(A))$ and $\Phi_V^t(A) = \Phi_E(t^{-1}(A))$ for all pointed subsets $A \in P(V_{G^*})$ and s, t the source and target maps of G.

Proposition 2.12 *The coequalizer $\rho_G : \mathcal{C} \to \mathcal{C}_G^{\mathrm{coeq}}(\Phi_E)$ of the functors Φ_V^s, Φ_V^t gives a category $\mathcal{C}_G^{\mathrm{coeq}}(\Phi_E)$ of resources that is optimal with respect to the property that resources associated to the systems $\Phi_E(s^{-1}(A))$ and $\Phi_E(t^{-1}(A))$ satisfy the conservation law at vertices*

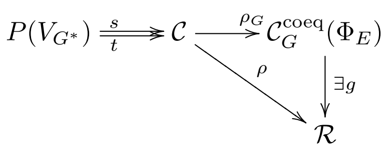
$$\rho_G(\Phi_E(s^{-1}(A))) = \rho_G(\Phi_E(t^{-1}(A))), \quad \forall A \in P(V_{G^*}). \tag{2.5}$$

The multiple coequalizer $\rho_G : \mathcal{C} \to \mathcal{C}_G^{\mathrm{coeq}}$ over the family $\{(\Phi_V^s, \Phi_V^t) \,|\, \Phi_E \in \Sigma_{\mathcal{C}}(E_{G^*})\}$ gives a category $\mathcal{C}_G^{\mathrm{coeq}}$ such that the conservation law (2.5) holds for all $\Phi_E \in \Sigma_{\mathcal{C}}(E_{G^*})$.

Proof. Consider the coequalizer $\mathcal{C}_G^{\mathrm{coeq}}(\Phi_E) := \mathrm{coeq}(\Phi_V^s, \Phi_V^t)$ of the functors

$$\Phi_V^s, \Phi_V^t : P(V_{G^*}) \rightrightarrows \mathcal{C},$$

with the functor $\rho_G : \mathcal{C} \to \mathcal{C}_G^{\mathrm{coeq}}$ satisfying $\rho_G \circ \Psi_V^s = \rho_G \circ \Psi_V^t$. This is characterized by the universal property given by the commutativity of the diagrams



for all small categories $\mathcal{R}$ and functors $\rho : \mathcal{C} \to \mathcal{R}$ such that $\rho \circ \Phi_V^s = \rho \circ \Phi_V^t$, and a functor $g : \mathcal{C}_G^{\mathrm{coeq}}(\Phi_E) \to \mathcal{R}$ with $g \circ \rho_G = \rho$.

By the result of [12] recalled above, we can describe the coequalizer $\rho_G : \mathcal{C} \to \mathcal{C}_G^{\mathrm{coeq}}$ as the quotient functor to $\mathcal{C}_G^{\mathrm{coeq}}(\Phi_E) = \mathcal{C}/_{\sim_{G,\Phi_E}}$ where $\sim_{G,\Phi_E}$ is the principal generalized congruence on $\mathcal{C}$ generated by the relations $\Phi_E(s^{-1}(A)) \sim \Phi_E(t^{-1}(A))$ for all $A \in P(V_{G^*})$ and the same equivalence on morphisms corresponding to pointed inclusions of sets in $P(V_{G^*})$.

The universal property of the coequalizer shows that the category $\mathcal{C}/_{\sim_{G,\Phi_E}}$ is the optimal choice of a category $\mathcal{R}$ of resources with a functor $\rho : \mathcal{C} \to \mathcal{R}$ from systems to resources that implements the conservation laws (2.5) of resources at vertices for the summing functor Φ_E. □

Definition 2.13 *If the category $\mathcal{C}_G^{\rm coeq}$ obtained as the multiple coequalizer $\rho_G : \mathcal{C} \to \mathcal{C}_G^{\rm coeq}$ in Proposition 2.12 admits a symmetric monoidal structure then we can consider the category of summing functors*

$$\Sigma_{\mathcal{C}}^{\rm coeq}(G) := \Sigma_{\mathcal{C}_G^{\rm coeq}}(E_{G^*}).$$

This category describes the imposition of constraints (2.5) at vertices.

An advantage of the coequalizer construction is that, instead of selecting a smaller subcategory of summing functors with fixed target category, it imposes the conservation law at vertices by suitably altering only the target category.

2.3 Categories of summing functors on networks

The examples of constructions of categories of summing functors on networks described in §2.2.1 and §2.2.2 via equalizers and coequalizers are special cases (realized by subcategories) of a more general setting that we introduce here. The subcategories obtained via equalizers and coequalizers correspond to choosing only those summing functors that are determined by certain specific types of constraints at vertices. We will then show in §2.3.2 another example of a construction of a category of summing functors on networks that also fits into the general framework discussed here, but which arises from different types of constraints coming from grafting operations.

We assume that $\mathcal{C}$ is either a category with zero object and sum, or more generally a symmetric monoidal category.

Given a directed graph $G : \mathbf{2} \to \mathcal{F}$, a subgraph is another functor $G' : \mathbf{2} \to \mathcal{F}$ with a natural transformation $\alpha : G' \hookrightarrow G$, meaning that $\alpha_V : V_{G'} \hookrightarrow V_G$ and $\alpha_E : E_{G'} \hookrightarrow E_G$ are inclusions. The case of pointed directed graphs is analogous with α_V, α_E inclusions of pointed sets.

Definition 2.14 *Given $G : \mathbf{2} \to \mathcal{F}$, let $P(G)$ be the category whose objects are the subgraphs $G' \hookrightarrow G$ with morphisms given by the inclusions. A network summing functor is a functor $\Phi : P(G) \to \mathcal{C}$ that maps the empty subgraph to the zero object and such that*

$$\Phi(G' \sqcup G'') = \Phi(G') \oplus \Phi(G'')$$

for non-intersecting subgraphs. The category $\Sigma_{\mathcal{C}}(G)$ consists of network summing functors with invertible natural transformations.

Remark 2.15 For $G^* : \mathbf{2} \to \mathcal{F}_*$ a pointed graph with base vertex v_* with looping edge e_*, the category $\Sigma_{\mathcal{C}}(G^*)$ consists of functors $\Phi : P(G^*) \to \mathcal{C}$ that map the pointed component $\Phi(\{v_*, e_*\}) = 0$ to the zero object of $\mathcal{C}$ and satisfy $\Phi(G' \cup G'') = \Phi(G') \oplus \Phi(G'')$ for $G', G'' \in P(G^*)$ with $G' \cap G'' = \{v_*, e_*\}$. For the graph G^* obtained by adding to a non-based graph G a separate component $\{v_*, e_*\}$, the categories $\Sigma_{\mathcal{C}}(G)$ and $\Sigma_{\mathcal{C}}(G^*)$ are equivalent, so we will use the same notation $\Sigma_{\mathcal{C}}(G)$.

The categories of summing functors $\Sigma_{\mathcal{C}}^{\rm eq}(G)$ and $\Sigma_{\mathcal{C}}^{\rm coeq}(G)$ considered in §2.2.1 and §2.2.2 are (sub)categories of network summing functors. Indeed, we can view a $\Phi \in \Sigma_{\mathcal{C}}^{\rm eq}(G)$ as an object in $\Sigma_{\mathcal{C}}(G)$ by precomposition with the functor $P(G^*) \to P(E_{G^*})$, hence $\Sigma_{\mathcal{C}}^{\rm eq}(G) \subset \Sigma_{\mathcal{C}}(G)$. In the same way a functor $\Phi \in \Sigma_{\mathcal{C}}^{\rm coeq}(G)$ can be seen as an object in the category $\Sigma_{\mathcal{C}_G^{\rm coeq}}(G)$. One can see from these examples that, in more concrete problems, one will want to restrict summing functors to some suitable subcategory of $\Sigma_{\mathcal{C}}(G)$ that corresponds to specific types of constraints one wants to impose dictated by the structure of the network (such as conservation laws at vertices in these examples).

2.3.1 Graphs in terms of vertices and flags

There are other variants of the standard categorical description of directed graphs of Definition 2.6 that can also be useful in our setting, especially for the formulation of §2.3.2 below. If one does not need the directed structure, but would like graphs to have some "external edges" (external ports, which in the non-directed case serve simultaneously as inputs and outputs), then the physics description of graphs in terms of vertices and half-edges (flags) instead of vertices and edges would be more suitable.

Definition 2.16 *Let $\mathbf{2}_F$ be the category with two objects V, F and non-identity morphisms $\partial : F \to V$ and $\iota : F \to F$ with $\iota^2 = 1_F$, as well as $\iota \circ \partial$. A finite graph is a functor $G : \mathbf{2}_F \to \mathcal{F}$ to the category of finite sets.*

Here $V_G := G(V)$ is the set of vertices and $F_G := G(F)$ is the set of half-edges. The morphism ∂ assigns to each half-edge the vertex it is attached to, and the involution ι glues together the loose ends of the half-edges. Here we do not assume that ι is fixed-point free: the fixed points of ι are the external edges of the graph, while the pairs of flags $f \neq f'$ with $f' = \iota(f)$ are the half-edges glued together to form an (internal) edge of G. The resulting graphs can have multiple and looping edges.

A pointed version can be obtained as in the previous case, by replacing the target category $\mathcal{F}$ with finite pointed sets $\mathcal{F}_*$. Since the induced morphisms determined by ∂ and ι have to be maps of pointed sets, we obtain that the base vertex v_* has a base external edge $f_* = \iota(f_*)$ attached to it. Given a graph $G : \mathbf{2}_F \to \mathcal{F}$ the associated based $G^* : \mathbf{2}_F \to \mathcal{F}_*$ simply has an added component consisting of v_* with the external edge f_*. Note that, in the case of the category of pointed graphs $G^* : \mathbf{2}_F \to \mathcal{F}_*$, one can use the base vertex with external edge as a way to incorporate data of an assigned external input to the network.

As in the case of the description of graphs of Definition 2.6, one can then consider categories of summing functors $\Sigma_{\mathcal{C}}(V_{G^*})$ and $\Sigma_{\mathcal{C}}(F_{G^*})$.

In the case of directed graphs, one can also accommodate external edges in two possible ways. One is simply to consider any univalent vertices as "external" vertices and the corresponding edges as "external edges", the other is to adapt the flag definition of graphs of Definition 2.16 to the directed case in the following way.

Definition 2.17 *Consider the category $\mathbf{2}^{i/o}$ with objects $\{V, E, F_i, F_o\}$ and morphisms freely generated by*

$$E \stackrel{f_i}{\to} F_i \stackrel{t}{\to} V \stackrel{s}{\leftarrow} F_o \stackrel{f_o}{\leftarrow} E \,. \tag{2.6}$$

A (finite) directed graph with external edges (also called an "open-ended" graph) is a functor $G : \mathbf{2}^{i/o} \to \mathcal{F}$, with $\mathcal{F}$ the category of finite sets, where the morphisms f_i, f_o are mapped to injective maps.

We interpret here the sets $E(G) := G(E)$ and $V(G) := G(V)$ as directed (internal) edges and vertices, and we interpret the sets $F_i(G) := G(F_i)$ and $F_o(G) := G(F_o)$ as the incoming/outgoing flags (oriented to/from the vertex). The morphisms $t : F_i(G) \to V(G)$ and $s : F_o(G) \to V(G)$ are the boundary morphisms that associate to a flag the corresponding vertex (target or source depending on orientation) and the morphisms $f_i : E(G) \to F_i(G)$ and $f_o : E(G) \to F_o(G)$ assign to an edge its two flags (half-edges), respectively attached to source and target vertex. The set $E_{ext}(G)$ of external edges of G is then given by the set

$$E_{ext}(G) = (F_i(G) \smallsetminus f_i(E)) \sqcup (F_o(G) \smallsetminus f_o(E)) \,.$$

The case of the category $\mathbf{2}$ and directed graphs $\mathcal{G} = \mathrm{Func}(\mathbf{2}, \mathcal{F})$ without external edges corresponds to the case where the outer arrows of (2.6) are identity maps. In this case edges attached to valence-one vertices are not considered external.

Definition 2.17 allows for directed cycles (for example, pairs of vertices with a directed edge between them in both directions). In the following, in general we will be restricting to acyclic graphs, for compatibility with the properad composition, see Lemma 2.19.

2.3.2 Constraints through grafting operations

We now describe another construction of an interesting subcategory of summing functors, where instead of simple conservation conditions at vertices one uses more interesting grafting operations, in a case where additional compositionality structure is present on the target category $\mathcal{C}$. This type of construction will be useful in the case where we consider resources given by certain classes of computational architectures (see §4.1 and §4.2).

The compositionality structures referred to above can be expressed in terms of the notion of *properad* [105] (see also [68]).

Definition 2.18 *Let* Cat *denote the category of small categories. A* properad *in* Cat *is a collection* $\mathcal{P} = \{\mathcal{P}(m,n)\}_{m,n\in\mathbb{N}}$ *of small categories with composition functors (grafting operations)*

$$\circ^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell} : \mathcal{P}(m,k)\times\mathcal{P}(n,r)\to\mathcal{P}(m+n-\ell,k+r-\ell)\,, \tag{2.7}$$

for non-empty $\{i_1,\ldots,i_\ell\}\subset\{1,\ldots,k\}$ *and* $\{j_1,\ldots,j_\ell\}\subset\{1,\ldots,n\}$*,* $i_s < i_{s+1}$ *and* $j_s < j_{s+1}$ *for* $s = 1,\ldots,\ell-1$*. These composition operations satisfy associativity and unity conditions. A* symmetric properad *also has symmetric group actions of* $\Sigma_m\times\Sigma_n$ *on* $\mathcal{P}(m,n)$ *with respect to which the compositions are bi-equivariant. We will assume properads to be symmetric.*

The unit $\mathbf{1}\in\mathcal{P}(1,1)$ of the properad satisfies $\mathbf{1}\circ^1_j P = P$ for all $P\in\mathcal{P}(n,r)$ and $P'\circ^i_1\mathbf{1} = P'$ for all $P'\in\mathcal{P}(m,k)$, for all $j\in\{1,\ldots,n\}$ and all $i\in\{1,\ldots,k\}$. We will not write out here explicitly the associativity condition for the properad composition laws (2.7), but it can be deduced directly from the definition of the composition law.

It is in general assumed that properads are symmetric, especially in the context of graphs, which would otherwise require additional data of planar structures compatible with composition. In the symmetric case an abstract set rather than an ordered set suffices for indexing.

For a more detailed discussion of the properties of properads and the compatibility between the properad composition and the monoidal structure in the case where $\mathcal{C}$ is unital symmetric monoidal, see §1.1.1 of [78]. An explicit example of properad in Cat and its properad composition is described in [78] in the form of a category of deep neural network architectures.

We consider here open-ended subgraphs $G'\in P(G)$ of the open-ended graph G that are *full*, in the sense that if a vertex is in the subgraph then all its incident half-edges are also in the subgraph, and if two vertices are in the subgraph then all internal edges between them are also in the subgraph. For an acyclic graph G, we also require the subgraphs to be convex, in the sense that if two vertices are in the subgraph, so are all the intermediate vertices along directed paths connecting them.

Given a directed graph G and two subgraphs $G', G''\in P(G)$ as above with $V_{G'}\cap V_{G''}=\emptyset$, let $E(G',G'')\subset E_G$ denote the set of edges with one endpoint in $V_{G'}$ and the other in $V_{G''}$, and let $G'\star G''$ denote the subgraph of G with $V_{G'\star G''} = V_{G'}\cup V_{G''}$ and $E_{G'\star G''} = E_{G'}\cup E_{G''}\cup E(G',G'')$. For the purpose of the following construction we assume that the graph G has a certain number $\deg^{\rm in}(G)\geq 1$ of incoming external legs and a number $\deg^{\rm out}(G)\geq 1$ of outgoing external legs. Similarly for a subgraph $G'\subset G$. Let $E(G', G\smallsetminus G')$ denote the set of edges in G with one end in $V_{G'}$ and the other end in $V_G\smallsetminus V_{G'}$.

We write $\deg^{\rm in}(G')$ (respectively, $\deg^{\rm out}(G')$) for the number of edges in $E(G', G\smallsetminus G')$ with target vertex (respectively, source vertex) in G', plus the number of external (half)edges of G with target (respectively, source) vertex in G'. Then the following is a direct consequence of the definitions.

Recall that, for a vertex $v\in V_G$ the corolla $C(v)$ consisting of v together with all the attached (half)edges, with $\deg^{\rm in}(v)$ incoming and $\deg^{\rm out}(v)$ outgoing (half)edges.

Lemma 2.19 *Let* $\mathcal{C}$ *be a symmetric monoidal category such that there is a family of full subcategories* $\mathcal{C}(n,m)$ *for* $n,m\in\mathbb{N}$ *with the properties:*

- $\mathrm{Obj}(\mathcal{C}) = \cup_{n,m\in\mathbb{N}}\mathrm{Obj}(\mathcal{C}(n,m))$*;*
- *the monoidal structure* $(\otimes,\mathbb{I})$ *satisfies*

$$\otimes : \mathcal{C}(m,k)\times\mathcal{C}(n,r)\to\mathcal{C}(m+n,k+r)\,;$$

- *the family* $\{\mathcal{C}(n,m)\}_{n,m\in\mathbb{N}}$ *is a properad in* Cat.

Let G be a directed acyclic graph. For two subgraphs $G', G'' \in P(G)$ as above with $V_{G'} \cap V_{G''} = \emptyset$, we say that $G' < G''$ if there are no directed paths from vertices of G'' to vertices of G'. Then there is a full subcategory $\Sigma^{\rm prop}_{\mathcal{C}}(G) \subset \Sigma_{\mathcal{C}}(G)$ given by the summing functors $\Phi : P(G) \to \mathcal{C}$ with the following properties:

1. *for all full convex open-ended subgraphs $G' \in P(G)$,*
$$\Phi(G') \in {\rm Obj}(\mathcal{C}(\deg^{\rm in}(G'), \deg^{\rm out}(G'))\,,$$
2. *for any vertex, $\Phi(\{v\}) = \Phi(C(v))$ where $C(v)$ is the corolla of the vertex v in G,*
3. *for any $G' < G'' \in P(G)$ with $V_{G'} \cap V_{G''} = \emptyset$,*
$$\Phi(G' \star G'') = \Phi(G') \circ_{E(G',G'')} \Phi(G''), \tag{2.8}$$
where $E(G', G'') \subset E_G$ is the set of edges with source endpoint in $V_{G'}$ and target in $V_{G''}$ and $\circ_{E(G',G'')}$ is the properad composition
$$\circ_{E(G',G'')} : \mathcal{C}(\deg^{\rm in}(G'), \deg^{\rm out}(G')) \times \mathcal{C}(\deg^{\rm in}(G''), \deg^{\rm out}(G''))$$
$$\to \mathcal{C}(\deg^{\rm in}(G' \star G''), \deg^{\rm out}(G' \star G''))\,.$$

Note that in (3) of Lemma 2.19 the properad composition requires $E(G', G'') \neq \emptyset$. In the case with $E(G', G'') = \emptyset$ one can replace the properad composition with the monoidal operation. This would correspond to generalizing properads to *props*, where composition along an empty overlap of outputs and inputs is also allowed.

In our formulation of Lemma 2.19, the requirement that the $\mathcal{C}(n, m)$ are full subcategories is motivated by the case of subcategories of a category of computational systems (automata) where one fixes the number of inputs and outputs. This is in contrast with the usual example of the category of vector spaces, with $\mathcal{C}(n, m)$ given by spaces of linear maps from the n-th to the m-th powers, which would not be full subcategories.

Corollary 2.20 *Let G be a directed acyclic graph. A network summing functor $\Phi \in \Sigma^{\rm prop}_{\mathcal{C}}(G)$ is completely determined by its value on corollas.*

Proof. At each vertex $v \in V_G$ consider the corolla $C(v)$. The functor Φ assigns values $\Phi_v := \Phi(C(v)) \in \mathcal{C}(\deg^{\rm in}(v), \deg^{\rm out}(v))$. Consider a first vertex $v \in V_G$ and the associated value Φ_v. Choose then a second vertex $w \in V_G$ with value Φ_w. If $v \leq w$, the subgraph $C(v) \star C(w)$ will have value $\Phi(C(v) \star C(w)) = \Phi_v \circ_{E(v,w)} \Phi_w$, with $E(v, w)$ the set of directed edges of G connecting v to w. Inductively, if $\Phi(G')$ has been constructed for all subgraphs $G' \subset G$ with up to n vertices that are lowersets for the partial order of the directed graph G, and $\#V_G > n$, then choose another vertex u of G not in G'. If $u \geq v$ for some $v \in G'$, the subgraph $G' \star \{u\}$ has $\Phi(G' \star \{u\}) = \Phi(G') \circ_{E(G',u)} \Phi_u$, and this determines the value on all lowerset subgraphs with $n + 1$ vertices. The order of choice of the new vertices does not matter because of the associativity condition of the properad operations, and the presence of external edges does not change the result because of the unity condition of the properad, since external edges are compositions with the properad unit. □

We will see a more concrete instance of this type of construction in §4.1 and §4.2. External edges and the properad unit are also further discussed in §2.1.2 and §2.1.3 of [78].

2.3.3 Inclusion-exclusion properties

In the case where the category $\mathcal{C}$ is an abelian category or a triangulated category, one can also make requirements on the dependence of the summing functors on subnetworks through imposing inclusion-exclusion behavior.

- If $\mathcal{C}$ is an *abelian category*, one can in particular consider those summing functors in $\Sigma_{\mathcal{C}}(G)$ that satisfy an inclusion-exclusion relation, in the form of exact sequences, namely summing functors such that, for all $G', G'' \in P(G)$, there is an exact sequence in $\mathcal{C}$

$$0 \to \Phi_G(G' \cap G'') \to \Phi_G(G') \oplus \Phi_G(G'') \to \Phi_G(G' \cup G'') \to 0.$$

- If $\mathcal{C}$ is a *triangulated category*, one can consider those summing functors in $\Sigma_{\mathcal{C}}(G)$ such that, for all $G', G'' \in P(G)$, one has a Mayer–Vietoris type distinguished triangle

$$\Phi_G(G' \cap G'') \to \Phi_G(G') \oplus \Phi_G(G'') \to \Phi_G(G' \cup G'') \to \Phi_G(G' \cap G'')[1].$$

This choice determines a subcategory $\Sigma_{\mathcal{C}}^{\rm incl/excl}(G) \subset \Sigma_{\mathcal{C}}(G)$ of summing functors that satisfy a form of inclusion-exclusion.

The category of computational systems described in §4.1 does not have the structure needed to formulate this kind of inclusion-exclusion properties, although it is suitable for the grafting conditions described in §2.3.2, but the category of information systems that we will discuss in §5.4 is an abelian category, so this type of summing functors will be relevant in that context.

3 Neural information networks and resources

In the previous section we have been referring to a category $\mathcal{C}$ which has zero object and sum or is a symmetric monoidal category as a "category of resources", with the category of summing functors representing a configuration space parameterizing all the possible assignments of resources to subsets of a set or to subnetworks of a network. In this section we explain more precisely what we mean by "resources". Our discussion here is based primarily on the "mathematical theory of resources" developed in [27] and [40]. This section serves as a general introduction to our understanding of resources, while in the following sections, §4 and §5, we provide some more explicit and directly relevant examples of such categories of resources.

In modeling of networks of neurons, one can consider three different but closely related aspects: the transmission of information with related questions of coding and optimality, the sharing of resources and related issues of metabolic efficiency, and the computational aspects. The third of these characteristics has led historically to the development of the theory of neural networks, starting with the McCulloch–Pitts model of the artificial neuron [84] in the early days of cybernetics research, all the way to the contemporary very successful theory of deep learning [52]. For the first two aspects mentioned above, a good discussion of the computational neuroscience background can be found, for instance, in [99]. One of our goals is to present ways of modeling the assignment to a network of resouces describing its computational capacity, in terms of concurrent and distributed computing architectures, consistently with informational and metabolic constraints.

3.1 Networks with informational and metabolic constraints

We consider here a kind of neuronal architecture consisting of populations of neurons exchanging information via synaptic connections and action potentials, subject to a tension of two different kinds of constraints: metabolic efficiency and coding efficiency for information transmission. As discussed in §4 of [99], metabolic efficiency and information rate are inversely related. The problem of optimizing both simultaneously is reminiscent of another similar problem of coding theory: the problem of simultaneous optimization, in the theory of error-correcting codes, between efficient encoding (code rate) and efficient decoding (relative minimum distance). For a discussion of error-correcting codes in the context of neural networks, see [72]. In order to model the optimization of resources as well as of information transmission, we rely on a categorical framework for a general mathematical theory of resources, developed in [27] and [40], and on a categorical formulation of information loss [5], [6], [76]. Before discussing the relevant categorical framework, we give a very quick overview of the main aspects of the neural information setting, for which we refer the readers to [99] for a more detailed presentation.

3.1.1 Types of neural codes

There are different kinds of neural codes. There are binary codes that account only for the on/off information of which neurons in a given population/network are firing. In these binary codes, each code word is a binary string of some length N, which represents the total number of time intervals Δt considered. There is one code word for each neuron in the given neuron population, with the i-th entry equal to 0 or 1 depending on whether that neuron has been firing during the i-th time interval. Thus, we can view the code words as a binary (and coarse-grained by the choice of Δt) representation of the spike train of the individual neurons. Comparing the i-th entry of all the code words shows which neurons in the population considered have been simultaneously firing during that time interval. This type of code allows for an interesting connection to homotopy theory through a reconstruction of the homotopy type of the stimulus space from the code, see [29], [73]. Different types of coding are given by rate codes, where the input information is encoded in the firing rate of a neuron, by spike timing codes, where the precise timing of spikes carries information, and by correlation codes that use both the probability of a spike and the probability of a specific time interval from the previous spike.

3.1.2 Spikes, coding capacity, and firing rate

Using a Poisson process to model spike generation, spikes are regarded as mutually independent, given a firing rate of y spikes per second. All long spike trains generated at that firing rate are equiprobable. The information contained in a spike train is computed by the logarithm of the number of different ways of rearranging the number n of spikes in the total number N of basic time intervals considered. The neural coding capacity (the maximum coding rate R for a given firing rate y) is given by the output entropy H divided by the basic time interval Δt. This can be approximated (§3.4 of [99]) by $R_{\max} = -y \log(y\Delta t)$.

3.1.3 Metabolic efficiency and information rate

One defines the metabolic efficiency of a transmission channel as the ratio $\epsilon = I(X, Y)/E$ of the mutual information $I(X, Y)$ of output Y and input X to the energy cost E per unit of time, where the energy cost is a sum of the energy required to maintain the channel and the signal power. The latter represents the power required to generate spikes at a given firing rate. The energy cost of a spike depends on whether the neuron axon is myelinated or not, and in the latter case on the diameter of the axon. A discussion of optimal distribution of axon diameters is given in §4.7 of [99].

This description of metabolic efficiency shows in particular that an assignment of informational resources (in the form of mutual information measurements) to a network also governs the assignment of metabolic resources, once the data about the channels that determine the energy costs E are assumed as known. This provides an example of interdependence between different types of resources, which we will be discussing more extensively in §4 and §5.

3.1.4 Connection weights and mutual information

Over a fixed time interval T subdivided into N discrete steps Δt, and a population of K neurons that respond to a stimulus, the output can be encoded as a $K \times N$ matrix $X = (x_{k,n})$, where the $x_{k,n}$ entry records the output of the k-th neuron during the n-th time interval in response to the stimulus. When this output is transmitted to a next layer of R cells (for example, in the visual system, the output of a set of cones transmitted to a set of ganglion cells) an $R \times K$ weight matrix $W = (w_{r,k})$ assigns weights $w_{r,k}$ to each connection so that the next input is computed by $y_{r,n} = \sum_{k=1}^K w_{r,k} x_{k,n}$. Noise on the transmission channel is modeled by an additional term, $\eta = (\eta_{r,n})$ given by a random variable so that $Y = WX + \eta$. The optimization with respect to information transmission is formulated as the weights W that maximize the mutual information $I(X, Y)$ of output and input.

We see here another example of the interdependence between different types of resources assigned to a network, where informational resources depend on underlying resources of weighted codes, as we will discuss more in detail in §5.

3.1.5 Resources and constraints

In all the examples described above, one can see certain kinds of *resources* associated to a network (energy and metabolic resources, neural codes, information) subject to *constraints*, which are either intrinsic to a certain kind of resurce or that involve the relation between different kinds of resources (such as the relation between metabolic efficiency and information rate). What we want to argue in the rest of this section is the fact that a categorical framework is especially suitable for describing resources and assignments of resources to networks, in the form of symmetric monoidal categories of resources and summing functors that describe the assignments to networks. The categorical language also provides a setting for describing constraints and relations between resources, in the form of functors between categories of resources and universal properties, which are a way of categorically describing optimality constraints.

3.2 The mathematical theory of resources

A general mathematical setting for a theory of resources was developed in [27] and [40]. We recall here the main setting and the relevant examples we need for the context of neural information.

A theory of resources, as presented in [27], is a symmetric monoidal category $(\mathcal{R}, \circ, \otimes, \mathbb{I})$, where the objects $A \in \mathrm{Obj}(\mathcal{R})$ represent resources. The product $A \otimes B$ represents the combination of resources A and B, with the unit object $\mathbb{I}$ representing the empty resource. The morphisms $f : A \to B$ in $\mathrm{Mor}_{\mathcal{R}}(A, B)$ represent possible conversions of resource A into resource B. In particular, no-cost resources are objects $A \in \mathrm{Obj}(\mathcal{R})$ such that $\mathrm{Mor}_{\mathcal{R}}(\mathbb{I}, A) \neq \emptyset$ and freely disposable resources are those objects for which $\mathrm{Mor}_{\mathcal{R}}(A, \mathbb{I}) \neq \emptyset$. The composition of morphisms $\circ : \mathrm{Mor}_{\mathcal{R}}(A, B) \times \mathrm{Mor}_{\mathcal{R}}(B, C) \to \mathrm{Mor}_{\mathcal{R}}(A, C)$ represents the sequential conversion of resources.

3.2.1 Examples of resources

Among the cases relevant to us are the two examples based on classical information mentioned in [27], and another example of [27] more closely related to the setting of [76].

- *Resources of randomness*: the category $\mathcal{R} = \mathrm{FinProb}$ has objects the pairs (X, P) of a finite set X with a probability measure $P = (P_x)_{x \in X}$ with $P_x \geq 0$ and $\sum_{x \in X} P_x = 1$, and with morphisms $\mathrm{Mor}_{\mathcal{R}}((X, P), (Y, Q))$ the maps $f : X \to Y$ satisfying the measure-preserving property $Q_y = \sum_{x \in f^{-1}(y)} P_x$, and with product $(X, P) \otimes (Y, Q) = (X \times Y, P \times Q)$ with unit $(\{*\}, 1_*)$ a point set with measure 1.
- *Random processes*: the category $\mathcal{R} = \mathrm{FinStoch}$ with objects the finite sets X and maps given by stochastic matrices $S = (S_{yx})_{x \in X, y \in Y}$ with $S_{yx} \geq 0$ for all $x \in X$ and $y \in Y$ and $\sum_{y \in Y} S_{yx} = 1$ for all $x \in X$.
- *Partitioned process theory*: the category considered in this case is the coslice category $\mathbb{I}/\mathcal{R}$ of objects of $\mathcal{R}$ under the unit object. This has objects given by the morphisms $f : \mathbb{I} \to A$, for $A \in \mathrm{Obj}(\mathcal{R})$, and morphisms

$$\mathrm{Mor}_{\mathbb{I}/\mathcal{R}}((f : \mathbb{I} \to A), (g : \mathbb{I} \to B)) = \{(\xi : A \to B) \in \mathrm{Mor}_{\mathcal{R}}(A, B) \,|\, \xi \circ f = g\}.$$

  The category $\mathcal{FP}$ of [76] has objects (X, P) the pairs of a finite set with a probability distribution $P = (P_x)_{x \in X}$ and morphisms $\mathrm{Mor}_{\mathcal{FP}}((X, P), (Y, Q))$ given by the stochastic maps $S = (S_{y,x})_{x \in X, y \in Y}$ such that $Q = SP$. It is the coslice category $\mathcal{FP} = \mathbb{I}/\mathrm{FinStoch}$ with FinStoch the category of stochastic processes as in the previous example.

In this last example, partitioned processes in [27] describe a theory of processes (resources and their conversions, described by a symmetric monoidal category $\mathcal{C}$) together with a subtheory of "free processes". No-cost resources are precisely those objects of $\mathcal{C}$ that have a morphism from the unit object, and "states" for this subtheory are described by processes with input the unit object.

3.2.2 Convertibility of resources

The question of convertibility of a resource A to a resource B is formulated as the question of whether the set $\mathrm{Mor}_{\mathcal{R}}(A,B) \neq \emptyset$. Thus, to the symmetric monoidal category $(\mathcal{R}, \circ, \otimes, \mathbb{I})$ of resources, one can associate a preordered abelian monoid $(R, +, \succeq, 0)$ on the set R of isomorphism classes of $\mathrm{Obj}(\mathcal{R})$, with $[A]+[B]$ the class of $A \otimes B$ with unit 0 given by the class of the unit object $\mathbb{I}$ and with $[A] \succeq [B]$ iff $\mathrm{Mor}_{\mathcal{R}}(A,B) \neq \emptyset$. The partial ordering is compatible with the monoid operation: if $[A] \succeq [B]$ and $[C] \succeq [D]$ then $[A]+[C] \succeq [B]+[D]$.

The maximal conversion rate $\rho_{A\to B}$ between resources $A, B \in \mathrm{Obj}(\mathcal{R})$ is given by

$$\rho_{A\to B} := \sup \left\{ \frac{m}{n} \,\middle|\, n\cdot[A] \succeq m\cdot[B],\ m,n \in \mathbb{N} \right\}, \tag{3.1}$$

where $n\cdot[A] \in R$ is the class of $A^{\otimes n}$. It measures the optimal (maximal) fraction of number of copies of resource B that can be produced by A.

Given an abelian monoid with partial ordering $(S, *, \geq, 1_S)$, an S-valued measuring of $\mathcal{R}$-resources is a monoid homomorphism $M : (R, +, 0) \to (S, *, 1_S)$ such that $M(A) \geq M(B)$ in S whenever $[A] \succeq [B]$ in R. (Here and below we write $M(A)$ as shorthand for $M([A])$.)

For $(S, *) = (\mathbb{R}, +)$ and $M : (R, +) \to (\mathbb{R}, +)$ a measuring monoid homomorphism, we have (Theorem 5.6 of [27])

$$\rho_{A\to B} \cdot M(B) \leq M(A),$$

that is, the optimal fraction of copies of resource B that one can obtain using resource A is not bigger than the value of A relative to the value of B.

3.2.3 Information loss

A characterization of information loss is given in [5] as a map $F : \mathrm{Mor}_{\mathrm{FinProb}} \to \mathbb{R}$ satisfying

1. additivity under composition $F(f \circ g) = F(f) + F(g)$;
2. convex linearity $F(\lambda f \oplus (1-\lambda) g) = \lambda F(f) + (1-\lambda) F(g)$ for $0 \leq \lambda \leq 1$ and for $\lambda f \oplus (1-\lambda) g : (X \sqcup Y, \lambda P \oplus (1-\lambda) Q) \to (X' \sqcup Y', \lambda P' \oplus (1-\lambda) Q')$ the convex combination of morphisms $f : (X, P) \to (X', P')$ and $g : (Y, Q) \to (Y', Q')$ in FinProb;
3. continuity of F over $\mathrm{Mor}_{\mathrm{FinProb}}$.

The Khinchin axioms for the Shannon entropy can then be used to show that an information-loss functional satisfying these properties is necessarily of the form $F(f) = C \cdot (H(P) - H(Q))$ for some $C > 0$ and for $H(P) = -\sum_{x\in X} P_x \log P_x$ the Shannon entropy. When working with the category $\mathcal{FP} = \mathbb{I}/\mathrm{FinStoch}$, a similar characterization of information loss using the Khinchin axioms for the Shannon entropy is given in §3 of [76].

3.3 Adjunction and optimality of resources

The discussion in this subsection is not directly needed for our main goal in this paper, but it is included here because it provides a better intuition on how to think of optimization processes in categorical terms.

Suppose then that we have a category $\mathcal{C}$ as above that models distributed/concurrent computational architecture (a category of transition systems or of higher dimensional automata, see §4 below). We also assume that we have a category $\mathcal{R}$ describing metabolic or informational resources. The description of the resource constraints associated to a given automaton is encoded in a strict symmetric monoidal functor $\rho : \mathcal{C} \to \mathcal{R}$. The property of being strict symmetric monoidal here encodes the requirement that independent systems combine with combined resources.

A stronger property would be to require that the functor $\rho : \mathcal{C} \to \mathcal{R}$ that assigns resources to computational systems has a left adjoint, a functor $\beta : \mathcal{R} \to \mathcal{C}$ such that for all objects $C \in \mathrm{Obj}(\mathcal{C})$ and $A \in \mathrm{Obj}(\mathcal{R})$ there is a bijection

$$\mathrm{Mor}_{\mathcal{C}}(\beta(A), C) \simeq \mathrm{Mor}_{\mathcal{R}}(A, \rho(C)). \tag{3.2}$$

The meaning of the left-adjoint functor and the adjunction formula (3.2) can be understood as follows. In general an adjoint functor is a solution to an optimization problem. In this case the assignment $A \mapsto \beta(A)$ via the functor $\beta : \mathcal{R} \to \mathcal{C}$ is an optimal way of assigning a computational system $\beta(A)$ in the category $\mathcal{C}$ to given constraints on the available resources, encoded by the object $A \in \mathrm{Obj}(\mathcal{R})$. The optimization is expressed through the adjunction (3.2), which states that any possible conversion of resources from A to the resources $\rho(C)$ associated to a system $C \in \mathrm{Obj}(\mathcal{C})$ determines in a unique way a corresponding modification of the system $\beta(A)$ into the system C. Note, moreover, that the system $\beta(A)$ is constructed from the assigned resources $A \in \mathrm{Obj}(\mathcal{R})$, and since some of the resources encoded in A are used for the manufacturing of $\beta(A)$ one expects that there will be a conversion from A to the remaining resources available to the system $\beta(A)$, namely $\rho(\beta(A))$. The existence of the left-adjoint $\beta : \mathcal{R} \to \mathcal{C}$ (hence the possibility of solving this optimization problem) is equivalent to the fact that the conversion of resources $A \to \rho(\beta(A))$ is the initial object in the category $A \downarrow \rho$. Here, for an object $A \in \mathrm{Obj}(\mathcal{R})$ the comma category $A \downarrow \rho$ of objects ρ-under A has objects the pairs (u, C) with $C \in \mathrm{Obj}(\mathcal{C})$ and $u : A \to \rho(C)$ a morphism in $\mathcal{R}$ and morphisms $\phi : (u_1, C_1) \to (u_2, C_2)$ given by morphisms $\phi \in \mathrm{Mor}_{\mathcal{C}}(C_1, C_2)$ such that one has the commutative diagram

$$\begin{array}{ccc} & A & \\ {\scriptstyle u_1}\swarrow & & \searrow{\scriptstyle u_2} \\ \rho(C_1) & \xrightarrow{\rho(\phi)} & \rho(C_2)\,. \end{array}$$

Freyd's adjoint functor theorem gives a condition for the existence of a left-adjoint functor for a continuous functor $\rho : \mathcal{C} \to \mathcal{R}$, in the form of a completeness condition on the category $\mathcal{C}$ and the existence of a *solution set* in the comma category $A \downarrow \rho$. We briefly discuss what this result means in our setting.

The functor $\rho : \mathcal{C} \to \mathcal{R}$ is continuous if it commutes with limits. This is a reasonable assumption to make regarding the functor that assigns to a computational system C in the category $\mathcal{C}$ its resources in the category $\mathcal{R}$. As discussed in §3 of [89], categorical limits are solutions to constrained optimization problems that generalize to the categorical setting the usual notion of infimum (indeed the categorical limit agrees with the notion of greatest lower bound in the case of a category given by a poset). Requiring that the functor that assigns resources to systems is continuous means requiring that it preserves the optimization properties encoded in categorical limits.

The completeness of the category $\mathcal{C}$ depends on which models of concurrent and distributed computing we are considering in the category $\mathcal{C}$. We will be working broadly with the framework of a category $\mathcal{C}$ of transition systems introduced in [110] as a model for computational architectures, see §4. However, one can focus on more specific categorical models of concurrency. For example, among the categories considered in [110], the category of synchronization trees has infinite products and pullbacks, hence it is also complete.

If our category $\mathcal{C}$ is complete, as in the cases mentioned above, and the functor $\rho : \mathcal{C} \to \mathcal{R}$ preserves infinite products and equalizers, then the comma category $A \downarrow \rho$ is also complete for all objects $A \in \mathrm{Obj}(\mathcal{R})$. In this case Freyd's adjoint functor theorem then shows that the existence of an initial object in the category $A \downarrow \rho$ (hence the existence of a left-adjoint functor $\beta : \mathcal{R} \to \mathcal{C}$ for $\rho : \mathcal{C} \to \mathcal{R}$) follows from the existence of a *solution set*, that is, a set $\{T_j = (u_j, C_j)\}_{j \in J}$ of objects of $A \downarrow \rho$ such that every object $T = (u, C) \in \mathrm{Obj}(A \downarrow \rho)$ admits a morphism $f_j : T_j \to T$ for some $j \in J$.

The existence of a solution set can be interpreted in the following way. If we fix the resources by choosing an object $A \in \mathrm{Obj}(\mathcal{R})$, there is a set $\{C_j\}_{j \in J}$ of systems in $\mathcal{C}$ together with conversion of resources $u_j : A \to \rho(C_j)$ with the property that, for any system $C \in \mathrm{Obj}(\mathcal{C})$ for which there is a possible conversion of resources $u : A \to \rho(C)$ in $\mathrm{Mor}_{\mathcal{R}}(A, \rho(C))$, there is one of the systems C_j and a modification of systems $\phi : C_j \to C$ in $\mathrm{Mor}_{\mathcal{C}}(C_j, C)$ such that the conversion of resources $u : A \to \rho(C)$ factors through the system C_j, namely $u = \rho(\phi) \circ u_j$. One can therefore think of the solution set $\{(u_j, C_j)\}_{j \in J}$ as being those systems in $\mathcal{C}$ that are optimal with respect to the resources A, from which any other system that uses less resources than A can be obtained via modifications.

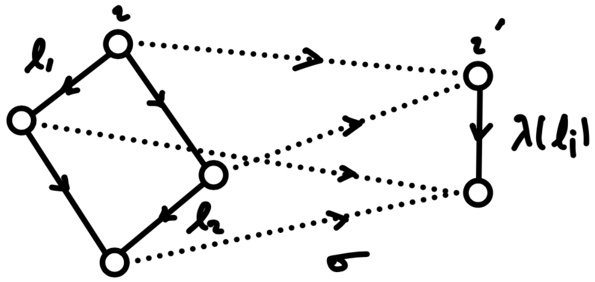


Figure 1: Example of a morphism of transition systems.

4 Networks with computational structures

In this section we focus on assignments of computational resources to a network, which we think of as computational models of individual nodes (neurons) of the network, together with prescriptions for their wiring together according to the network structure. As in the previous section, we aim at constructing a configuration space of all such possible assignments within which one can choose an initial assignment and prescribe a dynamical evolution. We will deal with the dynamical aspect in §6. Here we introduce a suitable category of computational resources, aimed at accommodating a sufficiently broad and flexible range of models of concurrent and distributed computing, in the form of automata describing transition systems. We then investigate the compositional structure that gives the compatibility of these assignments over the network. We discuss some related questions, including how to incorporate some computational models of neuromodulation based on a subcategory of the category of transition systems given by time-delay automata.

4.1 Transition systems: a category of computational resources

We consider here, as a special case of categories of resources, in the sense of [27] and [40] recalled above, a category of "reactive systems" in the sense of [110]. These describe models of computational architectures that involve parallel and distributed processing, including interleaving models such as synchronization trees and concurrency models based on causal independence. Such computational systems can be described in categorical terms, formulated as a category of transition systems [110]. The products in this category of transition systems represent parallel compositions where all possible synchronizations are allowed. More general parallel compositions are then obtained as combinations of products, restrictions and relabeling. The coproducts in the category of transition systems represent (non-deterministic) sums that produce a single process with the same computational capability of two or more separate processes.

In the most general setting, a category $\mathcal{C}$ of transition systems has objects given by data of the form $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ where S is the set of possible states of the system, ι is the initial state, $\mathcal{L}$ is a set of labels, and $\mathcal{T}$ is the set of possible transition relations of the system, $\mathcal{T} \subseteq S \times \mathcal{L} \times S$ (specified by pre state, label of the transition, and post state). A transition system $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ also has a set $S_F \subset S$ of final states. Such a system can be represented in graphical notation as a directed graph with vertex set S and with set of labeled directed edges $\mathcal{T}$. Morphisms $\mathrm{Mor}_{\mathcal{C}}(\tau, \tau')$ in the category $\mathcal{C}$ of transition systems are given by pairs (σ, λ) consisting of a function $\sigma : S \to S'$ with $\sigma(\iota) = \iota'$ and $\sigma(S_F) \subset S'_F$, and a (partially defined) function $\lambda : \mathcal{L} \to \mathcal{L}'$ of the labeling sets such that, for any transition $s_{in} \stackrel{\ell}{\to} s_{out}$ in $\mathcal{T}$, if $\lambda(\ell)$ is defined, then $\sigma(s_{in}) \stackrel{\lambda(\ell)}{\to} \sigma(s_{out})$ is a transition in $\mathcal{T}'$.

Heuristically, a morphism $(\sigma, \lambda) \in \mathrm{Mor}_{\mathcal{C}}(\tau, \tau')$ describes the fact that the system τ' can partially simulate the system τ, where "partially" is determined according to λ, see [87]. A simple explicit example of a morphism of transition systems is given graphically in Figure 1 (see [87]).

As shown in [110], the category $\mathcal{C}$ has a coproduct given by

$$(S, \iota, \mathcal{L}, \mathcal{T}) \sqcup (S', \iota', \mathcal{L}', \mathcal{T}') = (S \times \{\iota'\} \cup \{\iota\} \times S', (\iota, \iota'), \mathcal{L} \cup \mathcal{L}', \mathcal{T} \sqcup \mathcal{T}') \tag{4.1}$$

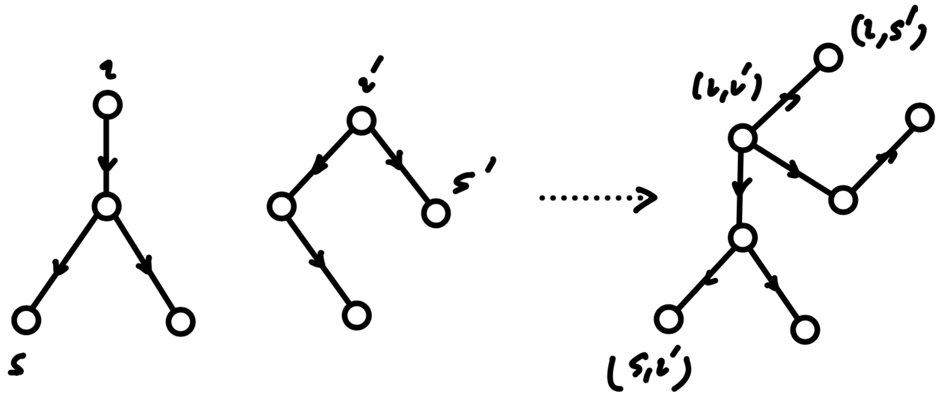


Figure 2: A simple example of coproduct of two transition systems.

$$\mathcal{T} \sqcup \mathcal{T}' := \{(s_{in}, \ell, s_{out}) \in \mathcal{T}\} \cup \{(s'_{in}, \ell', s'_{out}) \in \mathcal{T}'\},$$

where both sets are seen as subsets of

$$(S \times \{\iota'\} \cup \{\iota\} \times S') \times (\mathcal{L} \cup \mathcal{L}') \times (S \times \{\iota'\} \cup \{\iota\} \times S').$$

This coproduct $(S, \iota, \mathcal{L}, \mathcal{T}) \sqcup (S', \iota', \mathcal{L}', \mathcal{T}')$ satisfies the universal property of a categorical sum. The zero object is given by the stationary single-state system $S = \{\iota\}$ with empty labels and transitions. There is also a product structure on $\mathcal{C}$ given by

$$(S \times S', (\iota, \iota'), \mathcal{L} \times \mathcal{L}', \Pi),$$

where the product transition relations are determined by $\Pi = \pi^{-1}(\mathcal{T}) \cap \pi'^{-1}(\mathcal{T}')$, for the projections $\pi : S \times S' \to S$ and $\pi : \mathcal{L} \times \mathcal{L}' \to \mathcal{L}$ and $\pi' : S \times S' \to S'$ and $\pi' : \mathcal{L} \times \mathcal{L}' \to \mathcal{L}'$.

The coproduct of two transition systems is illustrated graphically in a simple example in Figure 2. As observed in §2.2.5 of [110], this categorical sum in the category $\mathcal{C}$ of transition systems represents a system that can behave as any one of its summands.

Note that (4.1) is a categorical coproduct only in the case of labeled transition systems with marked initial state. In the case where there is also a marked final state, this is no longer the case, but one can still define a monoidal structure.

A version of probabilistic transition systems is discussed in the Appendix, in §A.3.

4.2 Computational architectures in neuronal networks

We first review here some ideas about computational models for single neurons and how they can be made to fit with the very broad description of computational architectures provided by the category of transition systems.

In this context we can treat a computational model for a single neuron in terms of a sequence of simplifying steps. These follow the discussion in the introduction of [62].

- *Discretization in space* makes it possible to subdivide a neuron into separate "modules", and replace a model of the relevant quantities such as membrane voltage in terms of a set of PDEs into a model in terms of ODEs. This is a classical simplification of the problem, which leads to the well-known Hodgkin–Huxley model [61].
- *Discretization in time* further replaces the continuum-time ODE with a discrete dynamical system. We will discuss again this kind of step in relation to our categorical Hopfield network dynamics in §6.
- *Discretization in field values* then makes it possible to model the discrete dynamical system in terms of a finite state automaton.

If we follow this outline as in [62], then we would be assigning to single neurons (vertices $v \in V = V_G$ in the network) corresponding finite state automata. These are particular cases of the more general objects in the category of transition systems of [110] described in §4.1.

Another model of the computational structure of a single neuron is developed in [14]. In this model the input-output mapping complexity of neurons is investigated by identifying deep neural networks that can be trained to faithfully replicate the input-output function of various types of cortical neurons at millisecond spiking resolution. So for example a layer-5 cortical pyramidal cell requires a convolutional deep neural network with five to eight layers, while a minimal deep neural network with a single hidden layer suffices for the simple integrate-and-fire neuron model. In this case, the computational structures associated to (different types of) neurons are deep neural networks. Thus, in order to cast this model into our framework, one needs to formulate the right categorical structure describing compositional roles of neural networks and a relation to the category of transition systems described above. This will appear in a separate paper [78], so we will not include the discussion here, but we can direct the reader to [37], [38], [46] for some of the relevant categorical setting for deep neural networks.

There is also another possible approach to assigning a computational system to the individual neurons, as suggested in [13], by considering the system of ion-gated channels in the membrane as a concurrent computing system where synaptic inputs interact to modulate activity with shared resources (represented by different ion densities and thresholds), regarded as a system of interacting synaptic "programs". We do not develop this model in the present paper, but this would be a very natural approach in view of representing the entire computational architecture of the network in terms of concurrent/distributed computing. Such models would also fit within the category of transition systems described above, and with dynamical models of interacting neuron populations such as [66].

4.3 Computational architectures and network summing functors

We now look more closely at categories of network summing functors, as discussed in §2, where the target category is the category of transition systems of [110] that we recalled in §4.1 above. In particular we will discuss what specific conditions on network summing functors it is reasonable to require in such a model, or equivalenty what subcategory of $\Sigma_{\mathcal{C}}(G)$ one wants to focus on, with additional structure that takes into account local and larger-scale connections in the network. In particular, we show that a model of network summing functors based on grafting operations, similar to what we discussed more abstractly in §2.3.2 is especially suitable for assignments of computational resources to networks in the form of transition systems. A model of assignment of resources more directly built on the properad grafting operations of §2.3.2 will be discussed in a separate paper [78], in relation to the deep neural networks model of computational resources of individual neurons of [14].

4.3.1 Transition systems and network summing functor

Let $\mathcal{C}$ be the category of transition systems of [110] described in §4.1 Let $\mathcal{G} := \mathrm{Func}(\mathbf{2}, \mathcal{F})$ be the category of finite directed graphs. As before, for $G \in \mathrm{Obj}(\mathcal{G})$ we denote by G_* the associated pointed graph. For simplicity we write $\Sigma_{\mathcal{C}}(V_G)$ instead of $\Sigma_{\mathcal{C}}(V_{G_*})$ with the pointed structure implicitly understood.

Definition 4.1 *For $i = 1, 2$ let $\tau_i = (S_i, \iota_i, \mathcal{L}_i, \mathcal{T}_i)$ be objects in the category $\mathcal{C}$ of transition systems. Given a choice of two states $s \in S_1$ and $s' \in S_2$, the grafting of τ_1 and τ_2 is the object $\tau_{s,s'} = (S, \iota, \mathcal{L}, \mathcal{T})$ in $\mathcal{C}$ with $S = S_1 \sqcup S_2$, $\iota = \iota_1$, $\mathcal{L} = \mathcal{L}_1 \sqcup \mathcal{L}_2 \sqcup \{e\}$ and $\mathcal{T} = \mathcal{T}_1 \sqcup \mathcal{T}_2 \sqcup \{(s, e, s')\}$. Let $\mathcal{C}' \subset \mathcal{C}$ be the subcategory of transition systems τ that have a single final state $S_F = \{q\} \subset S$. For $\tau_i \in \mathrm{Obj}(\mathcal{C}')$, the grafting $\tau_1 \star \tau_2$ is simply defined as the grafting τ_{q_1,ι_2} with the final state of τ_1 grafted to the initial state of τ_2.*

A topological ordering ω of the vertices of a directed acyclic graph G is a linear ordering of the set of vertices such that, whenever there is a directed edge e with $s(e) = v$ and $t(e) = v'$ then $v \leq v'$ in the ordering, that is, a monotone map from the underlying poset of the vertices to a linear order.

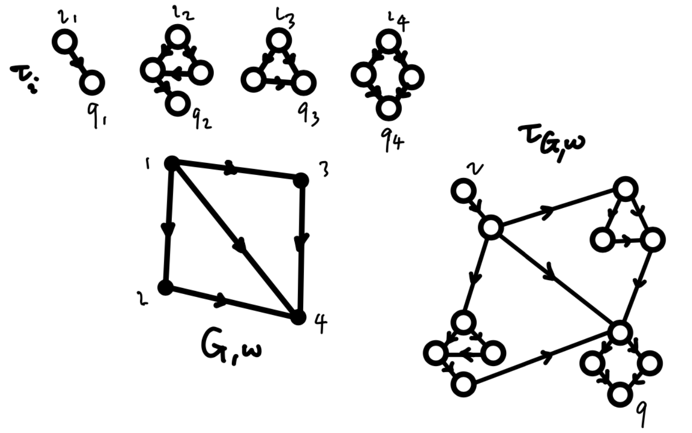


Figure 3: A simple example of the grafting operation of Lemma 4.2.

Lemma 4.2 *Let G be a finite acyclic directed graph with vertex set $V = V_G$. Let ω be a topological ordering of the vertex set V. Suppose given a collection $\{\tau_v\}_{v\in V}$ of objects in the subcategory $\mathcal{C}'$ of $\mathcal{C}$. There is a well-defined grafting $\tau_{G,\omega}$ of the τ_v that is also an object in $\mathcal{C}'$.*

Proof. For $v \in V$, we have $\tau_v = (S_v, \iota_v, \mathcal{L}_v, \mathcal{T}_v)$. Since τ_v is in $\mathcal{C}'$, the set S_v contains a unique final state q_v. Let v_{in} denote the first vertex and v_{out} the last vertex in the topological ordering ω. The object $\tau_{G,\omega} = (S, \iota, \mathcal{L}, \mathcal{T})$ has $S = \cup_{v\in V} S_v$ with initial state $\iota = \iota_{v_{in}}$ and final state $q = q_{v_{out}}$. The set of labels is given by $\mathcal{L} = \cup_{v\in V}\mathcal{L}_v \cup E$ with $E = E_G$ the set of edges of G and transitions $\mathcal{T} = \cup_{v\in V}\mathcal{T}_v \cup \{(q_{s(e)}, e, \iota_{t(e)}) \,|\, e \in E\}$ with $s(e), t(e)$ the source and target vertices of e. □

The grafting operation of Lemma 4.2 is illustrated in a simple example in Figure 3.

For an arbitrary finite directed graph G, a strongly connected component is a subset V' of the vertex set V_G such that each of the vertices in V' is reachable through an oriented path in G from any other vertex in V', and which is maximal with respect to this property. The strongly connected components determine a partition of V_G. The condensation graph $\bar{G}$ is a directed acyclic graph that is obtained from G by contracting each strongly connected component (consisting of the vertices of the component and all the edges between them) to a single vertex. Given two strongly connected components $X \neq Y$, there is an edge $e_{X,Y}$ connecting the corresponding vertices in the condensation graph $\bar{G}$ if there is an edge $e_{v,w}$ in G for some $v \in X$ and $w \in Y$.

There are algorithms that construct a topological ordering on a directed acyclic graph in linear time, such as the Kahn algorithm [64]. For a given directed graph G we write $\bar{\omega}$ for the topological ordering of its condensation graph $\bar{G}$ obtained through the application of a given such algorithm.

Definition 4.3 *Let G be a strongly connected graph and let $\{\tau_v\}_{v\in V_G}$ be a collection of objects $\tau_v = (S_v, \iota_v, \mathcal{L}_v, \mathcal{T}_v)$ in $\mathcal{C}'$ with q_v the respective final states. For a given pair (v_{in}, v_{out}) in $V_G \times V_G$ let $\tau_{G,v_{in},v_{out}} = (S, \iota, \mathcal{L}, \mathcal{T})$ be the object in $\mathcal{C}'$ with $S = \cup_{v\in V_G} S_v$, $\mathcal{L} = \cup_{v\in V_G}\mathcal{L}_v \cup E_G$, and $\mathcal{T} = \cup_{v\in V_G}\mathcal{T}_v \cup \{(q_{s(e)}, e, \iota_{t(e)})\}_{e\in E_G}$ and with initial and final state $\iota = \iota_{v_{in}}$ and $q = q_{v_{out}}$. Then set $\tau_G := \oplus_{(v_{in},v_{out})\in V_G\times V_G} \tau_{G,v_{in},v_{out}}$.*

Notice that this definition represents correctly what one heuristically expects to be the grafting for a strongly connected graph. In a transition system a state is reachable if there is a directed path of transitions from the initial state ι to that state. In particular a final state is assumed to be reachable. A transition system is reachable if every state is reachable. Since in the strongly connected case any vertex can be reached via a directed path from any other, then any of the initial states ι_v of the systems τ_v can be taken to be the initial state of the grafting, and any final state q_v can be taken as the final state of the grafting. The grafting τ_G for a strongly connected graph G represents a transition system that can behave as the grafting of the τ_v with any possible pair (ι_v, q_v) as the initial and final state.

In Lemma 4.2 (see also the example in Figure 3) we have described simple grafting operations at vertices. More generally, and more realistically, the grafting should also involve a matching

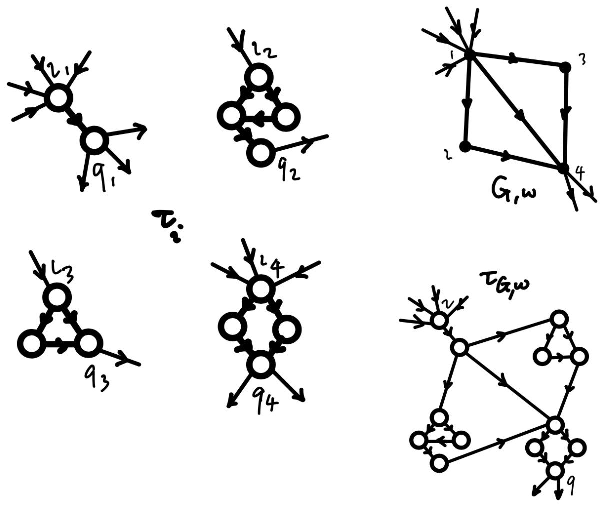


Figure 4: Grafting operation with matching external edges.

of external (half)edges and can be formulated following the setting of §2.3.1 and §2.3.2. The corresponding modifications of Lemma 4.2 is are straightforward. An example illustrating this form of grafting is given in Figure 4. An explicit example where the grafting is directly modeled on Lemma 2.19, with a category of deep neural networks, is discussed in [78].

In this case, we assign to the initial state ι and the final state q an in-degree and an out-degree, respectively. The meaning of these in/out degrees and the attached half-edges is that the output computed at the final state q is made available as pre state on all the outgoing external half-edges, and similarly, the initial state ι is made available as post state on each of the incoming external half-edges. When endowed with these additional data, we can organize the objects of the category $\mathcal{C}'$ into subsets $\mathcal{C}'(n,m)$ consisting of those transition systems τ with $n = \deg^{\rm in}(\iota)$ and $m = \deg^{\rm out}(q)$. The category $\mathcal{C}'$ then has a properad composition that matches outputs to inputs. We now construct an associated category of network summing functors that satisfy grafting conditions, as discussed in §2.3.2.

Proposition 4.4 *Given a network G, there is a faithful functor $\Upsilon : \Sigma_{\mathcal{C}'}(V_G) \to \Sigma^{\rm prop}_{\mathcal{C}'}(G)$, with $\mathcal{C}'$ the subcategory of transition systems of Definition 4.1, with the target category as introduced in §2.3.2, with the $\mathcal{C}'(n,m) \subset \mathcal{C}'$ and the properad composition as described here above.*

Proof. By Lemma 2.3, a summing functor $\Phi \in \Sigma_{\mathcal{C}'}(V_G)$ is completely determined by the assignment of the objects $\Phi(v) \in \mathcal{C}'$. The morphisms are invertible natural transformations that are in turn determined by isomorphisms of these objects. Given $\Phi \in \Sigma_{\mathcal{C}'}(V_G)$ we construct an associated summing functor, $\Upsilon(\Phi)$ in $\Sigma^{\rm prop}_{\mathcal{C}'}(G)$, where the composition operations on the target category $\mathcal{C}'$ are the grafting operations described above in Lemma 4.2 and Definition 4.3, in the modified form that accounts for matching of external edges at the grafting of final and initial state, as discussed above. For $G' \subseteq G$ we set $\Upsilon(\Phi)(G')$ to be equal to the object in $\mathcal{C}'$ obtained as the grafting $\tau_{\bar{G}',\bar{\omega}}$ as in Lemma 4.2 of the objects $\tau_{G'_i}$, with G'_i the strongly connected components of G', with $\tau_{G'_i}$ given by the grafting of Definition 4.3 of the $\Phi(v)$ associated to the vertices of G'_i, once matching of external edges is included (as in Figure 4). Since morphisms in $\Sigma_{\mathcal{C}'}(V_G)$ are given by isomorphisms of the $\Phi(v)$, these induce isomorphisms of the grafted objects, hence invertible natural transformations of the obtained summing functors $\Upsilon(\Phi)$. □

4.4 Larger-scale structures and distributed computing

We have shown in Proposition 4.4 how to obtain a functorial assignment of a computational structure in the category of transition systems of [110] to a network of neurons related by synaptic connections, assuming a computational model for individual neurons is given. This construction is based on a given model of local automata that implement the computational properties of individual neurons with their pre-synaptic and post-synaptic activity (for example the map-based model of [62] or the deep network model of [14]) and on the grafting of these automata into a larger computational structure where their inputs and outputs are connected according to the connectivity of the network.

As discussed in [90], there are larger-scale structures involved in the computational structure of neuronal arrangements beyond what is generated by the pre-synaptic and post-synaptic activity. In particular, non-local neuromodulation bridges between the microscopic and the larger-scale structures and plays a role in synaptic plasticity and learning. These are not captured by the construction of Proposition 4.4. Thus, such phenomena provide a reason why a suitable subcategory of the category $\Sigma_{\mathcal{C}}(G)$ of network summing functors may have to be larger than that accounting for summing functor built from some type of grafting operations (which reflect only the local connectivity of the network).

Neuromodulators are typically generated in neurons in the brainstem and in the basal forebrain and transmitted to several different brain regions via long-range connections. As shown in [90], this kind of larger-scale structure of neuromodulated plasticity, where the neuromodulatory signal is generated within the network, is better accounted for by a distributed computing model. The focus in [90] is on efficient simulation, in a distributed environment, of a neuromodulated network activity. Here we have a somewhat different viewpoint as we are interested in a computational architecture that can be realized by the network with its local and large-scale structure. Nonetheless, the model developed in [90] can be useful in identifying how to go beyond the local structure encoded in the construction given in Proposition 4.4.

4.4.1 Distributed computing model of neuromodulation

The distributed computing model considered in [90] can be summarized as follows:

- The network of neurons and synaptic connections is described by a finite directed graph G.
- The set of vertices $V = V_G$ is partitioned into N subsets V_i, the different machines $\mathbf{m}_i$ of the distributed computing system.
- The set of edges $E = E_G$ is partitioned into the machines $\mathbf{m}_i$ by the rule that an edge e belongs to $\mathbf{m}_i$ iff the target vertex $t(e)$ belongs to V_i.
- One additional vertex $v_{0,i}$ is added into each machine $\mathbf{m}_i$, which accounts for the neuromodulator transmission.
- There is a set $E_{0,i}$ of additional edges connected to the vertices $v_{0,i}$ in $\mathbf{m}_i$: the incoming edges $e \in E_{0,i}$ with $t(e) = v_{0,i}$ can have source vertex $s(e)$ anywhere in the graph G, not necessarily inside $\mathbf{m}_i$, while the outgoing edges $e \in E_{0,i}$ with $s(e) = v_{0,i}$ have their target vertex in the same machine, $t(e) \in V_i$.
- We obtain in this way a new directed graph G_0 obtained from G by adding the vertices $v_{0,i}$ and the edges in the sets $E_{0,i}$.
- The vertices that are sources of edges in $E_{0,i}$ with target $v_{0,i}$ are the neurons that release the neuromodulator, while the edges in $E_{0,i}$ outgoing from $v_{0,i}$ represent the synaptic connections that are neuromodulated. The nodes $v_{0,i}$ collect *globally* the spikes from the neuromodulator releasing neurons and transmits them *locally* to neuromodulated synapses.
- Each edge e in the sets $E_{0,i}$ carries a time delay information d_e (in multiples of the fixed time interval Δt of the discretized dynamics of the system).

If more than one type of neuromodulator is present at the same time, then each neuromodulator determines a (different) partition of G into machines $\mathbf{m}_i$, and a corresponding set of vertices $v_{0,i}$ and edges $E_{0,i}$. Thus one obtains a graph G_0 by adding all of these new vertices and edges for each neuromodulator present in the model. For simplicity we restrict to considering the case of a single modulator.

4.4.2 Network summing functors and automata with time delays

In models of distributed computing one considers in particular a generalization of finite state automata given by *timed automata*, see [1]. In general, these are described as finite state machines with a finite set of real-valued clocks, which can be independently reset with the transitions of the automaton. Transitions can take place only if the current values of the clocks satisfy certain specified constraints.

In order to model the time delays introduced in the neuromodulator model of [90] one does not need this very general form of timed automata. Indeed, it is better in this case to work with the class of *automata with time delay blocks*, developed in [24]. These automata generate a class of formal languages that strictly contains the regular languages and that is incomparable to the context-free languages (as it includes some non-context-free languages while it cannot represent some context-free ones).

In a finite state automaton with time delay blocks, the transitions are labeled by the usual label symbols of the underlying finite state machine, and by an additional symbol given by a non-negative integer number $n \in \mathbb{Z}_+$ which represents the time delay block of that transition. Thus, given a directed path in the directed graph of the finite state automaton starting at the initial state ι, given by a string $(a_1, n_1)(a_2, n_2) \cdots (a_m, n_m)$, the time-zero transition consists of the substring of a_i such that $n_i = 0$, the time-one transition consists of the substring of a_i with $n_i = 1$, and so on. Thus, at time zero the automaton carries out the computation corresponding to the string made by the a_i with $n_i = 0$ (which must be in the regular language of the underlying finite state automaton), and so on for the successive times. The sequence of integer times is usually assumed to be non-decreasing.

For example, an automaton with three states s_0, s_1, s_2, with initial state s_0 and transitions a between s_0 and s_1, b between s_1 and s_2 and c between s_2 and s_0 would produce the $\{(abc)^n : n \in \mathbb{N}\}$ language. However, if one introduces time delays, using timed transitions $(a, 0)$ between s_0 and s_1, $(b, 1)$ between s_1 and s_2 and $(c, 2)$ between s_2 and s_0, then only the symbol with delay $n = 0$ is deposited in the output until time resets to 1, then only the symbol with time 1, until the automaton returns to the state s_0 and time if reset, so that this timed automaton produces a timed language $\{(a, 0)^n (b, 1)^n (c, 2)^n : n \in \mathbb{N}\}$ and the associated untimed language (forgetting the time markings) is now $\{a^n b^n c^n : n \in \mathbb{N}\}$.

We can then modify the construction of Proposition 4.4 to accommodate this kind of model of neuromodulated networks.

Definition 4.5 *Let $\mathcal{C}^t \subset \mathcal{C}' \subset \mathcal{C}$ denote the time-delay subcategory of the category $\mathcal{C}$ of transition systems of §4.1, with objects $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ that have a unique final state q and whose label set is of the form $\mathcal{L} = \mathcal{L}' \times \mathbb{Z}_+$, where $\mathcal{L}'$ is a label set and $n \in \mathbb{Z}_+$ is a time delay block as above. When a time delay is not explicitly written in a transition in $\mathcal{T}$ it is assumed to mean that $n = 0$. These correspond to the usual transition with labeling set $\mathcal{L}'$. As in the case of the subcategory $\mathcal{C}' \subset \mathcal{C}$, we can consider a version of the category $\mathcal{C}^t$ where the objects τ are also endowed with incoming half-edges at the initial state and outgoing half-edges at the final state, with subcategories $\mathcal{C}^t(n, m)$ where $\deg^{\rm in}(\iota) = n$ and $\deg^{\rm out}(q) = m$.*

As described in §4.4.1, we define a distributed structure on a directed graph G as follows.

Definition 4.6 *A distributed structure $\mathbf{m}$ on a finite directed graph G is given by:*

1. *a partition into N machines $\mathbf{m}_i$ as described in §4.4.1,*
2. *two subsets of vertices $V_{s,i}, V_{t,i}$ inside the vertex set V_i of each machine $\mathbf{m}_i$*
3. *a directed graph G_0 with $G_0 \supset G$, obtained by adding*
   - *for all $i = 1, \ldots, N$, a new vertex $v_{0,i}$ to each vertex set V_i, with $V_{\mathbf{m}_i} = V_i \cup \{v_{0,i}\}$*
   - *for all $i = 1, \ldots, N$ and for each vertex $v \in V_{t,i}$ a new edge with source $v_{0,i}$ and target v,*
   - *for all $i, j = 1, \ldots, N$ and for each vertex $v \in V_{s,j}$ a new edge with source v and target $v_{0,i}$,*

- *a non-negative integer* $n_e \in \mathbb{Z}_+$ *assigned to each edge* $e \in E_{G_0}$*, with* $n_e = 0$ *if* $e \in E_G$*.*

Given a pair $(G, \mathbf{m})$ *of a directed graph with a distributed structure, we denote by* $\bar{G}_0(\mathbf{m})$ *the condensation graph obtained by contracting each of the subgraphs* G_i *given by the vertices* V_i *and the edges between them to a single vertex. (Note that the condensation graph* $\bar{G}_0(\mathbf{m})$ *is acyclic.)*

Definition 4.7 *Let* $\mathcal{G}^{\rm dist}$ *be the category with objects* $(G, \mathbf{m})$ *given by a finite directed graph with a distributed structure as in Definition 4.6, with the properties that the induced subgraphs* G_i *of* G_0 *with vertex set* $V_{\mathbf{m}_i}$ *are strongly connected.*

Morphisms $\alpha \in {\rm Mor}_{\mathcal{G}^{\rm dist}}(G, \mathbf{m}), (G', \mathbf{m}'))$ *are given by morphisms* $\alpha : G \to G'$ *of directed graphs that are compatible with the distributed structure, in the sense that the induced morphisms* $\alpha_i = \alpha|_{G_i} : G_i \to G'_{j(i)}$ *map the subgraphs* G_i *of the distributed structure of* G *to the subgraphs* G'_j *of the distributed structure of* G'*.*

Note that we use here, as morphisms of directed graphs the natural transformation of functors in $\mathrm{Func}(\mathbf{2}, \mathcal{F})$ (see Definition 2.6). These morphisms allow for identifications of edges, but not for contractions of edges to vertices. A slight variant of the category $\mathbf{2}$ that also allows for edge contractions is discussed in §2.1.1 of [78].

We then have the suitable modification of the functorial construction of Proposition 4.4 adapted to this setting, where we consider the category $\mathcal{C}^t$ with subcategories $\mathcal{C}^t(n, m)$ as in Definition 4.5 and the properad structure as in the case of $\mathcal{C}'$.

Proposition 4.8 *Given an object* $(G, \mathbf{m})$ *of* $\mathcal{G}^{\rm dist}$*, let* $P(G, \mathbf{m})$ *be the category of subgraphs with compatible distributed structure. Given a summing functor* $\Phi \in \Sigma_{\mathcal{C}^t}(V_G)$ *with values in the time-delay subcategory* $\mathcal{C}^t$*, consider the following procedure:*

- *consider the objects* $\Phi(v)$ *for* $v \in V_{\mathbf{m}_i}$*, the vertex set of the subgraph* G_i*, as in Definition 4.7;*
- *these determine the objects* τ_{G_i} *obtained by grafting as in Definition 4.3;*
- *for* $\bar{G}_0(\mathbf{m})$ *the condensation graph as in Definition 4.6, perform the grafting* $\tau_{\bar{G}_0(\mathbf{m}), \bar{\omega}}$*, as in Lemma 4.2, of the objects* τ_{G_i}*.*

This procedure determines, as in Proposition 4.4, a summing functor

$$\Upsilon(\Phi) \in \Sigma^{\rm prop}_{\mathcal{C}^t}(G, \mathbf{m}),$$

which assigns to an object $(G'\mathbf{m}')$ *in* $P(G, \mathbf{m})$ *the object in* $\mathcal{C}^t$ *given by the grafting* $\tau_{\bar{G}_0(\mathbf{m}), \bar{\omega}}$*.*

4.4.3 Topological questions

An interesting mathematical question is then to describe the topological structure, in terms of protocol simplicial complexes, of the distributed computing algorithm implementing a neuromodulated network, and to investigate how the topology of the resulting protocol simplicial complexes are related to other topological structures we have been considering in this paper. We leave this question to future work.

There is a further interesting aspect to the larger-scale structures of the network and its computational properties. As pointed out in [90], the usual analysis of networks in neuroscience is based on the abstract connectivity properties of the network as a directed graph without any information on its embedding in 3-dimensional space. Topologically it is well known that embedded graphs are at least as interesting as knots and links and capture subtle topological properties of the ambient space that are not encoded in the structure of the graph itself, but in the embedding. We will not be developing this aspect in the present paper, but it is an interesting mathematical question to identify to what extent invariants of *embedded graphs*, such as the fundamental group of the complement (as in the case of knots and links), can carry relevant information about the informational and computational structure of the network beyond the local connectivity structure.

5 Codes, probabilities, and information

In this section we show a toy model construction, where we use the setting of categories of network summing functors described in §2 to describe functorial assignments of codes to neurons in a network and of associated probabilities and information measures. This shows a possible way of describing informational constraints in a network of neurons.

5.1 Introducing neural codes

There is an additional part of the modeling of a neural information network which we have not introduced in our construction yet. Neurons transmit information by generating a spike train, with a certain firing rate. As discussed in [99], the spiking activity can be described in terms of a binary code, in the following way. Let $T > 0$ be a certain interval of observation time, during which one records the spiking activities. We assume it is subdivided into multiples of some unit of time Δt, with $n = T/\Delta t$ the number of basic time intervals considered. Assuming that Δt is sufficiently small, so that one does not expect a time interval of length Δt to contain more than one spike, one can assign a digital word of length n to an observation by recording a digit 1 for each time interval Δt that contained a spike and a 0 otherwise. When k observations are repeated, one obtains k binary words of length n, that is, a binary code $C \subset \mathbb{F}_2^n$. We assume that the neurons generate spikes at a given rate y of spikes per second. This rate is computed from observations as the number m of spikes observed per observation time, $y = m/T$.

5.1.1 Firing rates of codes

For sufficiently large T (hence for large n), the empirical estimate $y\Delta t = m/n$ of observing a spike in a time interval Δt will approximate a probability $0 < p < 1$. Thus, for large n the digits of the code words of C are drawn randomly from the distribution P on $\{0, 1\}$ that gives probability p to 1 and $1 - p$ to 0. This means that the relevant probability space to consider here is the following.

Shift spaces and subshifts of finite type are a class of symbolic dynamical systems used to model various types of dynamics, see [65]. In particular, given an alphabet A with $\#A = q$, the shift space $\Sigma_q^+ = A^{\mathbb{N}}$ is the space of all sequences $a_0a_1a_2 \ldots a_n \ldots$ with $a_i \in A$, endowed with the one-sided shift map $\sigma : \Sigma_q^+ \to \Sigma_q^+$ that maps $\sigma(a_0a_1a_2 \ldots) = a_1a_2a_3 \ldots$. The set Σ_q^+ can be topologized (as a Cantor set) with a basis for the topology given by the cylinder sets $\Sigma_q^+(w)$, with $w = w_0 \ldots w_m$ for some $m \geq 1$ a word in the alphabet A, where $\Sigma_q^+(w) = \{wa_{m+1}a_{m+2} \ldots a_n \ldots , a_i \in A\}$ is the set of infinite words starting with the word w.

Let (Σ_2^+, μ_P) denote the probability space with $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$ the shift space given by all the infinite sequences of zeroes and ones, and with μ_P the Bernoulli measure that assigns to the cylinder set $\Sigma_2^+(w_1, \ldots, w_n)$ of sequences starting with the word $w_1 \cdots w_n$ the measure $\mu_P(\Sigma_2^+(w_1, \ldots, w_n)) = p^{a_n(w)}(1 - p)^{b_n(w)}$, where $a_n(w)$ is the number of 1's in $w_1 \cdots w_n$ and $b_n(w) = n - a_n(w)$ is the number of zeros.

The observation that code words of C are drawn randomly from the distribution $P = (p, 1-p)$, corresponds to saying that C is in the Shannon Random Code Ensemble (SRCE), which is the set of codes with this property. Note that one commonly works with the SRCE for the uniform distribution with $p = 1/2$, but one can equally consider SRCEs for a given $P = (p, 1 - p)$ specified by the problem.

Lemma 5.1 *For large n the neural code C is a code in the Shannon Random Code Ensemble, generated by the probability space (Σ_2^+, μ_P). Moreover, with μ_P-probability one, codes obtained in this way represent neural codes with firing rate $y = p/\Delta t$.*

Proof. As above, the probability $0 < p < 1$ is the probability of observing a spike in a time interval, for sufficiently large n, with the code words of C drawn according to the distribution $P(1) = p$, $P(0) = 1 - p$. This means that the code words can be identified as parts of a sequence in Σ_2^+ generated with the stochastic process given by the Bernoulli measure μ_P. As observed above, this is the property that the code C belongs to the Shannon Random Code Ensemble with this Bernoulli measure. Consider sequences $w = w_1, w_2, \ldots , w_n, \ldots$ in the shift space Σ_2^+ with the

Bernoulli measure μ_P determined by P. Let $a_n(w)$ be the number of 1's in the first n digits of a sequence $w \in \Sigma_2^+$. Then μ_P-almost everywhere one has the limit

$$\lim_{n\to\infty} \frac{a_n(w)}{n} \stackrel{a.e.}{=} p.$$

This means that for random codes generated from sequences in Σ_2^+ drawn according to the Bernoulli measure, the ratio $m/n = a_n(w)/n$, which described the firing rate of the neural code, approaches p for sufficiently large n, hence with probability one, codes C obtained in this way can be regarded as possible neural codes associated to a neuron with firing rate $y = p/\Delta t$, obtained from $k = \#C$ observations of $n = T/\Delta t$ time intervals, where T and Δt are given. □

Thus, in order to incorporate the information of the possible neural codes produced by the nodes of the network, given the firing rates of the neurons at the nodes, we can assign to each node a probability space (Σ_2^+, μ_P) from which the neural codes are generated. This is entirely determined by assigning the finite probability $P = (p, 1-p)$ at the node.

5.1.2 Codes and finite probabilities

Models of neural codes typically assume that only the firing rates and the timing of spikes encode information, while other characteristics such as spike amplitudes do not contribute to encoding of the stimulus. The use of binary codes as discussed above is adequate for this type of models, as it records only the digital 0/1 information on whether a spike is detected in a given time interval Δt or not. However, it has been suggested that other kinds of information may be present in the neural codes that are not captured by the binary code detecting the presence and timing of spikes. This is the case, for example, with the proposal that "spike directivity" contributes to the neural encoding [4]. In order to allow for the possibility of additional data in the neural code, besides the 0/1 record of whether a spike is present or not in a given time interval, one can consider non-binary codes. Since the codes we are considering are unstructured rather than linear, we do not need to require that the number of letters q of the code alphabet is a prime power and that the ambient space the code sits in is a vector space over a finite field. Thus we can simply assume that a discretization of the additional data being recorded (such as spike directivity) is chosen with a set of q values, for some $q \in \mathbb{N}$, $q \geq 2$. The code is now constructed with words that record, for each of the n time intervals Δt, whether a spike is absent (a digit 0) or whether it is present (a non-zero digit) and what is the registered value of the other parameters, discretized over the chosen range of q possible values (including 0). In this more general setting, we then assign to each node of our neural information network a code C of length n on an alphabet of q letters (where in principle q may vary with the node, depending on different types of neurons present). It is not obvious in this more general setting that Bernoulli processes on shift spaces will be adequate to model these more general codes, but in first approximation we can assume the same model and consider these neural codes as codes in the Shannon Random Code Ensemble generated by a Bernoulli process on the space of sequences Σ_q^+ determined by a finite probability measure μ_P with $P = (p_1, \dots, p_q)$. We consider this setting in the following. It is easy to restrict to the original case by just restricting to $q = 2$ for all the codes. The firing rate of the neuron is still related to the probability distribution P. Indeed, for $w \in \Sigma_q^+$, let $a_n^0(w)$ be the number of zeroes in the first n digits of the sequence w and let $b_n(w) = n - a_n^0(w)$ be the number of the non-zero digits. The firing rate can now be seen as the ratio $b_n(w)/n$ which has a μ_P-almost everywhere limit

$$\lim_{n\to\infty} \frac{b_n(w)}{n} \stackrel{a.e.}{=} \sum_{i=1}^{q} p_i = 1 - p_0 \,.$$

To make more precise the construction of the probability distribution associated to a code C, we focus for simplicity on the case of binary codes, though the following discussion can be easily generalized to q-ary codes.

Recall that an $[n, k, d]_2$-code is a binary code $C \subset \mathbb{F}_2^n$ of length n, with cardinality $\#C = 2^k$, and with minimum distance $d = \min\{d_H(c, c') \,|, c \neq c' \in C\}$, the minimal Hamming distance $d_H(c, c') = \#\{i \in \{1, \dots, n\} \,|\, c_i \neq c'_i\}$ between code words of C.

Definition 5.2 *Let C be an $[n,k,d]_2$-code. For every code word $c \in C$ let $b(c)$ denote the number of digits of c that are equal to one. The probability distribution $P_C = (p, 1-p)$ of the code C is given by*

$$p = \sum_{c\in C} p(c), \qquad \text{with} \qquad p(c) = \frac{b(c)}{n \cdot \#C}. \tag{5.1}$$

If $a(c)$ denotes the number of letters in the code word $c \in C$ that are equal to zero, then clearly we also have $1 - p = \sum_{c\in C} a(c)/(n \cdot \#C)$.

Lemma 5.3 *Let C, C' be binary codes of equal length n, both containing the word with all digits equal to zero. Let $f : C \to C'$ be a surjective map that sends the zero word to itself, such that for all code words $c, c' \in C$ the Hamming distance satisfies $d(f(c), f(c')) \le d(c, c')$. Then the probability $P_{C'}$ is related to P_C by $p(f(c)) = \lambda(c)p(c)$, where $\lambda(c) \le 1$ is given by the ratio $\lambda(c) = b(f(c))/b(c)$.*

Proof. Since the Hamming distance is decreasing under the map f we have $b(f(c)) = d(f(c), 0) \le d(c, 0) = b(c)$. It is then clear that $p(c') = \sum_{c\in f^{-1}(c')} \lambda(c)p(c)$ is the probability distribution associated to the code C'. □

5.1.3 Categories of codes

We discuss here two possible constructions of a category of codes. The first one is modeled on the notion of decomposable and indecomposable codes variously considered in the coding theory literature (see for instance [98]). The second one is more directly suitable for modeling neural codes associated to populations of neurons and their firing activities.

Let C be an $[n,k,d]_q$-code over an alphabet $\mathfrak{A}$ with $\#\mathfrak{A} = q$, so that $C \subset \mathfrak{A}^n$ with $\#C = q^k$. Given two such codes, C an $[n,k,d]_q$-code over the alphabet $\mathfrak{A}$ and C' an $[n',k',d']_{q'}$-code over the alphabet $\mathfrak{B}$, a morphism is a function $\phi : C \to C'$, such that the image $\phi(C) \subset C'$ satisfies $d_{\mathfrak{B}^{n'}}(\phi(c_1), \phi(c_2)) \le d_{\mathfrak{A}^n}(c_1, c_2)$ for all code words $c_1, c_2 \in C$, in the respective Hamming distances. Note that here we do not define the morphisms $\phi : C \to C'$ as maps $\phi : \mathfrak{A}^n \to \mathfrak{B}^{n'}$ of the ambient spaces that map C inside C'. If we restrict to only considering codes over a fixed alphabet $\mathfrak{A}$, then there is a sum operation given by

$$C \oplus C' := \{(c, c') \in \mathfrak{A}^{n+n'} \,|\, c \in C,\ c' \in C'\}. \tag{5.2}$$

A code C is decomposable if it can be written as $C = C' \oplus C''$ for codes C', C'' and indecomposable otherwise [98]. If C is an $[n,k,d]_q$-code and C' is an $[n',k',d']_q$-code then $C \oplus C'$ is an $[n+n', k+k', \min\{d,d'\}]_q$-code. With this choice of objects and morphisms the resulting category does not have a zero object. If we identify the alphabet $\mathfrak{A}$ with a set of q digits $\mathfrak{A} = \{0, \dots, q-1\}$ we can consider, for each $n \in \mathbb{N}$, only those codes $C \subset \mathfrak{A}^n$ that contain the zero word $(0, \dots, 0)$ as one of the code words. We can interpret these codes as being the result of a number $\#C$ of observations of the spiking neuron, with each observation consisting of n time intervals Δt, where the observations stop when no more spiking activity is detected in the $T = n\Delta t$ observation time, that is, when the response to the stimulus has terminated, so the last code word is the zero word.

Lemma 5.4 *Let* Codes *be the category with objects the codes containing the zero word and morphisms $\phi : C \to C'$ as above. This category has a zero object given by the code $C = \{0\} \subset \mathfrak{A}$ consisting of the zero word of length one and a coproduct of the form (5.2).*

Our previous construction of the measure associated to a binary code in Definition 5.2 satisfies the following property with respect to the sum of codes.

Lemma 5.5 *The probability associated to the sum $C \oplus C'$ of (5.2) is given by $P_{C\oplus C'} = \lambda P_C + (1-\lambda)P_{C'}$ with $\lambda = n/(n+n')$.*

Proof. The probability associated to the code C is given by

$$P_C = (p, 1-p) \quad \text{with} \quad p = \sum_{c\in C} p(c) \quad \text{and} \quad p(c) = b(c)/(n \cdot \#C) \tag{5.3}$$

with $b(c)$ the number of letters equal to one in the code word c. Similarly for the probability $P(C')$. For code words (c,c') in $C\oplus C'$ with $c\in C$ and $c'\in C'$, we have length $n+n'$, cardinality $\#C\cdot\#C'$, and number of letters equal to one given by $b(c,c')=b(c)+b(c')$. Thus the probability $P_{C\oplus C'}$ has $p=\sum_{(c,c')}p(c,c')$ with

$$p(c,c')=\frac{b(c)+b(c')}{(n+n')\cdot\#C\cdot\#C'}=\frac{b(c)\cdot n}{n\cdot(n+n')\cdot\#C\cdot\#C'}+\frac{b(c')\cdot n'}{n'\cdot(n+n')\cdot\#C\cdot\#C'}$$

$$=p(c)\cdot\frac{n}{(n+n')\cdot\#C'}+p(c')\cdot\frac{n'}{(n+n')\cdot\#C}.$$

Thus, we have

$$\sum_{c,c'}p(c,c')=\sum_{c,c'}\frac{p(c)\cdot n}{(n+n')\cdot\#C'}+\sum_{c,c'}\frac{p(c')\cdot n'}{(n+n')\cdot\#C}$$

$$=\frac{n}{(n+n')}\cdot\sum_c p(c)+\frac{n'}{(n+n')}\cdot\sum_{c'}p(c')=1\,.$$

Thus the resulting probabilities are related by $P_{C\oplus C'}=\lambda P_C+(1-\lambda)P_{C'}$ with $\lambda=n/(n+n')$. □

The counting of ones in the digits of the code words, used to obtain the probabilities of Lemma 5.5, can be viewed as comparing each code word to the zero word through the Hamming distance. One can refine this by comparing all the code words with each other through the Hamming distance. Thus, one can also associate to a code C the pair $(\delta,1-\delta)$ where $\delta=\min\{d_H(c,c')\,|\,c\neq c'\}/n=d/n$ is the relative minimum distance. Note that the Shannon information $\mathcal{I}(\delta,1-\delta)=\delta\log_q\delta+(1-\delta)\log_q(1-\delta)$ and the associated q-ary entropy function $H_q(\delta)=\delta\log_q(q-1)-\delta\log_q\delta-(1-\delta)\log_q(1-\delta)$ describe the asymptotic behavior of the volumes of the Hamming balls, and determines the position of random codes with respect to the Hamming bound [28]. It is well known that codes in the SRCE populate the region of the space of code parameters at and below the Gilbert–Varshamov line defined in terms of the q-ary entropy function [28].

We consider then another possibile construction of a category of codes with a sum and zero object, which is simply induced by the same structure on pointed sets. We will see in the next subsection that this choice has better properties with respect to the assignment of probabilities to neural codes. Unlike the usual setting of coding theory, we allow here for the possibility of codes with repeated code words, that is, where some $c,c'\in C$ have zero Hamming distance. While this is unnatural from the coding perspective as it leads to ambiguous encoding, it is not unreasonable when thinking of codes that detect firing patterns of neurons, as the possibility exists of two measurements leading to the same pattern. Thus, we think here of codes as subsets $C\subset\mathfrak{A}^n$ with possible multiplicities assigned to the code words. We assume the zero word always has multiplicity one. Repeated code words arise in the categorical setting we describe here when coproducts are taken using (5.4) instead of (5.2). The following is simply the usual categorical structure on finite pointed sets.

Lemma 5.6 *A symmetric monoidal category* $\mathrm{Codes}_{n,*}$ *of pointed codes of length n is obtained with set of objects given by $[n,k,d]_q$-codes, with fixed alphabet $\mathfrak{A}$ with $\#\mathfrak{A}=q$ and fixed length n, that contain among their code words the constant 0-word $c_0=(0,0,\dots,0)$, and are not equal to the code $C=\{c_0,c_1\}$ consisting only of the constant 0-word and the constant 1-word $c_1=(1,1,\dots,1)$. As maps $f:C\to C'$ we consider functions mapping the 0-word to itself. The categorical sum is the wedge sum of pointed sets*

$$C\oplus C':=C\vee C'=C\sqcup C'/c_0\sim c_0' \tag{5.4}$$

and the zero object is the code $C=\{c_0\}$ consisting only of the 0-word c_0 of length n.

We exclude among the objects of the category of codes the code $C=\{c_0,c_1\}$ containing only the constant 0-word c_0 and the constant 1-word c_1 to ensure that any code that is not just $\{c_0\}$ contains at least a word with non-zero information. We regard the 0-word as the baseline corresponding to lack of any spiking activity, and we require that the presence of spiking activity carries some non-trivial information.

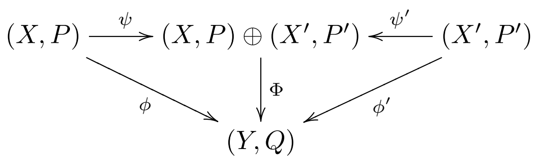
In the categorical setting we described here, a neural code associated to a network of neurons can be viewed as a summing functor $\Phi_E : P(V_{G^*}) \to \text{Codes}$ where V_G is the set of vertices of the network G, as discussed in the previous section. The code assigned to a vertex describes the spiking behavior of that neuron.

5.1.4 Codes and associated probabilities

It is convenient here to consider a slightly different version of the category of finite probabilities, with respect to the versions mentioned earlier. This is itself a variant, where pointed sets are considered, over a standard construction of a category of (finite) measure spaces. The morphisms (f, Λ) will be defined using functions $f : X \to Y$ of finite pointed sets and non-negative weights $\Lambda = \{\lambda_y\}$ on the fibers that serve the purpose of matching the probability measures. More precisely, we have the following.

Lemma 5.7 *A category $\mathcal{P}_f$ of finite probabilities with fiberwise measures as morphisms is obtained by considering as objects the pairs (X, P_X) of a finite pointed set X with a probability measure P_X such that $P_X(x_0) > 0$ at the base point. Morphisms $\phi : (X, P_X) \to (Y, P_Y)$ consist of a pair $\phi = (f, \Lambda)$ of a function $f : X \to Y$ of pointed sets, $f(x_0) = y_0$, with $f(\text{supp}(P_X)) \subset \text{supp}(P_Y)$, together with a collection $\Lambda = \{\lambda_y\}$ of measures λ_y on the fibers $f^{-1}(y) \subset X$, with $\lambda_{y_0}(x_0) > 0$, such that $P_X(A) = \sum_{y\in Y} \lambda_y(A \cap f^{-1}(y))\, P_Y(y)$. The category has a coproduct and a zero object.*

Proof. Note that the fiberwise measures λ_y are not assumed to be probability measures. While in the case of a surjection $f : X \to Y$ we can have $\sum_{x\in f^{-1}(y)} \lambda_y(x) = 1$, in the case of an injection $\iota : X \hookrightarrow Y$ scaling factors $\lambda_y(x) \geq 1$ will adjust the normalization so that $\sum_x P_X(x) = \sum_x \lambda_{\iota(x)}(x) P_Y(\iota(x)) = 1$ while $\sum_{y\in\iota(X)} P_Y(y) = 1 - P_Y(Y \smallsetminus \iota(X)) \leq 1$. Composition of morphisms $\phi = (f, \Lambda) : (X, P_X) \to (Y, P_Y)$ and $\phi' = (g, \Lambda') : (Y, P_Y) \to (Z, P_Z)$ is given by $\phi' \circ \phi = (g \circ f, \tilde{\Lambda})$ with $\tilde{\lambda}_{g(f(x))}(x) = \lambda_{f(x)}(x)\lambda'_{g(f(x))}(f(x))$. We want to show the existence of a unique (up to unique isomorphism) object $(X, P) \oplus (X', P')$ in $\mathcal{P}_f$ with morphisms $\psi : (X, P) \to (X, P) \oplus (X', P')$ and $\psi' : (X', P') \to (X, P) \oplus (X', P')$ such that for any given morphisms $\phi = (f, \Lambda) : (X, P) \to (Y, Q)$ and $\phi' = (g, \Lambda') : (X', P') \to (Y, Q)$, there exists a unique morphism $\Phi : (X, P) \oplus (X', P') \to (Y, Q)$ such that the diagram commutes:



We take $(X, P) \oplus (X', P')$ to be the object $(X \vee X', \tilde{P})$ where $X \vee X' = X \sqcup X'/x_0 \sim x'_0$ and $\tilde{P}(x) = P(x) \cdot \alpha_{X,X'}$ for all $x \in X \smallsetminus \{x_0\}$, $\tilde{P}(x') = P'(x') \cdot \beta_{X,X'}$ for all $x' \in X' \smallsetminus \{x'_0\}$, and $\tilde{P}(x_0 \sim x'_0) = \alpha_{X,X'} P(x_0) + \beta_{X,X'} P'(x'_0)$, with $\alpha_{X,X'} = N/(N+N')$ with $N = \#X$ and $N' = \#X'$ and $\beta_{X,X'} = 1 - \alpha_{X,X'} = N'/(N + N')$ so that $\sum_{a\in X\sqcup X'} \tilde{P}(a) = \alpha_{X,X'} \sum_{x\in X\smallsetminus\{x_0\}} P(x) + \beta_{X,X'} \sum_{x'\in X'\smallsetminus\{x'_0\}} P'(x') + \tilde{P}(x_0 \sim x'_0) = 1$. The morphisms $\psi = (\iota : X \hookrightarrow X \sqcup X', \Lambda = \alpha^{-1}_{X,X'})$ and $\psi' = (\iota' : X' \hookrightarrow X \sqcup X', \Lambda' = \beta^{-1}_{X,X'})$ and the induced morphism $\Phi = (F, \tilde{\Lambda}) : (X \sqcup X', \tilde{P}) \to (Y, Q)$ given by $F(\iota(x)) = f(x)$ and $F(\iota'(x')) = g(x')$ with $\tilde{\lambda}_y(x) = \alpha_{X,X'} \cdot \lambda_y(x)$ for $x \in f^{-1}(y)$ and $\tilde{\lambda}_y(x') = \beta_{X,X'} \cdot \lambda'_y(x')$ for $x' \in g^{-1}(y)$, give a commutative diagram as above. The coproduct constructed in this way is unique up to unique isomorphism, since if there is another object $(Z, \hat{P})$ with morphisms $\hat{\psi} : (X, P) \to (Z, \hat{P})$ and $\hat{\psi}' : (X', P') \to (Z, \hat{P})$ that satisfies the same universal property, there are unique maps $\Phi : (X \sqcup X', \tilde{P}) \to (Z, \hat{P})$ and $\hat{\Phi} : (Z, P) \to (X \sqcup X', \tilde{P})$ that make the respective diagrams commute so that $\Phi \circ \psi = \hat{\psi}$, $\Phi \circ \psi' = \hat{\psi}'$, $\hat{\Phi} \circ \hat{\psi} = \psi$, and $\hat{\Phi} \circ \hat{\psi}' = \psi'$. The object $(*, 1)$ with a single point with probability one is a zero object, with unique morphism $(f, \Lambda) : (*, 1) \to (X, P)$ given by $f(*) = x_0$ and $\lambda_{x_0}(*) = P(x_0)$ and unique morphism $(f, \Lambda) : (X, P) \to (*, 1)$ with $f(x) = *$ for all $x \in X$ and $\lambda_*(x) = P(x)$. □

Given a code C, we assign a finite probability P_C to the code, as in (5.1), which refines the binary probability $(p, 1-p)$ of (5.3). More precisely, we construct P_C as follows.

Definition 5.8 *The probability space P_C associated to a binary code C is given by*

$$P_C(c) = \begin{cases} \frac{b(c)}{n(\#C-1)} & c \neq c_0 \\ 1 - \sum_{c' \neq c_0} \frac{b(c')}{n(\#C-1)} & c = c_0 \end{cases} \tag{5.5}$$

with $b(c)$ the number of digits equal to 1 in the word c.

Lemma 5.9 *The assignment $C \mapsto P_C$ determines a functor $P : \mathrm{Codes}_{n,*} \to \mathcal{P}_f$ compatible with sums and zero objects.*

Proof. A map of codes $f : C \to C'$ induces a map $\phi = (f, \Lambda) : (C, P_C) \to (C', P_{C'})$ with $\Lambda = \{\lambda_{c'}(c) \,|\, c \in f^{-1}(c')\}$ given by $\lambda_{f(c)}(c) = \frac{P_C(c)}{P_{C'}(f(c))}$. This is well defined because by (5.5) the only code word with $b(c) = 0$ would be the 0-word c_0 and $b(c) = n$ only for the word c_1 with all digits equal to one, so as long as the code C does not contain only the words c_0 and c_1, we have both $P_C(c_0) \neq 0$ and $P_C(c) \neq 0$ for all $c \neq c_0$. The sum of codes is given by the wedge sum of pointed sets (5.4). The associated probability is given by

$$P_{C_1 \vee C_2}(c) = \frac{b(c)}{n(\#(C_1 \vee C_2) - 1)}$$

for $c \neq c_0$ and $1 - \sum_{c \neq c_0} P_{C_1 \vee C_2}(c)$ at the zero word. For $N = \#C_1 - 1$ and $N' = \#C_2 - 1$, we have $N + N' = \#(C_1 \vee C_2) - 1$ so that

$$P_{C_1 \vee C_2}(c) = \begin{cases} \frac{N}{N+N'} P_{C_1}(c) & c \in C_1 \smallsetminus \{c_0\} \\ \frac{N'}{N+N'} P_{C_2}(c) & c \in C_2 \smallsetminus \{c_0\} \\ \frac{N}{N+N'} P_{C_1}(c_0) + \frac{N'}{N+N'} P_{C_2}(c_0) & c = c_0, \end{cases}$$

hence $P_{C_1 \vee C_2}$ agrees with the probability $\tilde{P}$ of the direct sum $(C_1, P_1) \oplus (C_2, P_2) = (C_1 \vee C_2, \tilde{P})$ as in Lemma 5.7. The zero object $C = \{c_0\}$ is mapped to the zero object $(\{c_0\}, 1)$. □

5.2 Weighted codes and linear relations

In order to illustrate this general framework in a simple example, we show how a "linear neuron" toy model can be fit within the setting described in the previous subsections.

Of course, in reality the neuron is non-linear, and the non-linearities can be described in terms of a threshold function (such as a sigmoid, or piecewise linear, or step function). In this subsection we just look at the simplified linear case, while we will discuss how to formulate in our setting the case of non-linear neurons and threshold dynamics in §6.

Lemma 5.10 *A category of weighted codes $\mathcal{W}\mathrm{Codes}_{n,*}$ is obtained with objects given by pairs (C, ω) of pointed codes C of length n containing the zero word c_0 and a function $\omega : C \to \mathbb{R}$ assigning a (signed) weight to each code word, with $\omega(c_0) = 0$. Morphisms $\phi : (C, \omega) \to (C', \omega')$ are pairs $\phi = (f, \Lambda)$ of a pointed map $f : C \to C'$ mapping the zero word to itself and $f(\mathrm{supp}(\omega)) \subset \mathrm{supp}(\omega')$, and a collection $\Lambda = \{\lambda_{c'}(c)\}_{c \in f^{-1}(c')}$ satisfying $\omega(c) = \lambda_{f(c)}(c) \cdot \omega'(f(c))$ and $\lambda_{c_0}(c_0) = 0$. The category $\mathcal{W}\mathrm{Codes}_{n,*}$ has a sum given by $(C, \omega) \oplus (C', \omega') = (C \vee C', \omega \vee \omega')$ with $\omega \vee \omega'|_C = \omega$ and $\omega \vee \omega'|_{C'} = \omega'$ and with zero object $(\{c_0\}, 0)$.*

The argument is analogous to the case of the category $\mathcal{P}_f$, in fact simpler because in the case of weights instead of probabilities we do not have the normalization property of probability measures that needs to be preserved.

Consider then a pointed directed graph $G^* \in \mathrm{Func}(\mathbf{2}, \mathcal{F}_*)$ as before, and the categories of summing functors $\Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_{G^*})$ and $\Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(V_{G^*})$ with the source and target functors $s, t : \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_{G^*}) \rightrightarrows \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(V_{G^*})$. As discussed earlier, a summing functor Φ_G in the equalizer of the source and target functors

$$\Sigma^{\mathrm{eq}}_{\mathcal{W}\mathrm{Codes}_{n,*}}(G) := \mathrm{equalizer}(s, t : \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_{G^*}) \rightrightarrows \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(V_{G^*}))$$

is a summing functor $\Phi_G \in \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_{G^*})$ with the property that for all pointed subsets $A \subset V(G^*)$ the conservation law $\Phi_G(s^{-1}(A)) = \Phi_G(t^{-1}(A))$ holds, which we also write as before as $\oplus_{s(e)\in A}(C_e,\omega_e) = \oplus_{t(e)\in A}(C_e,\omega_e)$, where the sum is the categorical sum in $\mathcal{W}\mathrm{Codes}_{n,*}$ and $(C_e,\omega_e) = \Phi_G(\{e,e_*\})$.

Remark 5.11 If we assume that the directed graph G has a single outgoing edge at each vertex, $\{e \in E_G \,|\, s(e) = v\} = \{\mathrm{out}(v)\}$, then the equalizer condition becomes

$$(C_{\mathrm{out}(v)}, \omega_{\mathrm{out}(v)}) = \oplus_{t(e)=v}(C_e,\omega_e), \tag{5.6}$$

which is the formulation in our categorical setting of the linear neuron model.

In this model, we interpret the directed edges of the network as synaptic connections between neurons, the code C_e as determined by spiking potentials incoming along that edge from the neuron at the source vertex $s(e)$, and the weight ω_e is a measure of the efficacy of the synapses, depending on physiological properties such as number of synaptic vescicles in the presynaptic terminal and number of gated channels in the post-synaptic membrane, with the sign of ω_e describing whether the synapse is excitatory or inhibitory. In this interpretation, in particular, the sign of $\omega_e(c)$ depends only on the edge e and not on the code word c, so ω_e has constant excitatory or inhibitory sign on the entire code C_e and different amplitude on the different code words. On codes $C = \vee_e C_e$ the sign of $\omega = \vee_e \omega_e$ is no longer constant.

5.3 Information measures

The Shannon information of a finite measure

$$S(P) = -\sum_{x\in X} P(x)\log P(x)$$

satisfies the extensivity property

$$S(P') = S(P) + P\,S(Q)$$

for decompositions over subsystems $P' = (p'_{ij})$ with $p'_{ij} = p_j \cdot q(i|j)$, where

$$P\,S(Q) := \sum_j p_j S(Q|j) = -\sum_j p_j \sum_i q(i|j)\log q(i|j).$$

In fact, extensivity, together with other simple properties completely characterize axiomatically the Shannon entropy (Khinchin axioms).

Definition 5.12 *A thin category $\mathcal{S}$ is a category where, for any two objects $X, Y \in \mathrm{Obj}(\mathcal{S})$, the set $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ consists of at most one morphism.*

Up to equivalence a thin category $\mathcal{S}$ is the same as a partially ordered set (poset). Up to isomorphism a thin category is the same as a preordered set (proset), which satisfies the same properties as a partial order except for asymmetry (the property that $X \le Y$ and $Y \le X$ implies $X = Y$). We will write thin categories in the form $(S,\le)$, or $(S,\ge)$ for the opposite thin category.

Lemma 5.13 *Let $\mathcal{P}_{f,s}$ be the category of finite probabilities with fiberwise measures, where we only consider morphisms (f,Λ) with $f : X \to Y$ a surjection and where the $\lambda_y(x)$ for $x \in f^{-1}(y)$ are probability measures on the fibers. Consider the real numbers $(\mathbb{R},\ge)$ as a thin category with an object for each $r \in \mathbb{R}$ and a single morphism $r \to r'$ if and only if $r \ge r'$. The Shannon entropy is a functor $S : \mathcal{P}_{f,s} \to \mathbb{R}$.*

Proof. In the case of a morphism $(f,\Lambda) : (X,P) \to (Y,Q)$ in the category $\mathcal{P}_{f,s}$ where the map $f : X \to Y$ is a surjection and the fiberwise measures are probabilities $\Lambda = \{\lambda_y(x) \,|\, x \in f^{-1}(y)\}$ on each fiber, we have a special case of the extensivity property with $P(x) = \lambda_{f(x)}(x)\,Q(f(x))$ and we obtain

$$S(P) = -\sum_{y\in Y}\sum_{x\in f^{-1}(y)} \lambda_y(x)\,Q(y)\log(\lambda_y(x)\,Q(y))$$

$$= -\sum_{y\in Y}(\sum_{x\in f^{-1}(y)} \lambda_y(x))\, Q(y) \log Q(y) - \sum_{y\in Y} Q(y) \sum_{x\in f^{-1}(y)} \lambda_y(x)\, \log(\lambda_y(x))$$

$$= S(Q) + \sum_{y\in Y} Q(y)\, S(\Lambda|y) = S(Q) + Q\, S(\Lambda).$$

In particular, this implies that for these morphisms we have $S(P) \geq S(Q)$, and the difference $S(P) - S(Q) = \sum_{y\in Y} Q(y)\, S(\Lambda|y)$ measures the information loss along the morphism $(f,\Lambda) : (X,P) \to (Y,Q)$. □

However, when we consider more general morphisms (f,Λ) in the category $\mathcal{P}_f$, where the map f is not necessarily a surjection and the fiberwise measures $\Lambda = \{\lambda_y(x)\,|\, x \in f^{-1}(y)\}$ are not necessarily probabilities, the relation between the Shannon entropies is no longer a case of the usual extensivity property and does not always satisfy the same simple estimate. For example, consider the case of an embedding $j : X \hookrightarrow Y$ so that the values $\lambda_{j(x)}(x)$ are dilation factors that adjust the normalization of the measure $Q|_{j(X)}$. In this case we only have the relation

$$S(P) = -\sum_{y\in j(X)} \lambda_{j(x)}(x)\, Q(j(x)) \log(\lambda_{j(x)}(x)\, Q(j(x))) =$$

$$-\sum_{y\in j(X)} \lambda_{j(x)}(x)\, Q(j(x)) \log Q(j(x)) - \sum_{y\in j(X)} Q(j(x))\, \lambda_{j(x)}(x) \log(\lambda_{j(x)}(x)).$$

We can still obtain an estimate relating the Shannon entropies $S(P)$ and $S(Q)$, though not in the simple form of Lemma 5.13.

Lemma 5.14 *Given a summing functor $\Phi_X : \Sigma_{\mathcal{P}_f}(X)$ for a finite pointed set X, there exists constants $\lambda_{\min}, \lambda_{\max} \geq 1$ depending only on X such that $S(\Phi_X(A)) \leq \lambda_{\max} S(\Phi_X(A')) - \lambda_{\min} \log \lambda_{\min}$ for all inclusions $A \subset A'$ of pointed subsets of X.*

Proof. The summing functor $\Phi_X : P(X) \to \mathcal{P}_f$ assigns to pointed subsets $A \subset X$ probabilities P_A and to inclusions $j : A \hookrightarrow A'$ morphisms $(j,\Lambda) : (A,P_A) \to (A',P_{A'})$ with $\Lambda = \{\lambda_{j(a)}(a)\}_{a\in A}$ determined by $P_A(a) = \lambda_{j(a)}(a)\, P_{A'}(j(a))$. The functoriality of Φ_X ensures that the probabilities P_A and $P_{A'}$ are assigned with the consistency condition that $j(\mathrm{supp}(P_A)) \subset \mathrm{supp}(P_{A'})$. We then assign to pointed subsets $A \subset X$ the value of the Shannon entropy $S(P_A) = -\sum_{a\in A} P_A(a) \log P_A(a)$. A morphism in $P(X)$ is given by a pointed inclusion $j : A \hookrightarrow A'$, with $\Phi_X(j) = (j,\Lambda)$ the corresponding morphism in $\mathcal{P}_f$. Consider the inclusions $j_a : \{*\} \hookrightarrow \{*,a\}$ for $a \in X$ and the inclusions $\iota_{a,k} : \{*,a\} \hookrightarrow \{*,a\} \vee_{j=1}^k \{*,a_j\}$ in wedge sums of finite pointed sets. The corresponding morphisms in $\mathcal{P}_f$ have dilation factors $\lambda(j_a) \geq 1$ and $\lambda(\iota_{a,k}) \geq 1$ where these are the dilation factors of the embeddings in the coproduct of $\mathcal{P}_f$ as discussed in Lemma 5.7. Any $j : A \hookrightarrow A'$ inclusion of finite pointed subsets of X is a composition of these maps, hence its scaling factors are products of these factors. Thus, the bounds $\lambda_{\min} = \min \lambda_{j(a)}(a)$ and $\lambda_{\max} = \max \lambda_{j(a)}(a)$ over $a \in X$ and over all possible morphisms $j : A \hookrightarrow A'$ in $P(X)$ satisfy $\lambda_{\min}, \lambda_{\max} \geq 1$. The Shannon entropy satisfies

$$S(P) = -\sum_{j(a)\in j(A)} \lambda_{j(a)}(a)\, P_{A'}(j(a)) \log(\lambda_{j(a)}(a)\, P_{A'}(j(a)))$$

$$\leq -\lambda_{\max} \sum_{a'\in A'} P_{A'}(a') \log(P_{A'}(a')) - \lambda_{\min} \log \lambda_{\min}.$$

In particular, $S(P_A) \leq \lambda_{\max} S(P_{A'})$. □

In §5.4 below we discuss a better way of assigning probabilities and information structures to codes that bypasses the problem described here, and gives a good functorial construction that leads to information measures naturally associated to networks and their neural codes.

5.3.1 Category of simplices

As preliminary notation, we recall the construction of the category Δ of simplicial sets, which we will be using frequently in the rest of the paper, starting in the next subsection, §5.4.

Denote by $[n]$ for $n = 0, 1, 2, \ldots$ the totally ordered subset of integers $[n] = \{0, \ldots, n\}$. The simplex category $\triangle$ (not to be confused with the category Δ that we define below) has objects the sets $[n]$ and morphisms the nondecreasing maps $f : [n] \to [m]$.

Morphisms are generated by two classes of maps (see [50], pp. 14–15): ∂^i_n and σ^i_n, respectively given by the increasing injection $[n-1] \to [n]$ not taking the value i, and the nondecreasing surjection $[n+1] \to [n]$ taking the value i twice. Faces and degeneracies satisfy the relations

$$\partial^j_{n+1}\partial^i_n = \partial^i_{n+1}\partial^{j-1}_n \quad \text{for} \quad i < j;$$

$$\sigma^j_{n-1}\sigma^i_{n+1} = \sigma^i_n\sigma^{j+1}_{n+1} \quad \text{for} \quad i \leq j;$$

$$\sigma^j_{n-1}\partial^i_n = \begin{cases} \partial^i_{n-1}\sigma^{j-1}_{n-2} & \text{for} \quad i < j\,, \\ id_{[n-1]} & \text{for} \quad i \in \{j, j+1\}\,, \\ \partial^{i-1}_{n-1}\sigma^j_{n-2} & \text{for} \quad i > j+1. \end{cases}$$

A simplicial object of a category $\mathcal{C}$ is a functor $\triangle^{\rm op} \to \mathcal{C}$. In particular, a simplicial set is a functor $\triangle^{\rm op} \to \text{Sets}$ and a pointed simplicial set is a functor $\triangle^{\rm op} \to \text{Sets}_*$ to pointed set.

In the following we will always denote by Δ and Δ_* the categories of simplicial sets and of pointed simplicial sets

$$\Delta := \text{Func}(\triangle^{\rm op}, \text{Sets}), \qquad \Delta_* := \text{Func}(\triangle^{\rm op}, \text{Sets}_*)$$

with morphisms given by natural transformations of the functors.

The classical description of morphisms in $\triangle$ via generators (“i-th face maps”, “i-th degeneracy maps”) and relations recalled above produces explicit description of simplicial sets and their topological realizations.

The objects $[n]$ of $\triangle$ are realized by the standard simplices, denoted by $\Delta_n \subset \mathbb{R}^{n+1}$, namely the n-dimensional topological space

$$\Delta_n := \{(x_0, \ldots, x_n) | \sum_{i=0}^n x_i = 1, x_i \geq 0\}.$$

5.4 Summing functors and information measures

In this section we consider again the formalism of network summing functors introduced in §2, but we focus on the associated information structure, rather than on computational architectures as in §4. We start with a review of the cohomological information formalism.

5.4.1 Cohomological information theory

We adopt here the point of view of [10], [11], and especially [106], on a cohomological formulation of information measures. This will allow us to significantly improve the provisional construction described in §5.1.4 of probabilities assigned to networks via the corresponding neural codes. The main problem with the construction we described in §5.1.4 is that the category of probabilities we used does not have sufficiently good properties, with respect to information measures. This was shown in §5.3: while the Shannon entropy is functorial on the category of finite probability spaces with surjections with fiberwise probabilities as morphisms (as shown in Lemma 5.13), it is not functorial on the category of finite probabilities of Lemma 5.7, which is the target of the functors from codes discussed in §5.1.4. This is because, to have a sum and a zero object in this category, we need to allow for morphisms that are not surjections and fiberwise measures that are not probabilities, over which the Shannon entropy is not a monotone function (see Lemma 5.14). Note, however, that this still determines a symmetric monoidal structure that can be used as a category of resources.

To remedy this problem we now give a more refined construction, which uses network summing functors with a target category that is an abelian category describing probability data, as introduced in [106].

The most important aspects we want to retain of this general formalism of information structures and probabilities are the fact that there is a suitable category of random variables and functors $\mathcal{Q}$ from this category to simplicial sets that assign to a random variable X a corresponding simplicial set of probabilities $\mathcal{Q}_X$. There is then a functor $\mathcal{M}$ to vector spaces, that associates to $P \in \mathcal{Q}_X$ the real vector space of P-measurable functions. These vector spaces are in turn used to construct cochain complexes using a Hochschild-type resolution and Hochschild coboundary. This cochain complex is designed so that its cohomology describes classical information functionals. In a somewhat more detailed form, we summarize briefly the relevant parts of the setting of [106] that we need for our purposes.

- A *finite information structure* (S, M) is a pair of a thin category S, as in Definition 5.12 (the observables) and a functor $M : S \to \mathcal{F}$ to the category of finite sets.
- The category S has objects $X \in \mathrm{Obj}(S)$ given by random variables with values in a finite probability space and a morphism $\pi : X \to Y$ if the random variable Y is coarser than X (values of Y are determined by values of X), with the property that, if there are morphisms $X \to Y$ and $X \to Z$ then $YZ = Y \wedge Z$ (the random variable given by the joint measurement of Y and Z) is also an object of S.
- The category S has a terminal object $\mathbf{1}$ given by the random variable with value set $\{*\}$ a singleton.
- The functor $M : S \to \mathcal{F}$ maps a random variable X to the finite set given by its range of values M_X and morphisms $\pi : X \to Y$ to surjections $M(\pi) : M_X \to M_Y$. The value set $M_{X\wedge Y}$ is a subset of $M_X \times M_Y$.
- The category $\mathcal{IS}$ of finite information structures has objects the pairs (S, M) as above and morphisms $\varphi : (S, M) \to (S', M,')$ given by pairs $\varphi = (\varphi_0, \varphi^{\#})$ of a functor $\phi_0 : S \to S'$ and a natural transformation $\phi^{\#} : M \to M' \circ \phi_0$ such that $\phi_0(\mathbf{1}) = \mathbf{1}$ and $\phi_0(X \wedge Y) = \phi_0(X) \wedge \phi_0(Y)$ whenever $X \wedge Y$ is an object in S, and such that for all X the morphism $\phi^{\#}_X : M_X \to M'_{\phi_0(X)}$ is a surjection.
- The category $\mathcal{IS}$ has finite products $(S \times S', M \times M')$ with objects pairs (X, X') of random variables with value set $M_X \times M'_{X'}$ and coproducts $(S \vee S', M \vee M')$ with objects $\mathrm{Obj}(S \vee S') = \mathrm{Obj}(S) \vee \mathrm{Obj}(S') = \mathrm{Obj}(S) \sqcup \mathrm{Obj}(S')/\mathbf{1}_S \sim \mathbf{1}_{S'}$ and value set M_X or $M'_{X'}$ if $X \in \mathrm{Obj}(S)$ or $X' \in \mathrm{Obj}(S')$.
- A *probability functor* $\mathcal{Q} : (S, M) \to \Delta$ assigns to each object X a simplicial set $\mathcal{Q}_X$ of probabilities on the set M_X (which is a subset of the simplex Δ_{M_X} of all probability distributions on M_X) and to morphisms $\pi : X \to Y$ the morphism $\pi_* : \mathcal{Q}_X \to \mathcal{Q}_Y$ with $\pi_*(P)(y) = \sum_{x\in\pi^{-1}(y)} P(x)$.
- For each $X \in \mathrm{Obj}(S)$ there is a *semigroup* $\mathcal{S}_X = \{Y \in \mathrm{Obj}(S) \,|\, \exists \pi : X \to Y\}$ with the product $Y \wedge Z$, and a *semigroup algebra* $\mathcal{A}_X := \mathbb{R}[\mathcal{S}_X]$.
- There are associated *contravariant functors* $\mathcal{M}(\mathcal{Q}) : (S, M) \to \mathrm{Vect}$ that assign to objects $X \in \mathrm{Obj}(S)$ and probabilities $P_X \in \mathcal{Q}_X$ the vector space of real-valued (measurable) functions on (M_X, P_X) and to a morphism $\pi : X \to Y$ the map $\mathcal{M}(\mathcal{Q})(\pi) : f \mapsto f \circ \pi_*$.
- There is an *action* σ_α of the semigroup $\mathcal{S}_X$ on $\mathcal{M}(\mathcal{Q}_X)$ by
$$\sigma_\alpha(Y) : f \mapsto Y(f)(P_X) = \sum_{y\in M_Y \,:\, Y_* P_X(y)\neq 0} (Y_* P_X(y))^\alpha \, f(P_X|_{\pi^{-1}(y)})$$
for $Y \in \mathcal{S}_X$ and for an arbitrary $\alpha > 0$, with $Y_* P_X(y) = P_X(Y = y)$ the marginal law.
- There is an $\mathcal{A}_X$-*module* structure $\mathcal{M}_\alpha(\mathcal{Q}_X)$ on $\mathcal{M}(\mathcal{Q}_X)$, determined by the semigroup action σ_α.

- The category $\mathcal{A}$-Mod of modules over the sheaf of algebras $X \mapsto \mathcal{A}_X$ is an abelian category.
- There is a sequence $\mathcal{B}_n(X)$ of free $\mathcal{A}_X$-modules generated by symbols $[X_1 | \dots | X_n]$ with $\{X_1, \dots, X_n\} \subset \mathcal{S}_X$, and with boundary maps $\partial_n : \mathcal{B}_n \to \mathcal{B}_{n-1}$ of the Hochschild form

$$\begin{aligned} \partial_n[X_1 | \dots |X_n] = & \ X_1 [X_2 | \dots | X_n] \\ + & \sum_{k=1}^{n-1} (-1)^k [X_1 | \dots | X_k X_{k+1} | \dots | X_n] \\ + & \ (-1)^n [X_1 | \dots |X_{n-1}]. \end{aligned} \tag{5.7}$$

  The modules $\mathcal{B}_n(X)$ give a projective bar resolution of the trivial $\mathcal{A}_X$-module.
- There is a functor $C^\bullet(\mathcal{M}_\alpha(\mathcal{Q})) : (S, M) \to \mathrm{Ch}(\mathbb{R})$ to the category of *cochain complexes*, that assigns to $X \in \mathrm{Obj}(S)$ a cochain complex $(C^\bullet(\mathcal{M}_\alpha(\mathcal{Q}_X)), \delta)$ with $C^\bullet(\mathcal{M}_\alpha(\mathcal{Q}_X))^n = \mathrm{Hom}_{\mathcal{A}_X}(\mathcal{B}_n(X), \mathcal{M}_\alpha(\mathcal{Q}_X))$ (that is, natural transformations of functors $\mathcal{B}_n \to \mathcal{M}_\alpha(\mathcal{Q})$ compatible with the $\mathcal{A}$-action) and with coboundary δ given by the Hochschild-type coboundary

$$\begin{aligned} \delta(f)[X_1 | \dots | X_{n+1}] = & \ X_1(f)[X_2 | \dots | X_{n+1}] \\ + & \sum_{k=1}^{n} (-1)^k f[X_1 | \dots | X_k X_{k+1} | \dots | X_{n+1}] \\ + & \ (-1)^{n+1} f[X_1 | \dots |X_n]. \end{aligned} \tag{5.8}$$

- One writes $C^\bullet((S, M), \mathcal{M}_\alpha(\mathcal{Q})) := (C^\bullet(\mathcal{M}_\alpha(\mathcal{Q}_X)), \delta)$ and $H^\bullet((S, M), \mathcal{M}_\alpha(\mathcal{Q}))$ for the resulting cohomology. The zeroth cohomology is $\mathbb{R}$ when $\alpha = 1$ and zero otherwise. In the case of the first cohomology, any non-trivial 1-cocycle is locally a multiple of the Tsallis entropy

$$S_\alpha[X](P) = \frac{1}{\alpha - 1} \left(1 - \sum_{x \in M_X} P(x)^\alpha \right),$$

  for $\alpha \neq 1$ or of the Shannon entropy for $\alpha = 1$. The higher cohomologies similarly represent all possible higher mutual information functionals.

This functorial construction can be used to map networks to an *abelian* category of informational resources. According to what we discussed earlier in this section, we want an assignment of informational resources to networks that factors through an intermediate category of codes (or weighted codes). Thus, we revisit here the construction of §5.3, using the more sophisticated setting of cohomological information recalled above.

5.4.2 Network summing functors and information

We now return to the category of codes $\mathrm{Codes}_{n,*}$ introduced in Lemma 5.6 and the category of network summing functors $\Sigma^{eq}_{\mathrm{Codes}_{n,*}}(G)$. We show that there is an associated category of network summing functors obtained by mapping the summing functors $\Phi \in \Sigma^{eq}_{\mathrm{Codes}_{n,*}}(G)$ to summing functors in $\Sigma_{\mathcal{A}\text{-Mod}}(G)$, with $\mathcal{A}$-Mod the abelian category of sheaves of $\mathcal{A}_X$-modules as in [106], and to summing functors in $\Sigma_{\mathrm{Ch}(\mathbb{R})}(G)$ with values in cochain complexes. Summing functors in these categories satisfy the inclusion-exclusion relations of §2.3.3.

Lemma 5.15 *There is a contravariant functor $\mathcal{I} : \mathrm{Codes}_{n,*} \to \mathcal{IS}$ from the category $\mathrm{Codes}_{n,*}$ to the category $\mathcal{IS}$ of finite information structures that maps the coproduct $C \vee C'$ in $\mathrm{Codes}_{n,*}$ to the coproduct $(S, M) \vee (S', M')$ in $\mathcal{IS}$.*

Proof. Given a code $C \in \mathrm{Codes}_{n,*}$, with $\#C$ code words of length n including the base point given by the 0-word c_0, consider the set $\mathcal{I}(C) = S^C$ of all random variables $X : C \to \mathbb{R}$ with values in a finite subset of $\mathbb{R}$ and with $X(c_0) = 0$. One should think of such a variable as a probabilistic assignment of weights to the code words. A morphism in $\mathrm{Codes}_{n,*}$ is a function $f : C \to C'$ that maps the 0-word c_0 to itself. For $X' \in S^{C'}$ let $\mathcal{I}(f)(X') = X \in S^C$ be given by $X = X' \circ f : C \to \mathbb{R}$.

The object $\mathbf{1}$ is the random variable that maps the whole code to 0 and $\mathcal{I}(f)(\mathbf{1}) = \mathbf{1}$. Whenever $X'Y' = X' \wedge Y'$ is an object in $S^{C'}$ we have $X'Y' \circ f = X' \circ f \wedge Y' \circ f$ an object in S^C. We define the map on the value sets as the projection $\pi_f : M_{X'} = M_{X' \circ f}$ that maps $m \in M_{X'}$ to itself if $m = X'(f(c))$ for some $c \in C$ and to 0 otherwise. Note that 0 is always an element of both $M_{X'}$ and $M_{X' \circ f}$ because of the 0-word. The coproduct of codes $C \vee C'$ is obtained from the disjoint union of the two codes by identifying the respective 0-words. Under the functor $\mathcal{I}$, the code $C \vee C'$ is mapped to the category $S^{C \vee C'}$ of random variables with finite range $X^\vee : C \vee C' \to \mathbb{R}$ that map the 0-word to 0. Such a random variable $X^\vee$ applied to code words in C determines a random variable in S^C and applied to words in C' determines a random variable in $S^{C'}$, and in turn is determined by such random variables, which necessarily agree on the 0-word. The pair of $\mathbf{1}_{S^C}$ and $\mathbf{1}_{S^{C'}}$ gives $\mathbf{1}_{S^{C \vee C'}}$. Thus, we have $S^{C \vee C'} = S^C \vee S^{C'}$. □

Lemma 5.16 *Under the functor $\mathcal{MQ} : \mathcal{IS} \to \mathcal{A}$-Mod, the product $(S, M) \times (S', M')$ maps to the tensor product $\mathcal{M}_\alpha(\mathcal{Q}) \otimes \mathcal{M}_\alpha(\mathcal{Q}')$ of $\mathcal{A}$-modules and the coproduct $(S, M) \vee (S', M')$ in $\mathcal{IS}$ maps to the sum $\mathcal{M}_\alpha(\mathcal{Q}) \oplus \mathcal{M}_\alpha(\mathcal{Q}')$ of $\mathcal{A}$-modules.*

Proof. As shown in §2.12 of [106], at the level of the probability functors $\mathcal{Q}, \mathcal{Q}'$ we have $\mathcal{Q} \times \mathcal{Q}' : (S, M) \times (S', M') \to \Delta$ with $(\mathcal{Q} \times \mathcal{Q}')_{(X,X')}$ the simplicial set given by probabilities on $M_X \times M_{X'}$ that are products $P(x, x') = P(x)P'(x')$, so that $(\mathcal{Q} \times \mathcal{Q}')_{(X,X')} \simeq \mathcal{Q}_X \times \mathcal{Q}'_{X'}$, while $\mathcal{Q} \vee \mathcal{Q}' : (S, M) \vee (S', M') \to \Delta$ is defined on $X \in \mathrm{Obj}(S)$ as $\mathcal{Q}_X$ and on $X' \in \mathrm{Obj}(S')$ as $\mathcal{Q}'_{X'}$. When we consider the vector space of measurable functions we then obtain $\mathcal{M}_\alpha((\mathcal{Q} \times \mathcal{Q}')_{(X,X')}) = \mathcal{M}_\alpha(\mathcal{Q}_X \times \mathcal{Q}'_{X'}) \simeq \mathcal{M}_\alpha(\mathcal{Q}_X) \otimes \mathcal{M}_\alpha(\mathcal{Q}'_{X'})$. Similarly, the vector space of functions on the simplicial sets obtained from $\mathcal{Q} \vee \mathcal{Q}'$ splits as a direct sum of $\mathcal{M}_\alpha(\mathcal{Q}_X)$ for $X \in \mathrm{Obj}(S)$ and $\mathcal{M}_\alpha(\mathcal{Q}'_{X'})$ for $X' \in \mathrm{Obj}(S')$. □

The following is a direct consequence of the previous lemmas.

Corollary 5.17 *Composition with the functor $\mathcal{MQ} \circ \mathcal{I}$ maps summing functors $\Phi \in \Sigma^{\mathrm{eq}}_{\mathrm{Codes}_{n,*}}(G)$ to summing functors $\mathcal{MQ}(\mathcal{I}(\Phi)) \in \Sigma^{\mathrm{eq}}_{\mathcal{A}\text{-Mod}}(G)$.*

Similarly, we can consider composition with the functor that assigns to a finite information structure the corresponding cochain complex $C^\bullet((S, M), \mathcal{M}_\alpha(\mathcal{Q}))$ and its cohomology

$$H^\bullet((S, M), \mathcal{M}_\alpha(\mathcal{Q}))\,.$$

Proposition 5.18 *Let $\mathcal{K} := C^\bullet(\mathcal{M}_\alpha(\mathcal{Q})) : \mathcal{IS} \to \mathrm{Ch}(\mathbb{R})$ be the functor that maps finite information structures to their information cochain complex*

$$(S, M) \mapsto C^\bullet((S, M), \mathcal{M}_\alpha(\mathcal{Q})).$$

Composition with $\mathcal{K} \circ \mathcal{I}$ maps summing functors $\Phi \in \Sigma^{\mathrm{eq}}_{\mathrm{Codes}_{n,*}}(G)$ to summing functors $\mathcal{K}(\mathcal{I}(\Phi)) \in \Sigma^{\mathrm{eq}}_{\mathrm{Ch}(\mathbb{R})}(G)$, with $\mathcal{K}(\mathcal{I}(\Phi))(G') = C^\bullet((S, M)^{G'}, \mathcal{M}_\alpha(\mathcal{Q}))$ where we write $(S, M)^{G'} := \mathcal{I}(\Phi)(G')$ for $G' \subset G$. These satisfy the inclusion-exclusion property of §2.3.3, namely for $G_1, G_2 \subset G$ there is a short exact sequence of cochain complexes

$$0 \to \mathcal{K}(\mathcal{I}(\Phi))(G_1 \cap G_2) \to \mathcal{K}(\mathcal{I}(\Phi))(G_1) \oplus \mathcal{K}(\mathcal{I}(\Phi))(G_2) \to \mathcal{K}(\mathcal{I}(\Phi))(G_1 \cup G_2) \to 0,$$

with

$$\begin{array}{l} \mathcal{K}(\mathcal{I}(\Phi))(G_1 \cap G_2) = C^\bullet((S, M)^{G_1 \cap G_2}, \mathcal{M}_\alpha(\mathcal{Q})) \\ \mathcal{K}(\mathcal{I}(\Phi))(G_i) = C^\bullet((S, M)^{G_i}, \mathcal{M}_\alpha(\mathcal{Q})) \\ \mathcal{K}(\mathcal{I}(\Phi))(G_1 \cup G_2) = C^\bullet((S, M)^{G_1 \cup G_2}, \mathcal{M}_\alpha(\mathcal{Q})), \end{array}$$

hence a corresponding long exact sequence of information cohomologies.

Proof. Since Φ is a summing functor in $\Sigma^{\mathrm{eq}}_{\mathrm{Codes}_{n,*}}(G)$ the value of Φ on a subnetwork $G' \subset G$ reduces to the sum, in the category $\mathrm{Codes}_{n,*}$ of the codes $C_e = \Phi(e)$ associated to the edges $e \in E(G')$, that is, the coproduct $\bigvee_{e \in E(G')} C_e$. Thus, given $G_1, G_2 \subset G$ we have $\Phi(G_1 \cap G_2) = \bigvee_{e \in E(G_1 \cap G_2)} C_e$ as a subsummand of both $\Phi(G_1)$ and $\Phi(G_2)$, and each of these in turn gives a subsummand of $\Phi(G_1 \cup G_2)$. Applying Lemma 5.16 we then obtain an exact sequence of $\mathcal{A}$-modules

$$0 \to \mathcal{M}_\alpha(\mathcal{Q}^{G_1 \cap G_2}) \to \mathcal{M}_\alpha(\mathcal{Q}^{G_1}) \oplus \mathcal{M}_\alpha(\mathcal{Q}^{G_2}) \to \mathcal{M}_\alpha(\mathcal{Q}^{G_1 \cup G_2}) \to 0.$$

The $\mathcal{A}$-modules $\mathcal{B}_n$ are projective, hence $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B}_n, \cdot)$ is an exact functor, hence we obtain the short exact sequence of cochain complexes. □

5.4.3 Other functorial maps to information structures

In the previous subsection we focused on the category $\mathcal{C} = \mathrm{Codes}_{n,*}$ as we have done previously in §5, and a functor $\mathcal{I} : \mathrm{Codes}_{n,*} \to \mathcal{IS}$ from codes to information structures. The same construction and the result of Proposition 5.18 can be generalized to other categories $\mathcal{C}$ (as target of the summing functors), together with a functor $\mathcal{I} : \mathcal{C} \to \mathcal{IS}$ with the property that the sum $C_1 \oplus C_2$ in $\mathcal{C}$ maps to the coproduct $(S, M)^{C_1} \vee (S, M)^{C_2}$ of information structures.

Corollary 5.19 *Consider summing functors $\Phi \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$. Given a functor $\mathcal{I} : \mathcal{C} \to \mathcal{IS}$ preserving coproducts, the composition $\mathcal{K} \circ \mathcal{I}$ with $\mathcal{K} = C^\bullet(\mathcal{M}_\alpha(\mathcal{Q}))$ maps summing functors $\Phi \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$ to summing functors $\mathcal{K}(\mathcal{I}(\Phi)) \in \Sigma^{\rm eq}_{\mathrm{Ch}(\mathbb{R})}(G)$ with $\mathcal{K}(\mathcal{I}(\Phi))(G') = C^\bullet((S, M)^{G'}, \mathcal{M}_\alpha(\mathcal{Q}))$ satisfying the inclusion-exclusion property as in Proposition 5.18.*

In particular, one can consider the case where $\mathcal{C} = \mathcal{F}_*$ is the category of finite pointed set. As we will see in §7, this case corresponds to the Gamma-space that is the embedding of $\mathcal{F}_*$ in Δ_*, whose spectrum is the sphere spectrum. In this case, the functor $\mathcal{I} : \mathcal{F}_* \to \mathcal{IS}$ maps a finite pointed set $A \in \mathcal{F}_*$ to the information structure $(S, M)^A$ with $\mathrm{Obj}(S^A)$ the random variables $X : A \to \mathbb{R}$ with $X(a_0) = 0$ at the basepoint $a_0 \in A$. This satisfies $(S, M)^{A \vee A'} = (S, M)^A \vee (S, M)^{A'}$. In the case where the sets A describe the sets of vertices V_{G_*} or edges E_{G_*} of a network, we identify the resulting $C^\bullet((S, M)^{G'}, \mathcal{M}_\alpha(\mathcal{Q}))$ and its cohomology with the measuring of information content of the subnetwork $G' \subset G$. In the more general case of other categories $\mathcal{C}$ as in Corollary 5.19, the information complex $\mathcal{K}(\mathcal{I}(\Phi))(G') = C^\bullet((S, M)^{G'}, \mathcal{M}_\alpha(\mathcal{Q}))$ measures the information content of the resources $\Phi(G') \in \mathcal{C}$ assigned to the subnetwork G'.

5.5 Codes and simplicial sets

We discuss here how the simplicial sets associated to binary (convex) neural codes through the associated open covering and its nerve fit in the setting of information structures we introduced in §5.4. The convexity hypothesis for a code $C \subset \mathbb{F}_2^n$ consists of the requirement that the code words $c \in C$ can be realized as intersection patterns of a family $\{U_1, \ldots, U_n\}$ of convex open sets in some Euclidean space $\mathbb{R}^d$, see [31].

More precisely, in this setting, we have a code C with $N = \#C$ code words of n letters each, with alphabet $\{0, 1\}$. We consider a collection $\{U_\nu\}_{\nu=1}^n$ of open sets (receptive fields) associated to the n neurons ν. For each code word $c \in C$ we consider the letters $c_\nu = 1$. These are neurons that simultaneously fire in the reading represented by the code word c, hence receptive fields that overlap. This means that we have an intersection $\cap_{\nu : c_\nu = 1} U_\nu$ associated to each code word $c \in C$. One then considers a simplicial set associated to the code given by the nerve $\mathcal{N}(\mathcal{U}(C))$ of the collection $\mathcal{U}(C) = \{U_\nu\}$. This has a k-simplex for every non-empty $(k+1)$-fold intersection. We write these as Δ_c for the simplex associated to the intersection $\cap_{\nu : c_\nu = 1} U_\nu$. The code C is convex if the U_ν are convex.

Lemma 5.20 *Given a binary convex code C, there is a finite information structure (S, M) and a probability functor $\mathcal{Q}$ for which there is a random variable X in* $\mathrm{Obj}(S)$ *such that $\mathcal{Q}_X = \mathcal{N}(\mathcal{U}(C))$ is the nerve of the collection of open coverings $\mathcal{U}(C)$ associated to the code C.*

Proof. Given a code C as above, we write $C^\vee$ for the transpose code that has n code words with N letters each. We write the code words of $C^\vee$ as $\nu = (\nu_c)_{c=1}^N$. Consider then the set of real-valued random variables $X : C \times C^\vee \to \mathbb{R}$. In particular, consider the case of

$$X(c, \nu) = X_c(\nu) = \begin{cases} 0 & c_\nu = 0 \\ \alpha_\nu & c_\nu = 1 \end{cases}$$

where $\alpha_\nu \neq 0$ and $\alpha_\nu \neq \alpha_{\nu'}$ for $\nu \neq \nu'$. Consider a probability functor $\mathcal{Q}$, in the sense recalled in §5.4 mapping random variables X to simplicial sets $\mathcal{Q}_X \subset \Delta_{M_X}$, that maps $X : C \times C^\vee \to \mathbb{R}$ to the simplicial set $\mathcal{Q}_X = \cup_{c \in C} \Delta_{M_{X_c}}$. Note that the simplex $\Delta_{M_{X_c}}$ is just the simplex on a number of vertices equal to $\#\{\nu : c_\nu = 1\}$. □

We have obtained in this way a realization of the nerve simplicial set $\mathcal{N}(\mathcal{U}(C))$ through the information structures and probability functors construction of §5.4.

5.5.1 Nerves of coverings and functoriality

We can also consider the question of whether the assignment of the simplicial set $\mathcal{N}(\mathcal{U}(C))$ to a code C is functorial with respect to an appropriate choice of morphisms of codes.

In Lemma 5.6 we defined the category $\text{Codes}_{n,*}$ of binary codes with objects that are binary codes that include the zero word and morphisms that are maps of pointed sets between them. Here we consider a subcategory on the same objects with a subclass of morphisms. For simplicity we will neglect base points, and work with an un-based version Codes_n of the category of binary codes.

Since we want to think of our codes as neural codes that detect the spiking activity of a population of neurons over a span of time subdivided into basic intervals, we can regard codes as maps

$$C : X \times T_n \to \{0, 1\}, \tag{5.9}$$

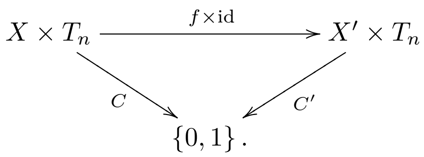
where $X \in \mathcal{F}$ is a finite set and

$$T_n = \{[t_0, t_0 + \Delta t], [t_0 + \Delta t, t_0 + 2\Delta t], \dots, [t_0 + (n-1)\Delta t, t_0 + n\Delta t]\}$$

is the set of basic intervals, identified with $T_n = \{1, 2, \dots, n\}$. Thus, the set T_n is fixed and dependent only on the choice of $n \in \mathbb{N}$. Code words in C are given by

$$c_x = C(\{x\} \times T_n), \quad \text{for } x \in X. \tag{5.10}$$

Proposition 5.21 *Let Codes'_n be the category of codes with objects the maps as in (5.9) and with morphisms $f \in \text{Mor}_{\text{Codes}'_n}(C, C')$, for $C : X \times T_n \to \{0,1\}$ and $C' : X' \times T_n \to \{0,1\}$, given by maps $f : X \to X'$ that fit in a commuting diagram*



There is a functor $F : \text{Codes}'_n \to \text{Codes}_n$ that is faithful when restricting to codes that have no repeated code words. The map

$$\mathcal{N}\mathcal{U} : \text{Codes}_n \to \Delta, \qquad C \mapsto \mathcal{N}(\mathcal{U}(C)),$$

that assigns to a code the simplicial set given by the nerve of the covering $\mathcal{U}(C)$ determined by the code defines a functor $\mathcal{N}\mathcal{U} \circ F : \text{Codes}'_n \to \Delta$.

Proof. We identify an object of Codes'_n with an object of Codes_n, by assigning to the map $C : X \times T_n \to \{0,1\}$ the set of code words $C = \cup_{x \in X} c_x = \cup_{x \in X} C(\{x\} \times T_n)$. Given a map $f : X \to X'$ we obtain a morphism $\phi_f : C \to C'$ in Codes_n by setting

$$\phi_f(c) = \phi_f(C(\{x\} \times T_n)) := C'(\{f(x)\} \times T_n).$$

In other words, the morphism ϕ_f places the word c_x of C in the position $f(x)$ in C'. Indeed, since $C' \circ (f, id) = C$, these code words agree, $c'_{f(x)} = c_x$, as binary words of length n. Suppose we only consider codes that have no repeated words (which implies we also consider only injective maps $f : X \to X'$). The identity $\phi_f(c) = \phi_g(c)$ for all $c \in C$ means that the code words $c'_{f(x)} = C'(\{f(x)\} \times T_n) = c_x$ and $c'_{g(x)} = C'(\{g(x)\} \times T_n) = c_x$ are the same for all $x \in X$. If $f(x) \neq g(x)$ for some $x \in X$, the code C' has repeated words. Thus, in the case of codes with no repeated words, we obtain a faithful functor $F : \text{Codes}'_n \hookrightarrow \text{Codes}_n$ that realizes Codes'_n as a subcategory of the category of codes Codes_n. To check the functoriality of the assignment $C \mapsto \mathcal{N}(\mathcal{U}(C))$, we can describe the simplicial set $\mathcal{N}(\mathcal{U}(C))$ in the following way. The set $\mathcal{N}(\mathcal{U}(C))_0$ of vertices of $\mathcal{N}(\mathcal{U}(C))$ is given by the subset of X

$$\mathcal{N}(\mathcal{U}(C))_0 = \{x \in X \mid C(x, i) = 1, \ \text{for some } i \in T_n\}.$$

The set $\mathcal{N}(\mathcal{U}(C))_k$ of k-simplices is given by the set

$$\mathcal{N}(\mathcal{U}(C))_k = \{\sigma = \{x_0, \ldots, x_k\} \subset X \,|\, \exists i \in T_n \text{ such that } C(x,i) = 1, \ \forall x \in \sigma\} .$$

Then we can associate to a morphism $f \in \mathrm{Mor}_{\mathrm{Codes}'_n}(C, C')$ an induced simplicial map $f_* : \mathcal{N}(\mathcal{U}(C)) \to \mathcal{N}(\mathcal{U}(C'))$ by setting

$$f_* : \mathcal{N}(\mathcal{U}(C))_k \to \mathcal{N}(\mathcal{U}(C'))_k , \qquad \sigma = \{x_0, \ldots, x_k\} \mapsto f_*(\sigma) = \{f(x_0), \ldots, f(x_k)\} .$$

Indeed, if there is an $i \in T_n$ such that $C(x,i) = 1$ for all $x \in \sigma$, then $C'(f(x), i) = C(x,i)$ by our choice of morphisms, so that we also have $f_*(\sigma) = \{f(x_0), \ldots, f(x_k)\} \in \mathcal{N}(\mathcal{U}(C'))_k$. □

5.6 Transition systems, codes, and information structures

We describe here a functorial mapping from the category $\mathcal{C}$ of transition systems to the category $\mathrm{Codes}_{n,*}$ of codes, describing codes generated by the automata in $\mathcal{C}$, and its composition with the functor $\mathrm{Codes}_{n,*}$ to the category $\mathcal{IS}$ of finite information structures, as in Lemma 5.15.

We consider again the category $\mathcal{C}$ of concurrent/distributed computational architectures given by transition systems, as in [110], recalled in §4.1 and §4. Also let $\mathcal{IS}$ denote the category of finite information structures of [106], recalled in §5.4.

As discussed in [110], in the category $\mathcal{C}$ of transition systems $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ one usually assumes that the set $\mathcal{T}$ of transitions always contains also the "idle transitions" of the form $(s, \star, s)$ with a special label symbol $\star \in \mathcal{L}$, which describe the case where the system at the state $s \in S$ does not update to a new state.

Recall that, given an automaton $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$, the formal language $\mathbb{L}(\tau)$ recognized by the automaton consists of all the sequences of composable transitions in the automaton τ, of arbitrary finite length,

$$(s_0, \ell_1, s_1)(s_1, \ell_2, s_2) \cdots (s_{n-1}, \ell_n, s_n), \qquad \text{with} \quad s_0 = \iota.$$

Lemma 5.22 *There is a contravariant functor $\mathcal{J} : \mathcal{C} \to \mathcal{IS}$ that assigns to a transition system $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ the finite information structure $\mathcal{J}(\tau) = (S, M)^{\mathbb{L}(\tau)}$, where $\mathbb{L}(\tau)$ is the language of the automaton τ, and the category $(S, M)^{\mathbb{L}(\tau)}$ has objects the random variables $X : \mathbb{L}(\tau) \to \mathbb{R}$ with finite range that map to 0 the language word consisting of the idle transition $(\iota, \star, \iota)$.*

Proof. A morphism $\phi : \tau \to \tau'$ consists of a pair $\phi = (\sigma, \lambda)$ of a function $\sigma : S \to S'$ with $\sigma(\iota) = \iota'$ and a (partially defined) function $\lambda : \mathcal{L} \to \mathcal{L}'$ such that, if $(s, \ell, s') \in \mathcal{T}$ and $\lambda(\ell)$ is defined, then one has $(\sigma(s), \lambda(\ell), \sigma(s')) \in \mathcal{T}'$. Let us consider here, for simplicity, the case where λ is globally defined. Such a morphism determines a function $\mathbb{L}(\tau) \to \mathbb{L}(\tau')$, by identifying words in the language $\mathbb{L}(\tau)$ with composable finite sequences of transitions in τ and mapping such a sequence via (σ, λ) to a corresponding sequence of composable transitions in τ', that is, to a word in the language $\mathbb{L}(\tau')$. When including idle transitions, one requires that morphisms $\phi = (\sigma, \lambda)$ in $\mathcal{C}$ not only have $\sigma(\iota) = \iota'$ but also $\lambda(\star) = \star'$, hence they map the idle transition $(\iota, \star, \iota)$ to the idle transition $(\iota', \star', \iota')$ and the word consisting of a concatenation of n idle transitions at the initial state is then mapped to itself. One then obtains a morphism $\mathcal{J}(\phi) : (S, M)^{\mathbb{L}(\tau')} \to (S, M)^{\mathbb{L}(\tau)}$ by precomposition with ϕ. □

Lemma 5.23 *For all $n \in \mathbb{N}$, there is a functor $C_{\mathbb{L},n} : \mathcal{C} \to \mathrm{Codes}_{n,*}$ from the category of transition systems $\mathcal{C}$ to the category of pointed binary codes, obtained by assigning to a system $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ the set $\mathcal{W}_n(\tau) \subset \mathbb{L}(\tau)$ of words of length n in the automaton language $\mathbb{L}(\tau)$, and then mapping the set $\mathcal{W}_n(\tau)$ to a binary code $C_{\tau,n}$ of length n, with code words $c(w)$, for $w \in \mathcal{W}_n(\tau)$ given by*

$$c(w)_i = \begin{cases} 0 & w_i = (s, \star, s) \text{ for some } s \in S \\ 1 & w_i \neq (s, \star, s) \, \forall s \in S \, , \end{cases} \tag{5.11}$$

detecting whether w_i is the idle transition or not.

Proof. Note that the code $C_{\tau,n}$ contains the zero word c_0 as the image of the word in $\mathcal{W}_n(\tau)$ consisting of a concatenation of n idle words $(\iota, \star, \iota)$. The code detects whether, in each of the

n discrete time intervals Δt, the system τ has moved from its current state to another state or has idled in the current state without activity. A morphism $\phi = (\sigma, \lambda) : \tau \to \tau'$ induces a map $\phi : \mathbb{L}(\tau) \to \mathbb{L}(\tau')$ that maps $\mathcal{W}_n(\tau)$ to $\mathcal{W}_n(\tau')$ and the word given by the concatenation of n idle transitions $(\iota, \star, \iota)$ to the concatenation of n idle transitions $(\iota', \star', \iota')$. It therefore induces a corresponding map $C_{\mathbb{L},n}(\phi) : C_{\tau,n} \to C_{\tau',n}$ that maps the code word $c(w)$ to the code word $c(\phi(w))$ mapping the zero word to itself. □

The next statement is then a direct consequence of Lemma 5.22, Lemma 5.23, and Lemma 5.15.

Proposition 5.24 *Let $\mathcal{I} : \mathrm{Codes}_{n,*} \to \mathcal{IS}$ be the contravariant functor from codes to finite information structures constructed in Lemma 5.15. Let $(S, M)^{C_{\mathbb{L}}(\tau)} = \mathcal{I}(C_{\mathbb{L},n}(\tau))$ be the image of an object $\tau = (\mathcal{S}, \iota, \mathcal{L}, \mathcal{T})$ of $\mathcal{C}$ under the composition $\mathcal{I} \circ C_{\mathbb{L},n}$ with the functor $C_{\mathbb{L},n}$ of Lemma 5.23. Let $(S, M)^{\mathbb{L}(\tau)} = \mathcal{J}(\tau)$ with the functor $\mathcal{J}$ as in Lemma 5.22. The category $(S, M)^{C_{\mathbb{L}}(\tau)}$ is the subcategory of $(S, M)^{\mathbb{L}(\tau)} = \mathcal{J}(\tau)$ whose objects are the random variables $X : \mathbb{L}(\tau) \to \mathbb{R}$ with finite range such that, when restricted to $\mathcal{W}_n(\tau) \subset \mathbb{L}(\tau)$ factor through the code $C_{\tau,n} = C_{\mathbb{L},n}(\tau)$,*

$$\begin{array}{ccc} \mathcal{W}_n(\tau) & \xrightarrow{X|_{\mathcal{W}_n(\tau)}} & \mathbb{R} \\ & \searrow \quad \nearrow & \\ & C_{\tau,n} & \end{array}$$

Equivalently, the random variables $X : \mathbb{L}(\tau) \to \mathbb{R}$ with finite range that are in the subcategory $(S, M)^{C_{\mathbb{L}}(\tau)}$ are those whose value on words in $\mathcal{W}_n(\tau)$ depends only on which transitions in these words are or are not idle, but does not depend on the specific non-idle transitions. This means that we can regard the set $\mathcal{W}_n(\tau)$ of words of length n in the automaton language $\mathbb{L}(\tau)$ as a natural refinement of the binary code $C_{\tau,n}$. In terms of networks of neurons, the binary code represents the neural code obtained by only retaining the information on whether a certain neuron in the network is firing or not during each of the n time intervals Δt, while the set $\mathcal{W}_n(\tau)$ also encodes more specific information on the output of the active neurons, with each interval of time Δt corresponding to a transition in the corresponding automata that simulate the computational activity of the neurons.

Note that there are different possible ways of constructing computational models of individual neurons, in the form of automata and computational architectures. For example, in [78] the categorical Hopfield equations introduced in this paper are analyzed in the case where computational models of the neuron are given by certain deep neural networks as in [14].

In particular, we can then apply the construction of cohomological information as in §5.4 and §8 either by applying probability functors $\mathcal{Q}$ to $(S, M)^{C_{\mathbb{L}}(\tau)}$ or to the larger category $(S, M)^{\mathbb{L}(\tau)}$.

5.7 Clique complexes and information structures

We have shown in §5.5 that the nerve simplicial set $\mathcal{N}(\mathcal{U}(C))$ of a (convex) code C can be recovered from the construction of §5.4 of the simplicial set $\mathcal{Q}_X$ associated to a random variable X in the information structure $(S, M)^C = \mathcal{I}(C)$ obtained from a binary code through the functor $\mathcal{I}$ of Lemma 5.15, for a particular choice of the probability functor $\mathcal{Q}$ and of the random variable.

We show here that in a similar way, for a particular choice of the probability functor $\mathcal{Q}$ and the random variable X, the simplicial set given by the clique complex $K(G)$ of the network G can be recovered from the construction of $\mathcal{Q}_X$ for X in $(S, M)^{\mathbb{L}(\tau_G)}$, where $\tau_G = \Upsilon(\Phi)(G)$, for some $\Phi \in \Sigma_{\mathcal{C}'}(G)$ and $\Upsilon(\Phi) \in \Sigma^{\mathrm{prop}}_{\mathcal{C}'}(G)$ obtained by grafting as in Proposition 4.4 of §4.3.1. This shows that the homotopy types obtained from the simplicial sets $\mathcal{Q}_X$ encompass both the usual homotopy types $\mathcal{N}(\mathcal{U}(C))$ detecting the nontrivial topological information carried by the receptive fields of neural codes and also the homotopy types $K(G)$ that detect the amount of non-trivial topology present in the activated network.

Proposition 5.25 *Consider the composition $\mathcal{J} \circ \Upsilon$ of the functors $\mathcal{J}$ of Lemma 5.22 with the functor Υ of Proposition 4.4. There is a choice of a probability functor $\mathcal{Q}$ and of a random variable X in the finite information structure $(S, M)^{\tau_G} = \mathcal{J} \circ \Upsilon(\Phi)(G)$ such that the resulting simplicial set $\mathcal{Q}_X$ is the (directed) clique complex $K(G)$ of the network G (see §7.4.1).*

Proof. For a network G and a summing functor $\Phi \in \Sigma_{\mathcal{C}'}(V_G)$, the functor $\mathcal{J} \circ \Upsilon$ determines a finite information structure $(S, M)^{\tau_G}$ whose objects are the random variables $X : \mathbb{L}(\tau_G) \to \mathbb{R}$ with finite range, where τ_G is the transition system in $\mathcal{C}'$ (or $\mathcal{C}^t$) obtained through the grafting procedure described in §4.3.1 applied to the systems $\tau_v = \Phi(v)$. For simplicity we consider the case where G itself is acyclic. The case for more general directed G is treated as in §4.3.1 by considering strong connected components G_i and the condensation acyclic graph $\bar{G}$ and is similar. For ω a topological ordering of the vertices of G as described in §4.3.1, we can write any word in the automaton language $\mathbb{L}(\tau_G)$ as a sequence

$$w_{i_0} e_{i_0} \cdots w_{i_{k-1}} e_{i_{k-1}} w_{i_k}, \quad \text{for some } k \in \mathbb{N}, \quad \text{with } \; w_\ell \in \tau_{v_\ell} = \Phi(v_\ell) \tag{5.12}$$

and with the e_{i_r} given by edges in G, with vertices along the path satisfying $v_i \leq v_j$ in the order ω for $i \leq j$. In other words, a sequence of transitions in the automaton τ_G consists of a sequence that alternates transitions in G (along the directed edges of a path in G) and transitions inside the automata τ_v associated to the vertices along the path. For $\sigma = \{v_{i_0}, \ldots, v_{i_k}\}$ we write $\sigma \in \text{supp}(X)$ if X takes non-zero values on all the words (5.12). We write $X|_\sigma$ for the restriction of the random variable X to the set of words of the form (5.12) for the ordered sequence of vertices in σ. We consider a probability functor $\mathcal{Q}$ given by $\mathcal{Q}_X = \cup_{\sigma \in \text{supp}(X)} \Delta_{M_{X|_\sigma}}$. We restrict then to those random variables $X : \mathbb{L}(\tau_G) \to \mathbb{R}$ that are supported on the subset of words in $\mathbb{L}(\tau_G)$ of the form (5.12), where the set $\sigma = \{v_{i_0}, \ldots, v_{i_k}\}$ of vertices is a k-clique of $K(G)$. We write Δ_σ for the k-simplex in $K(G)$ that corresponds to this clique. Note that by construction Δ_σ is in fact a directed clique in the ordering ω. We further consider, among these random variables, an $X : \mathbb{L}(\tau_G) \to \mathbb{R}$ such that X takes on exactly $k+1$ different non-zero values on each set of words (5.12) for each k-clique σ. For such a random variable, we then obtain that $\mathcal{Q}_X = \cup_\sigma \Delta_{M_{X|_\sigma}} = K(G)$ is the (directed) clique complex of G. □

6 Categorical Hopfield dynamics

The setting we described in the previous sections for modeling neural information networks, based on categories of network summing functors and symmetric monoidal categories of systems and resources, should be regarded as a static setting, like the kinematic description of a physical system, the overall configuration space, while we did not yet introduce an adequate modeling of dynamics. This is the topic we discuss in this section. Our model is based on the traditional way of describing dynamics of networks in terms Hopfield networks, where nodes have a dynamics governed by excitatory or inhibitory synaptic connections with certain thresholds.

It is important to note that the threshold-linear dynamics of Hopfield networks, which is what we formulate here in our categorical setting, is a *non-linear* model of the neuron, unlike the linear model we discussed briefly in §5.2.

Formulating a Hopfield network type of dynamics directly in the setting of categories of summing functors makes it possible to simultaneously include in the dynamics all the different levels of structures we have been analyzing in the previous sections, with their functorial relations: the network together with its associated codes and weights, the associated computational systems, the associated resources and constraints, both metabolic and informational. All of the structure evolves then according to an overall dynamics that functions in functorially related ways at the various different levels.

6.1 Continuous and discrete Hopfield dynamics

Typically, the Hopfield network models are either formulated in a discrete form with binary neurons and the dynamics in the form

$$\nu_j(n+1) = \begin{cases} 1 & \text{if } \sum_k T_{jk} \nu_k(n) + \eta_j > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{6.1}$$

or in the continuum form with neurons firing rates as variables and a threshold-linear dynamics of the form

$$\frac{dx_j}{dt} = -x_j + \left(\sum_k W_{jk} x_k + \theta_j \right)_+ \tag{6.2}$$

where W_{jk} are the real-valued connection strengths, θ_j are constant external inputs, and

$$(\cdot)_+ = \max\{0, \cdot\} \tag{6.3}$$

is the threshold function that introduces the non-linearities in the equation. For a detailed analysis of the dynamics of the continuum Hopfield networks see [30], [34], [86].

Here we consider a version of the Hopfield networks dynamics that can be formulated in a categorical setting and that can be applied to the setting of categories of network summing functors that we described in the previous sections.

6.2 Categorical threshold non-linearity

A main step in constructing the categorical version of the Hopfield dynamics is to have an appropriate way of describing the non-linearities through a threshold function. We do this using the measuring monoids $(R, +, \succeq, 0)$ associated to the symmetric monoidal categories of resources $(\mathcal{R}, \otimes, \mathbb{I})$, as recalled in §3.2.2.

We assume here that $\mathcal{C}$ is a symmetric monoidal category, which we write additively with $\oplus$ and 0, in order to maintain in the following the similarity of notation with the classical threshold function (6.3). Let $\rho : \mathcal{C} \to \mathcal{R}$ be a monoidal functor from the category $\mathcal{C}$ to a symmetric monoidal category of resources, as discussed in the previous section and let $(R, +, \succeq, 0)$ be the preordered monoid associated to the category $\mathcal{R}$.

One could just assume here, for simplicity, that $\mathcal{R} = \mathcal{C}$. We allow for another $\mathcal{R}$ to express the possibility that the threshold in $\mathcal{C}$ is measured with respect to another type of resources $\mathcal{R}$ that is related to $\mathcal{C}$ functorially. For example, we may be interested in viewing the dynamics at the level of a category of codes, with a threshold measured in terms of information associated to codes functorially as in §5.

Proposition 6.1 *Let $\mathcal{C}$ and $\mathcal{R}$ be unital symmetric monoidal categories with a monoidal functor $\rho : \mathcal{C} \to \mathcal{R}$ as above. Let $\hat{\mathcal{C}}$ denote the category with the same objects as $\mathcal{C}$ and with morphisms the invertible morphisms of $\mathcal{C}$. There is a threshold endofunctor $(\cdot)_+ : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ that acts on objects as*

$$(C)_+ = \begin{cases} C & \text{if } [\rho(C)] \succeq 0 \ \text{ in } (R, +, \succeq, 0) \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Composition with this threshold endofunctor induces an endofunctor of the categories of summing functors $\Sigma_{\mathcal{C}}(X)$, for finite pointed sets X.

Proof. The class $[\rho(C)]$ in R only depends on the isomorphism class $[C]$, as the functor $\rho : \mathcal{C} \to \mathcal{R}$ induces a corresponding semigroup homomorphism. Thus, if $\phi : C \to C'$ is an isomorphism, the image $(\phi)_+$ is either ϕ itself if $[\rho(C)] = [\rho(C')] \succeq 0$ or the identity morphism id_0 otherwise. This determines $(\cdot)_+$ as an endofunctor of $\hat{\mathcal{C}}$. Note that $(\cdot)_+$ is in general not an endofunctor of $\mathcal{C}$, and also that $(\cdot)_+$ need not be a monoidal functor. Suppose given a summing functor $\Phi \in \Sigma_{\mathcal{C}}(X)$. Since we are working here in the setting of unital symmetric monoidal categories, rather than categories with sums and zero object, we define $\Sigma_{\mathcal{C}}(X)$ as in Definition 2.5. Thus, Φ is defined by the collection of objects $\{\Phi(x)\}_{x \in X \smallsetminus \{*\}}$ of $\mathcal{C}$. Thus, we can assign to Φ a new summing functor $(\Phi)_+$ which is determined by the values $(\Phi(x))_+$ in $\mathcal{C}$ for $x \in X$. Morphisms $\phi : \Phi \to \Psi$ in the category of summing functors are a collection $\phi_x : \Phi(x) \to \Psi(x)$ of isomorphisms in $\mathcal{C}$. Composing with the threshold endofunctor $(\cdot)_+$ of $\hat{\mathcal{C}}$ then gives the corresponding isomorphisms $(\phi_x)_+ : (\Phi(x))_+ \to (\Psi(x))_+$, hence the corresponding invertible natural transformation $(\phi)_+ : (\Phi)_+ \to (\Psi)_+$. □

In the case where $\mathcal{C}$ is a commutative monoidal category, the argument above can be adapted to the other possible definition of summing functors, as in Definition 2.2 and Corollary 2.4.

6.3 Discrete Hopfield dynamics in categories of summing functors

As above, let $\mathcal{C}$ be a symmetric monoidal category, written additively with $\oplus$ and 0, and let $\rho : \mathcal{C} \to \mathcal{R}$ be a monoidal functor from the category $\mathcal{C}$ to a symmetric monoidal category of resources, as discussed in the previous section and let $(R, +, \succeq)$ be the preordered semigroup associated to the category $\mathcal{R}$. For simplicity of notation, we will write $r_C \in R$ for the class $[\rho(C)]$ used in our definition of the threshold functor (6.4).

Let $\Sigma_{\mathcal{C}}(X)$ be the category of summing functors. For a directed graph G, we focus here on the subcategory of the category $\Sigma_{\mathcal{C}}(G)$ of network summing functors given by the equalizer $\Sigma^{\rm eq}_{\mathcal{C}}(G)$ of the source and target functors $s, t : \Sigma_{\mathcal{C}}(E_{G^*}) \rightrightarrows \Sigma_{\mathcal{C}}(V_{G^*})$. In principle, the construction we present below can be adapted, mutatis mutandis, to other subcategories of network summing functors, but we focus here on discussing only one case. For simplicity of notation we just write $\Sigma_{\mathcal{C}}(E)$ and $\Sigma_{\mathcal{C}}(V)$ for these two categories of summing functors. We can then define a dynamical system with threshold-dynamics on $\Sigma^{\rm eq}_{\mathcal{C}}(G)$ in the following way.

Let $\mathcal{E}(\mathcal{C}) = {\rm Func}(\mathcal{C}, \mathcal{C})$ be the category of monoidal endofunctors of $\mathcal{C}$, with morphisms given by natural transformations. The sum of endofunctors is defined pointwise by $(F \oplus F')(C) = F(C) \oplus F'(C)$ for all $C \in {\rm Obj}(\mathcal{C})$.

Assume given a graph G and $E = E_{G^*}$ as above. Let $\mathcal{P}(E) \times \mathcal{P}(E)$ be the product category with objects given by pairs of objects (A, B) with pointed subsets $A \subset E$ and $B \subset E$ and morphisms given by pairs of inclusions $A \hookrightarrow A'$ and $B \hookrightarrow B'$.

Let $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ be a functor satisfying the summing properties $T_{A \cup A', B} = T_{A,B} \oplus T_{A',B}$, for $A \cap A' = \{e_*\}$ in E_{G^*} and for all $B \in \mathcal{P}(E)$, and $T_{A, B \cup B'} = T_{A,B} \oplus T_{A,B'}$ for $B \cap B' = \{e_*\}$ and for all A. In particular, we write $T_{ee'}$ for the case where $A = \{e, e_*\}$ and $B = \{e', e_*\}$. By the same argument as in Lemma 2.3, the endofunctors $T_{ee'}$ completely determine $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ because of the summing properties.

Let $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E)$ denote the category of functors $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ with the summing properties as above, with morphisms given by the invertible natural transformations. Similarly, we define $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(V)$ for $V = V_{G^*}$ with source and target functors $s, t : \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E) \rightrightarrows \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(V)$ given by $T^s_{A,B} = T_{s^{-1}(A), s^{-1}(B)}$ and $T^t_{A,B} = T_{t^{-1}(A), t^{-1}(B)}$, for $A, B \in \mathcal{P}(E)$. Let $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$ denote the equalizer of the functors

$$s, t : \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E) \rightrightarrows \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(V).$$

Definition 6.2 *Let $\Phi_0 \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$ be an initial choice of a summing functor $\Phi_0 : \mathcal{P}(E) \to \mathcal{C}$ with conservation law at vertices. We write $X_e(0) := \Phi_0(e)$ where $\Phi_0(e)$ stands for the object in $\mathcal{C}$ that is the image under Φ_0 of the pointed subset $\{e, *\}$ of E_{G^*}. The choice of a functor $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ as above, together with the initial $\Phi_0 \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$ determine a dynamical system*

$$X_e(n+1) = X_e(n) \oplus (\oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+ \tag{6.5}$$

where $\Theta_e = \Psi(e)$ are the values at $\{e, *\}$ of a fixed summing functor $\Psi \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$ and with $(\cdot)_+$ the threshold functor of Proposition 6.1.

Lemma 6.3 *For $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ in the equalizer $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$ and $\Phi_0 : \mathcal{P}(E) \to \mathcal{C}$ and $\Psi : \mathcal{P}(E) \to \mathcal{C}$ in the equalizer $\Sigma^{\rm eq}_{\mathcal{C}}(G)$, the dynamics (6.5) defines a sequence Φ_n of summing functors in $\Sigma^{\rm eq}_{\mathcal{C}}(G)$.*

Proof. If $r_{X_e(n)}$ never satisfies the threshold condition then the dynamics is trivial and just gives the constant Φ_0 functor. Assuming a non-trivial dynamics, the right-hand side of (6.5) defines the values $\Phi_{n+1}(e)$ at the subsets $\{e, *\}$ of E_{G^*} of the new functor Φ_{n+1}. Indeed, we have shown in Proposition 6.1 that the threshold functor is an endofunctor of the category of summing functors, so the right-hand side determines a unique summing functor, with values completely specified by the $\Phi_{n+1}(e)$, through the summing property $\Phi_{n+1}(A) = \oplus_{e \in A} \Phi_{n+1}(e)$. We need to check that the resulting Φ_{n+1} still satisfies the conservation law at vertices, so that it defines a summing functor in the equalizer $\Sigma^{\rm eq}_{\mathcal{C}}(G)$. We have

$$\bigoplus_{s(e)=v} \Phi_{n+1}(e) = \bigoplus_{s(e)=v} \Phi_n(e) \oplus \bigoplus_{e' \in E} \bigoplus_{s(e)=v} T_{ee'}(\Phi_n(e')) \oplus \bigoplus_{s(e)=v} \Psi(e),$$

The first and last term on the right-hand side are respectively equal to $\oplus_{t(e)=v}\Phi_n(e)$ and $\oplus_{t(e)=v}\Psi(e)$. Since $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ is in the equalizer $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$, the endofunctors $T_{ee'}$ of $\mathcal{C}$ satisfy $\oplus_{s(e)=v}T_{ee'}(C) = \oplus_{t(e)=v}T_{ee'}(C)$ for all $C \in \mathrm{Obj}(\mathcal{C})$, hence also the second term in the above sum is equal to $\oplus_{e'\in E} \oplus_{t(e)=v} T_{ee'}(\Phi_n(e'))$, hence we obtain that

$$\bigoplus_{s(e)=v} \Phi_{n+1}(e) = \bigoplus_{t(e)=v} \Phi_{n+1}(e)$$

which implies that Φ_{n+1} is in the equalizer $\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$. □

We should think of the equation (6.5) as the categorical version of a finite-difference form of the Hopfield network equations

$$\frac{x_j(t+\Delta t) - x_j(t)}{\Delta t} = (\sum_k T_{jk} x_k(t) + \theta_j)_+ \,, \tag{6.6}$$

where for simplicity we can assume discretized time intervals $\Delta t = 1$. Usually, in the Hopfield network dynamics, one introduces an additional "leak term" $-x_j(t)$ on the right-hand side of the equation, to ensure that a neuron firing rate would decay exponentially to zero if the threshold term is zero, so that the corresponding difference equation would look like

$$\frac{x_j(t+\Delta t) - x_j(t)}{\Delta t} = -x_j(t) + (\sum_k T_{jk} x_k(t) + \theta_j)_+ \,. \tag{6.7}$$

An analog of equation (6.6) (with $\Delta t = 1$) in the categorical setting would be of the form

$$X_e(n+1) = (\oplus_{e'\in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+ \tag{6.8}$$

for which again the result of Lemma 6.3 holds. A categorical analog of (6.7) for $\Delta t << 1$ can be formulated as

$$X_e(n+1) \oplus X_e(n) = (\oplus_{e'\in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+ \,. \tag{6.9}$$

In this case however, one cannot directly apply the argument of Lemma 6.3 anymore. One can still seek solutions of (6.9) where all the Φ_n are in $\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$, if they exist.

Note that, in the case where the symmetric monoidal category $\mathcal{C}$ of resources has a zero object, one does have projection maps $X_e(n+1) \oplus X_e(n) \to X_e(n+1)$ and $X_e(n+1) \oplus X_e(n) \to X_e(n)$, obtained by applying the unique morphism $X_e(n) \to 0$ and $X_e(n+1) \to 0$. However, this does not suffice to extend the argument of Lemma 6.3 to this case. Moreover, if 0 is a zero object, then the threshold-nonlinearity becomes trivial and the equation reduces to a linear dynamics.

For the purpose of this discussion, we will only consider the equation of the form (6.8), where we can incorporate a diagonal term so as to include the case (6.5), so that Lemma 6.3 applies.

Lemma 6.4 *The categorical Hopfield network dynamics (6.5), (6.8) induces a discrete dynamical system τ on the simplicial set given by the nerve $\mathcal{N}(\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G))$ and its realization, the classifying space $|\mathcal{N}(\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G))| = B\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$.*

Proof. The functoriality of the nerve construction, seen as a functor $\mathcal{N} : \mathrm{Cat} \to \Delta$ from the category of small categories to the category of simplicial sets implies that the endofunctor $\mathcal{T}$ that assigns to an object Φ in $\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$ the object $\mathcal{T}(\Phi)$ determined by the equation (6.5) or (6.8) induces a simplicial self-map $\mathcal{T}_{\mathcal{N}}$ of the nerve $\mathcal{N}(\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G))$ and a corresponding self-map $\mathcal{T}_B$ of the realization $B\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$ as a topological space. Thus, the categorical dynamical system (6.5) or (6.8) determines a classical discrete dynamical system on the topological space $B\Sigma^{\mathrm{eq}}_{\mathcal{C}}(G)$ given by the orbits under the iterates $\mathcal{T}_B^n$. □

We will return to comment more extensively on this topological model of the categories of summing functors and the dynamics in §7 below.

6.4 Category of weighted codes and ordinary Hopfield dynamics

The goal of the very general categorical form of Hopfield dynamics introduced in the previous section is to model dynamics of different types of resources associated to a network. For this reason, we have formulated the equations (6.5), (6.8) in such a way that the dynamical variable is an assignment of resources of type $\mathcal{C}$ to a network, that is, a summing functor. This setting is very general in the sense that the equations allow for an arbitrary choice of an initial assignment Φ_0, a constant term Ψ (which is a choice of another summing functor) and an endofunctor T that generates the dynamics.

Since we want this broad setting to be a generalization of the usual Hopfield equations on networks, we need to check a basic consistency with the original equations, namely we need to show that those can be re-obtained as a *special case* of the categorical Hopfield dynamics described above, for a very special choice of the category $\mathcal{C}$ and the data of the equation.

Thus, we now check that, in the case where the category $\mathcal{C}$ is a version of the category of weighted codes considered in §5, with a particular choice of the functor T in the categorical Hopfield equation, the categorical Hopfield dynamics recovers the usual Hopfield network dynamics (in a discretized finite-difference form) on associated total weights. To this purpose we restrict to the case with only non-negative weights, which in the resulting Hopfield network dynamics would be interpreted as activity levels.

Definition 6.5 *Let $\mathcal{W}\mathrm{Codes}^+_{n,*}$ be the category of weighted codes, where we only consider non-negative weights, that is, objects (C,ω) have $\omega(c) \geq 0$ for all $c \in C$ and morphisms $\phi = (f,\Lambda) : (C,\omega) \to (C',\omega')$ with the weights satisfying $\sum_{c\,:\,f(c)=c'} \lambda_{c'}(c) \leq 1$, for all $c' \in C'$. These conditions are well behaved under composition of morphisms.*

Lemma 6.6 *The assignment $\alpha(C,\omega) = \sum_{c\in C} \omega(c)$ defines a functor*

$$\alpha : \mathcal{W}\mathrm{Codes}^+_{n,*} \to \mathbb{R},$$

where we view $(\mathbb{R}, \leq)$ as a thin category, compatible with sums.

Proof. For $\phi = (f,\Lambda) : (C,\omega) \to (C',\omega')$ a morphism in $\mathcal{W}\mathrm{Codes}^+_{n,*}$ we have $\alpha(C,\omega) = \sum_{c\in C} \omega(c) = \sum_{c\in C} \lambda_{f(c)}(c)\, \omega'(f(c)) \leq \sum_{c'\in C'} \omega'(c') = \alpha(C',\omega')$, hence $\alpha(\phi)$ is the unique morphism in $(\mathbb{R}, \leq)$ between $\alpha(C,\omega)$ and $\alpha(C',\omega')$. The functor α maps the sum $(C,\omega) \oplus (C',\omega') = (C \vee C', \omega \vee \omega')$ to $\alpha(C,\omega) + \alpha(C',\omega') \in \mathbb{R}$, and maps the object $(\{c_0\}, 0)$ to $0 \in \mathbb{R}$. □

In order to distinguish, in our setting, between inhibitory and excitatory effects in the Hopfield dynamics (6.5), (6.6) and (6.7), we can consider the possibility of a term $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ in the equation that has values either in the category $\mathcal{E}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ (excitatory case) or in the category $\mathcal{E}^o(\mathcal{C})$ of *contravariant* endofunctors, determined by a collection of functors $T_{e,e'} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ (inhibitory case). More precisely, we can consider the following setting.

Definition 6.7 *Let $\mathcal{C} = \mathcal{W}\mathrm{Codes}^+_{n,*}$ and let $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ and $T^o : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}^o(\mathcal{C})$ be, respectively, functors in the equalizers $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$ and $\Sigma^{(2)}_{\mathcal{E}^o(\mathcal{C})}(G)$, where $\mathcal{E}(\mathcal{C})$ and $\mathcal{E}^o(\mathcal{C})$ are, respectively, the categories of covariant and contravariant endofunctors of $\mathcal{C}$. The functors T and T^o are, respectively, linear-excitatory and linear-inhibitory if for all $e, e' \in E$ there is a covariant or contravariant endofunctor, $\tau_{ee'}$ and $\tau^o_{ee'}$, respectively, of the thin category $(\mathbb{R}, \leq)$ such that the diagrams of functors commute*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T_{ee'}} & \mathcal{C} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ (\mathbb{R}, \leq) & \xrightarrow{\tau_{ee'}} & (\mathbb{R}, \leq) \end{array} \qquad \begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{T^o_{ee'}} & \mathcal{C} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ (\mathbb{R}, \geq) & \xrightarrow{\tau^o_{ee'}} & (\mathbb{R}, \leq) \end{array}$$

where $\tau_{ee'}$ and $\tau^o_{ee'}$ act linearly on $\mathbb{R}$, $\tau_{ee'}(r) = t_{ee'} \cdot r$ for some $t_{ee'} \in \mathbb{R}^* = \mathbb{R} \smallsetminus \{0\}$ and for all $r \in \mathbb{R}$, and similarly for $\tau^o_{ee'}$ and the corresponding $t^o_{ee'}$, where by covariance/contravariance $t_{ee'} > 0$ and $t^o_{ee'} < 0$.

We focus on the linear-inhibitory case. The excitatory case is analogous. Linear-inhibitory functors T also satisfy the following properties.

Lemma 6.8 *Let $\mathcal{C} = \mathcal{W}\mathrm{Codes}^+_{n,*}$ with $\rho : \mathcal{C} \to \mathcal{R}$ a functor to a symmetric monoidal category of resources and $(R, +, \succeq, 0)$ the associated monoid. Assume that there exists a measuring monoid homomorphism $M : (R, +, \succeq, 0) \to (\mathbb{R}, +, \geq, 0)$ satisfying $M \circ \rho = \alpha : \mathcal{C} \to \mathbb{R}$, and such that $M(r) \geq 0$ in $\mathbb{R}$ iff $r \succeq 0$ in R. A linear-inhibitory functor $T^o : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}^o(\mathcal{C})$ satisfies*

1. *By contravariance of $\tau^o_{ee'}$ and linearity, all the multiplicative factors satisfy $t^o_{ee'} < 0$.*
2. *For any object $(C, \omega) \in \mathrm{Obj}(\mathcal{C})$ such that $r_{\rho(C,\omega))} \succeq 0$ in $(R, +, \succeq, 0)$, and for all $e, e' \in E$, we have $0 \succeq r_{\rho(T^o_{ee'}(C,\omega))}$ in $(R, +, \succeq, 0)$.*
3. *The ratio $Mr_{\rho(T^o_{ee'}(C,\omega))}/Mr_{\rho(C,\omega))}$ is independent of the object (C, ω) and equal to $t^o_{ee'} < 0$.*

Proof. By contravariance of $\tau^o_{ee'}$ we have $\tau^o_{ee'}(r) \geq \tau^o_{ee'}(s)$ when $r \leq s$, hence if $\tau^o_{ee'}(r) = t^o_{ee'} \cdot r$ is linear, the multiplicative factor satisfies $t^o_{ee'} < 0$. The measuring homomorphism preserves the order relation so $r_{\rho(C,\omega))} \succeq 0$ implies $0 \succeq r_{\rho(T^o_{ee'}(C,\omega))}$ since $Mr_{\rho(C,\omega))} \geq 0$ implies $Mr_{\rho(T^o_{ee'}(C,\omega))} = \alpha T^o_{ee'}(C, \omega) = \tau^o_{ee'}\alpha(C, \omega) = t^o_{ee'} \cdot Mr_{\rho(C,\omega))} \leq 0$. The ratio

$$Mr_{\rho(T^o_{ee'}(C,\omega))}/Mr_{\rho(C,\omega))} = \alpha T^o_{ee'}(C, \omega)/\alpha(C, \omega) = t^o_{ee'}$$

is independent of the object (C, ω). □

Lemma 6.9 *Let $\rho : \mathcal{W}\mathrm{Codes}^+_{n,*} \to \mathcal{R}$ be a functor to a symmetric monoidal category of resources, with $(R, +, \succeq, 0)$ the associated semigroup, with a measuring semigroup homomorphism $M : (R, +, \succeq, 0) \to (\mathbb{R}, +, \geq, 0)$ as in Lemma 6.8. Let $T^o : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}^o(\mathcal{C})$ be a linear-inhibitory functor in the equalizer $\Sigma^{(2)}_{\mathcal{E}^o(\mathcal{C})}(G)$. Let Θ_e in (6.5) be such that $\theta_e = \alpha(\Theta_e) > 0$. The Hopfield dynamics (6.5) on $\Sigma^{\mathrm{eq}}_{\mathcal{W}\mathrm{Codes}^+_{n,*}}(G)$ induces the finite differences Hopfield network equation on the total weights*

$$\alpha_{n+1}(e) = \alpha_n(e) + \left(\sum_{e'} t^o_{ee'}\, \alpha_n(e') + \theta_e \right)_+ , \tag{6.10}$$

with inhibitory connections $t^o_{ee'} < 0$ and with $(x)_+ = \max\{0, x\}$.

Proof. Given a summing functor $\Phi : \mathcal{P}(E) \to \mathcal{W}\mathrm{Codes}^+_{n,*}$, with $(C_e, \omega_e) = \Phi(e)$, we define the total weight as a functor $\alpha_\Phi : \mathcal{P}(E) \to \mathbb{R}$, with $\alpha_\Phi(A) = \sum_{e \in A} \sum_{c \in C_e} \omega_e(c)$, so that $\alpha(A \cup A') = \alpha(A) + \alpha(A')$ for $A \cap A' = \{e_*\}$ and with $\alpha_\Phi(j : A \hookrightarrow A')$ a morphism in $(\mathbb{R}, \leq)$ since $\alpha_\Phi(A) \leq \alpha_\Phi(A')$ under the assumption that all the weights are non-negative. The total weight $\alpha_\Phi : \mathcal{P}(E) \to \mathbb{R}$ is the composite of the functor $\Phi : \mathcal{P}(E) \to \mathcal{W}\mathrm{Codes}^+_{n,*}$ with the functor $\alpha : \mathcal{W}\mathrm{Codes}^+_{n,*} \to \mathbb{R}$ of Lemma 6.6. Similarly, we associate to functors $T^o : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}^o(\mathcal{W}\mathrm{Codes}^+_{n,*})$ and $\Phi_0 : \mathcal{P}(E) \to \mathcal{W}\mathrm{Codes}^+_{n,*}$ the composites $\tau = \alpha \circ T^o : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}$ and $\alpha_0 = \alpha \circ \Phi_0 : \mathcal{P}(E) \to \mathbb{R}$. By applying the functor $\alpha : \mathcal{W}\mathrm{Codes}^+_{n,*} \to \mathbb{R}$ to the equation (6.5) we then obtain an equation of the form

$$\alpha_{n+1}(e) = \alpha_n(e) + \left(\sum_{e'} \alpha(T^o_{ee'}(\Phi_n(e'))) + \theta_e \right)_+ ,$$

where $\theta_e = \alpha(\Theta_e) > 0$. (The positivity of $\alpha(\Theta_e)$ is assumed in order to have a non-trivial dynamics.) The hypothesis of linearity of T ensures that $\alpha(T^o_{ee'}(\Phi_n(e'))) = \tau^o_{ee'}\alpha(\Phi_n(e')) = t^o_{ee'}\alpha_n(e')$. The condition that $\sum_{e'} T^o_{ee'}(\Phi_n(e')) + \Theta(e) \succeq 0$ in $(R, + \succeq)$ is satisfied iff there is a morphism in the monoidal category $\mathcal{R}$ of resources from $\rho(\sum_{e'} T^o_{ee'}(\Phi_n(e')) + \Theta(e))$ to the unit of $\mathcal{R}$. By the properties of the measuring semigroup homomorphism M this condition is satisfied iff $\alpha(\sum_{e'} T^o_{ee'}(\Phi_n(e')) + \Theta(e)) \geq 0$ in $\mathbb{R}$, hence it matches the condition that $\sum_{e'} t^o_{ee'}\, \alpha_n(e') + \theta_e \geq 0$, so that we obtain the equation (6.10). □

7 Gamma-spaces and Gamma networks

In the previous sections we have assigned resources in a category $\mathcal{C}$ to networks through a category of network summing functors $\Sigma_{\mathcal{C}}(G)$, or some suitable subcategory. As we discussed in §4, these categories of network summing functors are obtained as simple modifications of the original definition of [96] of categories of summing functors $\Sigma_{\mathcal{C}}(X)$, for X a finite pointed set. We have interpreted such categories $\Sigma_{\mathcal{C}}(G)$ of network summing functors as a configuration space of all possible consistent assignments of resources of type $\mathcal{C}$ to subnetworks of the network G. In §6 we have also described how to introduce a form of dynamics on this configuration space, through our categorical formulation of the Hopfield equations.

As we observed in Lemma 6.4, this categorical configuration space with the associated categorical dynamical system has a topological model provided by the nerve of the category of summing functors, together with the induced dynamics, given by a discrete dynamical system on a topological space. The latter can then be studied by the usual tools of dynamics on topological spaces.

Thus, while the category $\Sigma_{\mathcal{C}}(X)$ of summing functors represents the parameterizing space of all consistent assignments of resources to a system and its subsystems, the nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of the category $\Sigma_{\mathcal{C}}(X)$ organizes the data of these assignments of resources to subsets in a topological structure that keeps track of all equivalence relations between them, determined by the invertible natural transformations that are the morphisms of $\Sigma_{\mathcal{C}}(X)$ and their compositions. Thus, we view the topological space $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ as an actual geometric incarnation of our configuration space $\Sigma_{\mathcal{C}}(X)$.

Note that the geometric realization $|\mathcal{N}(\mathcal{A})|$ of the nerve of a category $\mathcal{A}$ is the classifying space $B\mathcal{A}$ of the category. This can be described (see [109]) as parameterizing sheaves of $\mathcal{A}$-sets with representable stacks, where an $\mathcal{A}$-set is a functor from $\mathcal{A}^{\rm op}$ to Sets and it is representable if it is of the form $F_A : B \mapsto {\rm Hom}_{\mathcal{A}}(B, A)$.

In the case of categories of summing functors $\Sigma_{\mathcal{C}}(X)$ for finite pointed sets X, the equivalence of categories between $\Sigma_{\mathcal{C}}(X)$ and $\hat{\mathcal{C}}^n$ with $\#X = n + 1$, shows that the nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$, when considered for all possible X, describe topological information about the category $\mathcal{C}$ in the form of a delooping of the infinite loop space given by (a completion of) the classifying space $B\mathcal{C}$, see [20] for a detailed discussion of this delooping construction. The point we want to stress here is that the collection of the nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$, for finite pointed sets X, only encode topological information about the category $\mathcal{C}$. This changes, however, when we consider network summing functors in $\Sigma_{\mathcal{C}}(G)$, as these also contain information on the structure of the network and subnetworks. We will describe in this section the original Segal construction of Gamma-spaces, which accounts for the collection of the nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ and their relations under maps of pointed sets, and we will introduce a corresponding notion of Gamma networks that is based instead on the nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(G))$ for finite directed graphs G.

7.1 Gamma-spaces

A *Gamma-space* (see [96]) is a functor $\Gamma : \mathcal{F}_* \to \Delta_*$ from the category $\mathcal{F}_*$ of finite pointed sets to the category Δ_* of pointed simplicial sets.

In the original construction of Segal [96], the source category of Gamma-spaces was taken to be the category (called Γ^0 in [96] and identified here with $\mathcal{F}_*^{\rm op}$) where the objects are finite pointed sets as in $\mathcal{F}_*$ but with morphisms given by the preimages under a map of pointed sets. This means that for pointed finite sets X and Y a morphism $\phi : Y \to X$ is a collection $\{S_y\}_{y\in Y}$ of subsets of X, given by $S_y = f^{-1}(y)$, for a map of pointed sets $f : X \to Y$ (a morphism in $\mathcal{F}_*$). However, we follow here the later use (see for instance the discussion in §XIV.3 of [92]) and we define Gamma-spaces as functors from the opposite of this category, for which we use the same notation $\mathcal{F}_*$ that we used in the previous sections, which is just the category of finite pointed sets with base-point-preserving maps. Working with this version of Gamma-spaces as *covariant* functors of pointed maps will be more convenient for us.

It is shown in [96] that to any category $\mathcal{C}$ with a categorical sum and a zero object, one can associate a Gamma-space $\Gamma_{\mathcal{C}} : \mathcal{F}_* \to \Delta_*$, which assigns to a pointed set X the nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of the category $\Sigma_{\mathcal{C}}(X)$ of summing functors.

Note that with this choice of $\mathcal{F}_*$ rather than Segal's Γ^0 as the source category of a Gamma-space, the morphism $\Gamma_{\mathcal{C}}(f)$ associated to a map of pointed finite sets $f : X \to Y$ is obtained using the pushforward map $f_* : \Sigma_{\mathcal{C}}(X) \to \Sigma_{\mathcal{C}}(Y)$ on summing functors defined by setting

$$f_*\Phi(B) = \Phi(f^{-1}(B \smallsetminus \{*\}) \cup \{*\}), \quad \text{for } B \in P(Y), \tag{7.1}$$

with $\Phi \in \Sigma_{\mathcal{C}}(X)$, so that $f_*\Phi : P(Y) \to \mathcal{C}$ is a summing functor, see §XIV.4 of [92].

As we will discuss more in details in §7.2, it is also shown in [96] that a Gamma-space $\Gamma : \mathcal{F}_* \to \Delta_*$ extends to an endofunctor $\Gamma : \Delta_* \to \Delta_*$ and the latter determines an associated spectrum with spaces $X_n = \Gamma(S^n)$ and structure maps $S^1 \wedge \Gamma(S^n) \to \Gamma(S^{n+1})$. More generally, one can consider categories $\mathcal{C}$ that are unital symmetric monoidal categories. It was shown in [101] that the Segal construction of Γ-spaces, seen as a functor $\Gamma : \mathcal{M} \to \mathbb{S}$ from the category $\mathcal{M}$ of small symmetric monoidal categories to the category $\mathbb{S}$ of connective spectra determines an equivalence of categories between the localization of the first category, obtained by inverting those morphisms that are sent to weak homotopy equivalences, and the stable homotopy category of connective spectra. Additionally, by this result of [101], all connective spectra can be obtained from Gamma-spaces. Moreover, the smash product of spectra has a very natural and simple description in terms of Gamma-spaces, as shown in [70].

The Γ-space construction is functorial. A strict symmetric monoidal functor $\rho : \mathcal{C} \to \mathcal{C}'$ of small symmetric monoidal categories induces a functor $\rho : \Sigma_{\mathcal{C}}(X) \to \Sigma_{\mathcal{C}'}(X)$ between the respective categories of summing functors given by composition $(\Phi_X : P(X) \to \mathcal{C}) \mapsto \rho \circ \Phi_X : P(X) \to \mathcal{C}'$. The fact that ρ is strict shows the summing property is preserved under $\Phi_X \mapsto \rho \circ \Phi_X$. This functor in turn determines a natural transformation $\rho : \Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{C}'}$ of the corresponding Γ-spaces.

The construction of Γ-spaces $\Gamma_{\mathcal{C}}$ was extended from the case of categories $\mathcal{C}$ with sums and zero object as in [96] to the case of unital symmetric monoidal categories in [101], [103]. In this more general setting, $\Gamma_{\mathcal{C}}$ is first defined as a pseudo-functor $\Gamma_{\mathcal{C}} : \Gamma^0 \to \mathrm{Cat}$ that assigns to a finite set X its category of summing functors $\Sigma_{\mathcal{C}}(X)$ as in Definition 2.5. This is a pseudo-functor since compatibility with composition of morphisms and identity morphisms is only satisfied up to canonical isomorphisms, involving the associators, unitors, and braiding, see the Appendix of [103]. One then obtains an actual functor by applying the Kleisli construction of [100]. We will not discuss this case in detail, but we refer the reader to the Appendix of [103] for a more precise treatment.

For our purposes we only need to know that the Γ-space formalism applies to unital symmetric monoidal categories and that, for example, a functor $\rho : \mathcal{C} \to \mathcal{C}'$ as above from a symmetric monoidal category of computational architectures to another associated category of resources, induces a corresponding natural transformation $\rho : \Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{C}'}$ of the associated Γ-spaces. Note that here one needs to pay attention to the distinction between lax monoidal functors and strict monoidal functors as we mentioned in §2.1, in relation to the setting considered in [103] for the 2-category of unital symmetric monoidal categories.

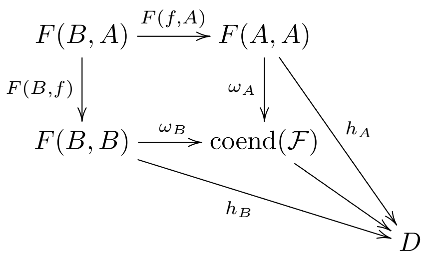
7.2 Gamma-spaces as endofunctors of simplicial sets

The extension of a Gamma-space $\Gamma_{\mathcal{C}} : \mathcal{F}_* \to \Delta_*$ to an endofunctor $\Gamma_{\mathcal{C}} : \Delta_* \to \Delta_*$ is obtained in the following way.

Given a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, a *cowedge* for F is a dinatural transformation (a natural transformation for both entries of F) from F to the constant functor on an object $D \in \mathrm{Obj}(\mathcal{D})$, that is, a family of morphisms $h_A : F(A, A) \to D$ such that, for all morphisms $f : A \to B$ in $\mathcal{C}$ one has a commutative diagram

$$\begin{array}{ccc} F(B,A) & \xrightarrow{F(f,A)} & F(A,A) \\ {\scriptstyle F(B,f)}\downarrow & & \downarrow{\scriptstyle h_A} \\ F(B,B) & \xrightarrow[h_B]{} & D \end{array}$$

The *coend* $\mathrm{coend}(F)$ is an initial object in the category of *cowedges* for F, that is, for every morphism $f : A \to B$ there is a unique arrow $\mathrm{coend}(F) \to D$ that gives a commutative diagram



It is customary to use for the coend the notation

$$\int^{C\in\mathcal{C}} F(C,C) := \mathrm{coend}(F).$$

Let $[n] = \{0, \dots, n\}$ denote the finite pointed set in $\mathcal{F}_*$ with $n+1$ elements. Given a pointed simplicial set K with K_n the pointed set of n-simplexes of K, the extension of a Gamma-space $\Gamma_\mathcal{C} : \mathcal{F}_* \to \Delta_*$ to an endofunctor of Δ_* is given by the coend

$$\Gamma_\mathcal{C} : K \mapsto \int^{[n]\in\mathcal{F}_*} K_n \wedge \Gamma_\mathcal{C}([n]). \tag{7.2}$$

The smash product $K_n \wedge \Gamma_\mathcal{C}([n])$ has the effect of attaching a copy of the simplicial set $\Gamma_\mathcal{C}([n])$ to each element of the set K_n, and the coend takes care of the fact that these attachments are made compatibly with the face and degeneracy maps of the simplicial set K. By comparison with the geometric realization of the pointed simplicial set K, where one takes the coend

$$|K| = \int^{[n]\in\mathcal{F}_*} K_n \wedge \Delta_n, \tag{7.3}$$

we see that in (7.2) the functor $\Gamma_\mathcal{C}$ acts on the simplicial set K by replacing all the n-simplexes Δ_n of K with copies of $\Gamma_\mathcal{C}([n])$.

The spectrum associated to the Gamma-space $\Gamma_\mathcal{C} : \mathcal{F}_* \to \Delta_*$ is then the collection of $X_n = \Gamma_\mathcal{C}(S^n)$ with S^n the n-sphere, with the structure maps $S^1 \wedge \Gamma_\mathcal{C}(S^n) \to \Gamma_\mathcal{C}(S^{n+1})$.

7.3 Gamma-spaces and homotopy types

By Proposition 4.9 of [15], the endofunctor $\Gamma_\mathcal{C} : \Delta_* \to \Delta_*$ determined by a Gamma-space $\Gamma_\mathcal{C}$ preserves weak homotopy equivalences, hence it descends to a map of homotopy types. Given a Gamma-space, we associate to any pointed simplicial set a collection of homotopy types defined as follows.

Definition 7.1 Let $\Gamma_\mathcal{C}$ be the Gamma-space associated to a category $\mathcal{C}$ with $\Gamma_\mathcal{C}(X) = \mathcal{N}(\Sigma_\mathcal{C}(X))$ for a finite pointed set X. Consider its extension $\Gamma_\mathcal{C} : \Delta_* \to \Delta_*$ to an endofunctor of pointed simplicial sets as above. Given a pointed simplicial set K, the family of homotopy types associated to K by $\Gamma_\mathcal{C}$ is the collection of pointed simplicial sets $\{\Gamma_\mathcal{C}(\Sigma^n(K))\}_{n\in\mathbb{N}}$ up to weak homotopy equivalence, with $\Sigma^n(K)$ the n-fold suspension. We refer to this collection of homotopy types as the "representation of K under $\Gamma_\mathcal{C}$" and in particular to the homotopy type $\Gamma_\mathcal{C}(K)$ as the "primary representation".

The point of view we have in mind here is to view a Gamma-space $\Gamma_\mathcal{C}$, seen as an endofunctor of Δ_* as a machine that encodes input simplicial sets (or input homotopy types) into output simplicial sets (output homotopy types) where the encoding is done via a combination of the input data with data from the category of resources $\mathcal{C}$.

This can be seen more precisely by comparing, as in §7.2 above, the two coend constructions of (7.2) and (7.3). If we have an input simplicial set, which we think of in terms of its realization

$$|K| = \int^{[n]\in\mathcal{F}_*} K_n \wedge \Delta_n \,,$$

the Gamma-space $\Gamma_{\mathcal{C}}$ transforms it into the simplicial set

$$\Gamma_{\mathcal{C}}(K) = \int^{[n]\in\mathcal{F}_*} K_n \wedge \Gamma_{\mathcal{C}}([n]),$$

where we have substituted, as basic building blocks, the simplices Δ_n with the simplicial sets $\Gamma_{\mathcal{C}}([n])$, which now depend on the category $\mathcal{C}$.

The following subsection provides examples of how this encoding of homotopy types into other homotopy types via a Gamma-space $\Gamma_{\mathcal{C}}$ affects their topological complexity. We will return to interpret this in terms of our model of neural information networks in §8.7.

7.4 Spectra and homotopy types

We analyze here a few examples of input data K and how these simplicial data are encoded into the $\Gamma_{\mathcal{C}}(K)$ and $\Gamma_{\mathcal{C}}(\Sigma^n(K))$ by a Gamma-space $\Gamma_{\mathcal{C}}$. The purpose of this choice of examples is to illustrate how the encoding by $\Gamma_{\mathcal{C}}$ preserves certain properties of connectedness. Indeed the statements presented in this section can be regarded as illustrating the principle that the presence of non-trivial homotopy groups in the output representation detects the presence of non-trivial homotopy groups in the input, at least within certain ranges.

The specific examples are chosen so that the input data are certain simplicial sets associated to networks. The reason for this choice will become more evident in §7.5 and 8.7.

7.4.1 Gamma-space representation of clique complexes

Suppose given an undirected graph G, which we assume has no looping edges and no parallel edges. The *clique complex* (clique simplicial set) $K(G)$ is the simplicial complex obtained from G by filling with an n-simplex each n-clique in G, that is, each subgraph Δ_n of G that is a complete graph on $n+1$ vertices.

In the case of a directed graph G, one can similarly consider a *directed clique complex* $K(G)$ (as in [91], [51], [82]) where an n-simplex is added to an n-clique of the graph G only when the n-clique is directed. Here one also assumes no looping edges and no parallel edges. Parallel edges are anyway collapsed to a single edge in the clique construction.) Thus, the skeleta are given by $Sk_\ell(K(G)) = \cup_{n\leq \ell} K(G)_n$, with the set of n-simplexes given by

$$K(G)_n = \{(v_0,\dots,v_n)\,|\, v_i\in V_G \text{ such that } \forall i<j,\ \exists e_{ij}\in E_G\},$$

where e_{ij} is a directed edge with $s(e_{ij}) = v_i$ and $t(e_{ij}) = v_j$. In particular, a directed n-clique $\sigma = (v_0,\dots,v_n)$ as above is an n-clique (complete graph on $n+1$ vertices) such that there is a single source and a single sink vertex and an ordering of the vertices such that if $v_i < v_j$ there is a directed path of edges from v_i to v_j, see [82]. (The no looping edges condition ensures that the single sink property holds.) This $K(G)$ is also referred to as the *directed flag complex*.

Here and elsewhere in this paper we will consider constructions that give rise to simplicial complexes, and we will then consider associated simplicial sets. While a simplicial complex has unordered vertices hence it does not directly define a simplicial set, which requires an ordering, one can use the nerve of the poset of simplices to obtain, functorially, an associated simplicial set, whose geometric realization is homeomorphic to the realization of the barycentric subdivision of the simplicial complex. We will use here the notation $K(G)$ for both the simplicial complex and the simplicial set obtained in this way.

Under the endofunctor of simplicial sets defined by the Gamma-space, the clique simplicial sets $K(G)$ associated to directed networks G (or to subnetworks of a fixed network) are mapped to the simplicial set $\Gamma_{\mathcal{C}}(K(G))$ obtained as in (7.2) by gluing to each directed n-clique Δ_n of G a copy of the simplicial set $\Gamma_{\mathcal{C}}(\Delta_n)$.

Proposition 7.2 *Let $\Gamma_{\mathcal{C}}$ be the Gamma-space associated to a category $\mathcal{C}$, extended to an endofunctor of pointed simplicial sets. Let $K(G)$ be the clique complex of a directed graph. Suppose that the simplicial set $K(G)$ is m-connected for some $m\geq 0$. Then its primary representation $\Gamma_{\mathcal{C}}(K(G))$ is also m-connected. Moreover, if $X_n = \Gamma_{\mathcal{C}}(S^n)$ is the spectrum determined by the Gamma-space, and $X_n \wedge K(G)$ is ℓ-connected for some $\ell \leq 2m+n+3$, then $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$ is also ℓ-connected.*

Proof. Let K be a simplicial set and Γ be an endofunctor of simplicial sets given by a Gamma-space. By Corollary 4.10 of [15], if K is m-connected for some $m \geq 0$, then so is $\Gamma(K)$. Moreover, if K, K' are connected simplicial sets, it follows from Proposition 5.21 of [70] that if K' is m-connected and K is n-connected, then the map $\Gamma(K') \wedge K \to \Gamma(K' \wedge K)$ is $2m+n+3$-connected, hence it induces an isomorphism on homotopy groups π_i with $i < 2m+n+3$ and a surjection on π_{2m+n+3}. When applied to $K' = S^n$ with $\Gamma_{\mathcal{C}}(S^n) = X_n$, this gives the second part of the statement. □

It follows from Proposition 7.2 that nontrivial homotopy groups of $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$ imply corresponding nontrivial homotopy groups for X_n and $K(G)$. Note that the converse implication does not hold: nontrivial homotopy groups of X_n and $K(G)$ do not necessarily imply nontrivial corresponding homotopy groups of $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$ under the map $X_n \wedge K(G) \to \Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$ as in Proposition 7.2.

This shows that enough non-trivial topology is required in the clique complex $K(G)$ to generate enough non-trivial topology in the simplicial sets $\Gamma_{\mathcal{C}}(K(G))$ and that enough non-trivial topology in both the clique complex $K(G)$ and the K-theory spectrum of the category $\mathcal{C}$ are needed to generate enough non-trivial topology in the simplicial sets $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$. Thus, a sufficiently rich class of homotopy types produced by the Gamma-space can be obtained as representation of an "activated subnetwork" $G' \subset G$ only if both $K(G')$ and the spectrum X_n of the Gamma-space have sufficiently rich homotopy types. (We will return to this interpretation more precisely in §7.5 and 8.7.) The existence of such non-trivial homotopy-type representations constrains both the topology of the clique complex of the activated network and the K-theory spectrum of the category $\mathcal{C}$.

7.4.2 The case of random graphs

In the case of a non-oriented graph G with no multiple edges and no looping edges, we can still define the clique complex $K(G)$ as the simplicial complex with all complete subgraphs of G as its simplices, as we noted at the beginning of §7.4.1. Note that topologically the case of directed and non-directed graphs can behave differently, since the forgetful functor from directed to ordinary graphs does not preserve homotopy groups. Here it is more convenient to work with ordinary graphs as we will be using results on random graphs that are proven in that setting. Again the goal here is to provide a class of examples relevant to the discussion in §7.5 and 8.7 below.

A detailed analysis of the topology of clique complexes of random graphs (in the non-directed sense specified above) is given in [63]. We only refer here to the results of [63] that are immediately relevant in our context.

Proposition 7.3 *Let $\Gamma_{\mathcal{C}}$ be the Gamma-space associated to a category $\mathcal{C}$, extended to an endofunctor of pointed simplicial sets. Let G be an Erdős–Rényi graph $G = G(N,p)$, where $N = \#V(G(N,p))$ and $0 < p < 1$ is the probability with which edges are independently inserted.*

1. *Let $p = p(N)$ be a function of the form*
$$p = \left(\frac{(2k+1)\log N + \omega(N)}{N} \right)^{1/(2k+1)} \tag{7.4}$$
where $\omega(N) \to \infty$. Then the simplicial set $\Gamma_{\mathcal{C}}(K(G(N,p)))$ is almost always k-connected.

2. *If $G(N,p)$ is such that $p^{k+1}N \to 0$ but $p^k N \to \infty$, then $\Gamma_{\mathcal{C}}(\Sigma^n(K(G(N,p))))$ is almost always homotopy equivalent to the space X_{k+n} of the spectrum of $\Gamma_{\mathcal{C}}$.*

Proof. It is shown in Theorem 3.4 of [63] that for p as in (7.4) with $\omega(N) \to \infty$ then the clique simplicial complex $K(G(N,p))$ is almost always k-connected. This means that the probability that $K(G(N,p))$ is k-connected, with $p = p(N)$ as in (7.4), goes to 1 when $N \to \infty$. By Theorem 3.5 of [63], if $G(N,p)$ is such that $p^{k+1}N \to 0$ but $p^k N \to \infty$, then $K(G(N,p))$ almost always retracts onto a sphere S^k, hence $\Gamma_{\mathcal{C}}(K(G(N,p)))$ is homotopy equivalent to the space $X_k = \Gamma_{\mathcal{C}}(S^k)$ of the spectrum of $\Gamma_{\mathcal{C}}$ and similarly for $\Gamma_{\mathcal{C}}(\Sigma^n(K(G(N,p)))) \simeq \Gamma_{\mathcal{C}}(S^n \wedge S^k) = X_{n+k}$. □

The two cases for random graphs described in Proposition 7.3 represent situations where for sufficiently large probability p the (non-oriented) clique simplicial complex $K(G(N,p))$ and its image $\Gamma_{\mathcal{C}}(K(G(N,p)))$ have no non-trivial topology up to level k, or the situation where the topology of $\Gamma_{\mathcal{C}}(\Sigma^n(K(G(N,p))))$ exactly captures the topology of the K-theory spectrum of the category $\mathcal{C}$ at level k.

7.4.3 Feedforward networks

Another explicit case we want to consider, which will be relevant for the discussion in §8, is the case of a feedforward network G, in particular in the form of multilayer perceptrons.

The topology of directed clique complexes for feedforward networks was analyzed in [25]. The kind of networks considered in [25] are fully-connected feedforward neural networks, that is, multilayer perceptrons. The work of [25] also analyzes a different kind of topological invariant, given by the path homology, but for our purposes it is the clique complex that is most relevant.

The result of [25] on the case of the directed clique complex is based on the simple observation that a multilayered perceptron does not have any "skip connections", that is, any edges that connect a node in a layer at level i to a node in a layer at level $i+j$ with $j \geq 2$. In particular, this means that there cannot be any cliques of order $j \geq 2$. In particular, this means that the topology of the clique simplicial set $K(G)$ is just the topology of G itself, with possible nontrivial homotopy groups only in degree zero and one.

Thus, the case of feedforward networks is essentially trivial from the point of view of the possible homotopy types $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$, as these depend only on the number of loops of G and on the K-theory spectrum of $\mathcal{C}$ without any higher-rank contributions from $K(G)$.

The fact that feedforward networks behave poorly in this respect, in the sense that they do not generate interesting homotopy types when mapped through a Gamma-space is interesting. Indeed, it is well known that feedforward networks also behave poorly with respect to measures of informational complexity like integrated information. The relation to integrated information will be discussed in §8.

7.5 Gamma networks

In the previous sections we have simply used Gamma-spaces $\Gamma_{\mathcal{C}}$, as functors from finite pointed sets (and, by extension, from pointed simplicial sets) to pointed simplicial sets to discuss how topological properties of certain types of input simplicial sets arising from networks are mapped under these functors. However, as discussed at the beginning of this section, the functors $\Gamma_{\mathcal{C}}$ only depend on the target category $\mathcal{C}$ and the topology of its classifying space $B\mathcal{C}$.

For our purposes, we need to generalize the notion of Gamma-space so that it also encodes data from networks. We do this through our previously discussed notion of network summing functors.

As in the previous sections, we identify finite directed graphs G with objects in the category of functors $\mathcal{G} = \mathrm{Func}(\mathbf{2}, \mathcal{F})$, with $\mathcal{F}$ the category of finite sets, and pointed finite directed graphs G_* as objects in $\mathcal{G}_* = \mathrm{Func}(\mathbf{2}, \mathcal{F}_*)$.

Definition 7.4 *A* Gamma network *is a functor*

$$\mathcal{E} : \mathrm{Func}(\mathbf{2}, \mathcal{F}_*) \to \Delta_* \,.$$

As in the case of Gamma-spaces, we can see that categories $\mathcal{C}$ with sum and zero object (or more generally unital symmetric monoidal categories) are a source of Gamma networks. In particular, we focus here on two constructions of Gamma networks that use the data of a category $\mathcal{C}$ of resources. The first construction uses a Gamma-space, together with a functor from graphs to simplicial sets, while the second construction replaces categories of summing functors with categories of network summing functors. In the first case (see Lemma 7.5) we first assign to a network its clique complex and then use that as input for a Gamma space, while in the second (see Lemma 7.6) one takes the network directly as input of a Gamma network. An advantage of the latter is that it does not require first to perform a clique decomposition, which is computationally complicated. On the other hand it is preferable to assign resources to cliques, for example in the setting discussed in §5.7.

Lemma 7.5 *There is a covariant functor $K : \mathcal{G} \to \Delta$ (or $K : \mathcal{G}_* \to \Delta_*$ in the pointed case) that assigns to a graph G its clique simplicial set $K(G)$. Given a category of resources $\mathcal{C}$ and the associated Gamma-space $\Gamma_{\mathcal{C}}$, seen as an endofunctor $\Gamma_{\mathcal{C}} : \Delta_* \to \Delta_*$, we obtain by precomposition a Gamma network of the form*

$$\mathcal{E}^K_{\mathcal{C}} := \Gamma_{\mathcal{C}} \circ K : \mathcal{G}_* \to \Delta_* . \tag{7.5}$$

This Gamma network takes graphs as input. If we work with directed graphs $\mathcal{G}_* = \mathrm{Func}(\mathbf{2}, \mathcal{F}_*)$ then we consider the directed clique complex $K(G)$, while if we consider non-directed graphs then we take as $K(G)$ the non-directed clique complex. The construction works similarly in both cases. The simplicial set $\mathcal{E}^K_{\mathcal{C}}(G)$ that we obtain associated to the graph is the coend

$$\mathcal{E}^K_{\mathcal{C}}(G) = \int^{[n]\in\mathcal{F}_*} K(G)_n \wedge \Gamma_{\mathcal{C}}([n]),$$

namely, as observed in the previous section, it is the simplicial set obtained by gluing in a copy of $\Gamma_{\mathcal{C}}([n])$ at every n-simplex of $K(G)$, that is, at every n-complete graph in G. In other words, given a graph G, we consider a decomposition of G into cliques. The clique covering problem for a graph is computationally NP-hard but an optimal partition into cliques can be found in polynomial time for graphs with bounded clique-width [35]. If $X \subset V_G$ is a subset of vertices corresponding to one of the cliques in the decomposition, we consider all possible assignments of resources of type $\mathcal{C}$ to the nodes in this clique. This is described by the category of summing functors $\Sigma_{\mathcal{C}}(X)$. The output simplicial set $\mathcal{E}^K_{\mathcal{C}}(G)$ is obtained by considering the geometric model $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of each of these configuration spaces of resource assignments, and gluing them together according to the way the cliques fit together in the graph G (and the corresponding simplices in the clique complex $K(G)$).

We can then interpret the examples discussed in §7.4 as describing how a Gamma network of the form (7.5) encodes an input of the form $K(G)$ (typically the activated subnetwork of a given network, in response to an external stimulus) into a new homotopy type $\mathcal{E}^K_{\mathcal{C}}(G)$ that reflects to some extent the connectivity properties of $K(G)$ but that also reflects the topology of the category $\mathcal{C}$ describing the type of resources that the network carries.

We describe another class of interesting Gamma networks, that also depend on a category of resources $\mathcal{C}$. These are obtained in the same way as the classical Gamma-spaces, but replacing summing functors with network summing functors.

Lemma 7.6 *Let $\mathcal{C}$ be a category of resources and, for $G \in \mathcal{G}_*$, let $\Sigma_{\mathcal{C}}(G)$ denote the associated category of network summing functors as in Definition 2.14, with invertible natural transformations as morphisms. The assignment*

$$G \mapsto \mathcal{E}_{\mathcal{C}}(G) = \mathcal{N}(\Sigma_{\mathcal{C}}(G)) \tag{7.6}$$

determines a Gamma network.

Proof. The construction works exactly as the original case of the Gamma-spaces $\Gamma_{\mathcal{C}} : \mathcal{F}_* \to \Delta_*$ recalled in §7.1, namely, given a natural transformation $\alpha : G \to G'$ between functors $G, G' \in \mathrm{Func}(\mathbf{2}, \mathcal{F})$, we take $\alpha_*\Phi : P(G') \to \mathcal{C}$, for $\Phi \in \Sigma_{\mathcal{C}}(G)$, to be defined as $\alpha_*\Phi(H) = \Phi(\alpha^{-1}(H))$, for $H \in P(G')$, where $\alpha^{-1}(H) : \mathbf{2} \to \mathcal{F}$ is the functor with $V_{\alpha^{-1}(H)} = \alpha_V^{-1}(H)$ and $E_{\alpha^{-1}(H)} = \alpha_E^{-1}(H)$ and source and target morphisms induced by those of G. Note that if we write everything in terms of the associated pointed graph G_*, then $\alpha_*\Phi : P(G'_*) \to \mathcal{C}$ is defined as in (7.1). □

The class of Gamma networks obtained as in Lemma 7.6 model a somewhat different idea about how networks generate associated homotopy types, with respect to the construction of Lemma 7.5. In the cases of Lemma 7.5 there is an underlying functorial construction from graphs to simplicial sets, at the level of *input* of the Gamma-space (through the clique complex, or in principle through other relevant constructions of a similar nature). On the other hand, in the construction of Lemma 7.6 the input is only the network itself and the Gamma network $\mathcal{E}_{\mathcal{C}}$ assigns to it the nerve of the category of network summing functors (or a suitably chosen subcategory). Thus, in the first case a network is first decomposed into cliques and the configuration space of assignment of resources is built from the resources associated to the individual cliques through a gluing procedure, while in the second case there is no a priori decomposition of the network and the resulting configuration space counts all assignments of resources according to the choice of the type of network summing functors used. These two examples illustrate possible different viewpoints that can be used separately or combined (the smash product of Gamma networks is still a Gamma network as for Gamma-spaces), depending on the type of model of networks with resources that one wants to consider.

The way one should interpret this, in terms of the model of networks with resources that we are describing, is the following. There is an overall network G with an associated configuration space describing all the assignments of resources of a given type $\mathcal{C}$ to the network. On this configuration space there is a way of describing the dynamics that governs such assignments of resources. When responding to an external stimulus, only a certain subnetwork $G' \subset G$ becomes activated. This means that the actual configuration space involved in describing the response to a given stimulus is a subset of the overall configuration space, which is determined by the value on this subnetwork G' of the appropriate Gamma network functor, $\mathcal{E}_{\mathcal{C}}(G')$. Thus, this is a way to account in a consistent way for a setting where the actual network (or part of network) involved varies according to the stimulus.

7.5.1 Gamma networks, codes, and nerves of coverings

We present here another example of Gamma networks, of the type described in Lemma 7.5, but with a different functor from networks to simplicial sets, based on associated neural codes and nerves of coverings. For simplicity we do not explicitly introduce base points.

Definition 7.7 *Let $\mathcal{G} := \mathrm{Func}(\mathbf{2}, \mathcal{F})$ be the category of finite directed graphs and let $\mathcal{C}$ be a category of resources. Let $\Delta_{\mathcal{G},\mathcal{C}}$ denote the category with objects given by pairs (G, Φ) with $G \in \mathrm{Obj}(\mathcal{G})$ and $\Phi \in \Sigma_{\mathcal{C}}(V_G)$. Morphisms $\alpha \in \mathrm{Mor}_{\Delta_{\mathcal{G},\mathcal{C}}}((G,\Phi),(G',\Phi'))$ are morphisms $\alpha : G \to G'$ in $\mathcal{G}$ (natural transformations in $\mathrm{Func}(\mathbf{2}, \mathcal{F})$) such that $\Phi'(\alpha_V(v)) = \Phi(v)$, with $\alpha_V : V_G \to V_{G'}$ the natural transformation α at the object $V \in \mathbf{2}$.*

We consider here in particular the case where $\mathcal{C} = \mathrm{Codes}_n$. We write $\mathcal{C}'$ for the category of codes Codes'_n discussed in Proposition 5.21.

By Lemma 2.3 a summing functor $\Phi \in \Sigma_{\mathcal{C}}(V_G)$ is completely determined by the assignment of an object $\Phi(v)$ for each $v \in V_G$. The objects $\Phi(v)$ are (binary) codes C_v of length n. If we think of the set of vertices V_G of the network as neurons and of codes as neural codes generated by spiking activity of neurons over a fixed set T_n of n basic time intervals, then we can restrict our attention to the case where $\Phi(v)$ consists of a single code word c_v with binary entries describing whether the neuron v is spiking or not during each time interval in T_n. (If we want to include base points, then we would have two code words $\Phi(v) = \{c_v, c_0\}$, with c_0 the zero word. We will ignore base points to simplify the discussion.)

Definition 7.8 *We refer to summing functors $\Phi \in \Sigma_{\mathrm{Codes}_n}(V_G)$ with the property that $\Phi(v) = c_v$ consists of a single binary code word of length n as* elementary*. We write $\Delta'_{\mathcal{G},\mathrm{Codes}_n}$ for the subcategory of $\Delta_{\mathcal{G},\mathcal{C}}$ with objects (G, Φ) where the summing functor $\Phi \in \Sigma_{\mathrm{Codes}_n}(V_G)$ is elementary.*

Lemma 7.9 *With $\mathcal{C} = \mathrm{Codes}_n$ and $\mathcal{C}' = \mathrm{Codes}'_n$ as above, there is a functor $C : \Delta'_{\mathcal{G},\mathcal{C}} \to \mathcal{C}'$ that assigns to a pair (G, Φ) with Φ elementary, the map $C : V_G \times T_n \to \{0, 1\}$ with $C(v, i) = \Phi(v)_i$, the i-th letter of the binary code word $c_v = \Phi(v)$.*

Proof. Consider a morphism $\alpha : (G, \Phi) \to (G', \Phi')$ in $\Delta'_{\mathcal{G},\mathcal{C}}$. Since we assume both Φ and Φ' are elementary, and we have $\Phi = \Phi' \circ \alpha_V$, we obtain that the morphism α and the induced map $\alpha_V : V_G \to V_{G'}$ give a morphism of the category $\mathcal{C}' = \mathrm{Codes}'_n$, since $C = C' \circ \alpha_V$. □

We obtain in this way another example of Gamma network, similar to the case discussed in Lemma 7.5. This is a more general form of Gamma networks, where we allow the input category to be given by $\Delta'_{\mathcal{G},\mathcal{C}}$ instead of just $\mathcal{G}$, so that the choice of an elementary $\Phi \in \Sigma_{\mathcal{C}}(V_G)$ is assumed here as part of the input data. The following statement is a direct consequence of Lemma 7.9 and Proposition 5.21.

Proposition 7.10 *The composite $\mathcal{NU} \circ C$ of the functor C of Lemma 7.9 and the functor $\mathcal{NU}$: $\mathrm{Codes}'_n \to \Delta$ of Proposition 5.21, gives a functor $\Xi = \mathcal{NU} \circ C : \Delta'_{\mathcal{G},\mathcal{C}} \to \Delta$. Composition with any Gamma-space $\Gamma_{\mathcal{R}}$, associated to a category of resources $\mathcal{R}$, determines a Gamma network*

$$\mathcal{E}^{\Xi}_{\mathcal{R}} = \Gamma_{\mathcal{R}} \circ \Xi : \Delta'_{\mathcal{G},\mathcal{C}} \to \Delta \,.$$

Note that incorporating a choice of a summing functor $\Phi \in \Sigma_{\mathcal{C}}(V_G)$ as part of the input data is consistent with settings such as our Hopfield equations, where solutions depend on the choice of a summing functor specifying the initial condition for the evolutionary equation.

8 Gamma networks and integrated information

Integrated information was introduced in neuroscience as a measurement of causal influence structures and informational complexity in neuronal networks [9], [104]. In neuroscience, integrated information was proposed as a possible quantitative measurement of consciousness. (For a general discussion of this point of view on consciousness, see [67], [81].) There are several slightly different versions of integrated information: for a comparative analysis, see [83]. We adopt here the geometric version of integrated information developed in [88], based on information geometry [2], which we recall in §8.1.

Our main results in this section are the construction of a cohomological form of integrated information, and using this to show that there is a way to keep track of the change of integrated information along the orbits of our categorical Hopfield dynamics, and under composition of a probability functor on random graphs with a Gamma-space (with the latter seen as an endofunctor of simplicial sets). We show that composition with a Gamma-space increases integrated information by an amount describable in terms of Shannon entropy.

8.1 Information geometry and integrated information

The geometric version of integrated information of [88] is constructed in the following way. Suppose given a stochastic dynamical system, where the state of the system at (discrete) time n is described by a set of random variables $\{X_i = X_i^{(n)}\}_{i=1}^N$ which correspond to a partition of the system into N subsystems, and the state at time $n+1$ by a set $\{Y_i = X_i^{(n+1)}\}_{i=1}^N$. The full system including all the mutual influences between these two sets of variables, understood in a statistical sense, is described by a probability distribution $P(X,Y)$. Integrated information is meant to capture the difference between this distribution and an approximation $Q(X,Y)$ where only certain kinds of mutual influences are retained. These are usually taken to be the interdependencies between the variables at the same time and between each X_i and the corresponding Y_i, while one removes the dependencies of the Y_i from the X_j with $j \neq i$. More precisely this condition of removal of dependencies is described by the requirement that the measure $Q(X,Y)$ satisfies, for all $i = 1,\dots,N$ of the given partition, the condition

$$Q(Y_i|X) = Q(Y_i|X_i). \tag{8.1}$$

The discrepancy between $P(X,Y)$ and $Q(X,Y)$ is measured by their Kullback–Leibler divergence

$$\mathrm{KL}(P(X,Y)||Q(X,Y)) = \sum_{x,y} P(x,y) \log \frac{P(x,y)}{Q(x,y)}, \tag{8.2}$$

where (x,y) varies over the set of values of (X,Y), which we assume finite here.

The best approximation to the full system probability $P(X,Y)$ by a measure $Q(X,Y)$ in the class of measures satisfying (8.1) can be described using information geometry. Given a partition λ

$$\{(X,Y)\} = \sqcup_{i=1}^N \{(X_i,Y_i)\}$$

of the random variables X,Y, one considers the space Ω_λ of all probability measures $Q(X,Y)$ that satisfy the constraint (8.1) for the partition λ. For a given $P(X,Y)$, a minimizer $Q_\lambda(X,Y) \in \Omega_\lambda$

of the Kullback–Leibler divergence (8.2) is obtained via the *projection theorem* of information geometry [2].

The setting of information geometry that is used for obtaining geometrically the minimizer probability

$$Q^*_\lambda(X,Y) = \mathrm{argmin}_{Q\in\Omega_\lambda}\ \mathrm{KL}(P(X,Y)||Q(X,Y)) \tag{8.3}$$

is summarized as follows (see §3.2 and §3.4 and in particular Theorem 3.8 and Corollary 3.9 of [2]).

A *divergence function* is a function $D(P||Q)$ on pairs of probability distributions (which we assume finite here), with the property that the quadratic term $g^{(D)}$ in the expansion

$$D(P+\xi||P+\eta) \sim \frac{1}{2}\sum_{i,j} g^{(D)}_{ij}(P)\xi^i\eta^j + \text{ higher order terms}$$

is positive definite, that is, a Riemannian metric, and the cubic term

$$h^{(D)}_{ijk} = \partial_i g^{(D)}_{jk} + \Gamma^{(D)}_{jk,i}$$

determines a connection $\nabla^{(D)}$ with Christoffel symbols $\Gamma^{(D)}_{ij,k} = \Gamma^{(D)}_{ji,k}$. Similarly, the dual divergence $D^*(P||Q) := D(Q||P)$ determines the same metric $g^{(D^*)} = g^{(D)}$ and a connection $\nabla^{(D^*)}$ that is dual to $\nabla^{(D)}$ under $g^{(D)}$. The duality condition for connections ∇, ∇^* with respect to a metric g means that, for any triple of vector fields V, W, Z, one has $Z\, g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y)$. In particular, in §3.2 of [2] conditions are given under which, for a smooth function $f(x)$, the expression

$$D_f(P||Q) = \sum_i P_i\, f(\frac{Q_i}{P_i}) \tag{8.4}$$

defines a divergence, with the associated metric $g^{(D_f)}$ proportional to the Fisher–Rao information metric g_{FR} (see Theorem 2.6 of [2]). In particular, for $f(x) = x\log x$ one has $D_f(P||Q) = \mathrm{KL}(Q||P)$ and for $f(x) = -\log(x)$ one has $D_f(P||Q) = \mathrm{KL}(P||Q)$.

Suppose given the triple $(g^{(D_f)}, \nabla^{(D_f)}, \nabla^{(D_f^*)})$ associated to a divergence D_f as above. One can consider, in the space of probabilities P, either $\nabla^{(D_f)}$-geodesics or $\nabla^{(D_f^*)}$-geodesics, that is, paths $\gamma(t)$ that are solutions to the geodesic equation

$$\ddot{\gamma}(t)^k + \sum_{ij} \Gamma^k_{ij}(\gamma(t))\ \dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0,$$

with Γ^k_{ij} the Christoffel symbols of the corresponding connection.

An important property of the divergence functions $D(P||Q)$ is the Pythagorean relation (Theorem 3.8 of [2]). Namely, if P, Q, R are three probability distributions, consider the $\nabla^{(D)}$-geodesic from P to Q and the $\nabla^{(D^*)}$-geodesic from Q to R. If these two geodesics meet orthogonally at Q, then the divergences satisfy the *Pythagorean relation*

$$D(P||R) = D(P||Q) + D(Q||R). \tag{8.5}$$

A consequence of this relation is the orthogonal projection theorem of information geometry (Corollary 3.9 of [2]). Namely, given P and a submanifold Ω of the space of probabilities, a point $Q^* \in \Omega$ satisfies

$$Q^* = \mathrm{argmin}_{Q\in\Omega}\, D(P||Q)$$

if and only if the $\nabla^{(D)}$-geodesic from P to Q^* meets Ω orthogonally at Q^*.

Consider the minimizer probability (8.3) obtained as above. Then the geometric integrated information, for a given partition λ, is defined as

$$\mathrm{II}_\lambda(P(X,Y)) := \mathrm{KL}(P(X,Y)||Q^*_\lambda(X,Y)) = \min_{Q\in\Omega_\lambda} \mathrm{KL}(P(X,Y)||Q(X,Y)), \tag{8.6}$$

with a further minimization over the choice of the partition,

$$\mathrm{II}(P(X,Y)) := \min_\lambda \mathrm{KL}(P(X,Y)||Q^*_\lambda(X,Y)) = \min_{Q\in\cup_\lambda\Omega_\lambda} \mathrm{KL}(P(X,Y)||Q(X,Y)). \tag{8.7}$$

The partition λ realizing the minimum is referred to as the "minimal information partition". Note that this notion is slightly different in other versions of the integrated information where one minimizes in information measure over partitions with a normalization factor that corrects for the asymmetry between the sizes of the pieces of the partition; see [83] for a comparative discussion of these different versions.

It is customary to use the letter Φ to denote integrated information (also referred to as the Φ-function). However, since in this paper we have been using the letter Φ for our summing functors, we will use the notation of (8.7) for integrated information.

8.2 Feedforward networks and integrated information

To see an explicit and relevant example of the behavior of integrated information, consider again the case of a feedforward network with the architecture of a multilayer perceptron as in §7.4.3. The fact that feedforward networks behave poorly with respect to integrated information was discussed in [9], using a slightly different form of integrated information. We show here that indeed, with the notion of geometric integrated information of [88] we also see a similar phenomenon.

Lemma 8.1 *Let G be a multilayer perceptron. Consider the set S of binary random variables $X : V_G \to \{0,1\}$ on the nodes V_G, detecting whether a node is activated or not. The network is subject to a dynamics that updates the state $X(v)$ of a node v through a function*

$$X_{t+1}(v) = \sigma(X_t(v') \,|\, \exists e \in E_G \,:\, v' = s(e), v = t(e))$$

of the $X_t(v')$ at all vertices that feed into v. Let $P(X_t, X_{t+1})$ be their joint probability distribution. There is a partition λ of S, with $X_i = X|_{S_i}$ such that the distribution $P(X_t, X_{t+1})$ satisfies $P(X_{t+1,i}|X_t) = P(X_{t+1,i}|X_{t,i})$, hence the integrated information vanishes, $\mathrm{II}(P(X_t, X_{t+1})) = 0$.

Proof. Consider the input nodes $v_1, \ldots, v_r$ of the multilayer perceptron G. These nodes have outgoing edges to the next layer nodes but no incoming edges from inside the system. If the state $X(v_i)$ of the input nodes is assigned at $t = 0$, it remains unchanged during the rest of the time evolution. Thus, we can choose a partition λ of the set S into 2^r subsets determined by the possible values of $X(v_i)$ at the input nodes $v_1, \ldots, v_r$. All these subsets S_i are preserves by the time evolution. Thus, the probability $P(X_{t+1,i}|X_t)$ of those variables $X_{t+1,i}$ in S_i given the state X_t at time t only depends on $X_{t,i}$ as only these variables have causal influence under the time evolution on the $X_{t+1,i}$. So we have $P(X_{t+1,i}|X_t) = P(X_{t+1,i}|X_{t,i})$, hence the probability distribution $P(X_t, X_{t+1})$ already lies in the manifold Ω_λ, hence $\mathrm{II}_\lambda(P(X_t, X_{t+1})) = 0$. □

Note that the source of the vanishing of integrated information for multilayered perceptrons is different from the source of the vanishing of the topological invariants in §7.4.3. Here the fact that $\mathrm{II}(P(X_t, X_{t+1})) = 0$ is caused by the input nodes that do not get any incoming input from the rest of the system, while in §7.4.3 the vanishing of the higher $\pi_i(K(G))$ of the clique complex is caused by the lack of skip connections between layers.

8.3 Kullback–Leibler divergence and information cohomology

Like the Shannon entropy, the Kullback–Leibler divergence can be interpreted as a 1-cocycle in information cohomology, see §3.7 of [107].

Just like the Tsallis entropy provides a one-parameter family of entropy functionals that recover the Shannon entropy for $\alpha \to 1$, a similar one-parameter deformation of the Kullback–Leibler divergence can be defined as

$$\mathrm{KL}_\alpha(P||Q) = \frac{1}{1-\alpha} \sum_i P_i \left((\frac{P_i}{Q_i})^{1-\alpha} - 1 \right). \tag{8.8}$$

This clearly satisfies $\mathrm{KL}_\alpha(P||Q) \to \mathrm{KL}(P||Q) = \sum_i P_i \log(\frac{P_i}{Q_i})$ for $\alpha \to 1$.

Consider information structures (S, M) and (S', M') and a joint random variable (X, Y) with values in a finite set $M_{XY} \subset M_X \times M'_Y$, where $X \in \mathrm{Obj}(S)$ and $Y \in \mathrm{Obj}(S')$. Also consider a pair of probability functors $\mathcal{Q} : (S, M) \times (S', M') \to \Delta$ and $\mathcal{Q}' : (S, M) \times (S', M') \to \Delta$, where the simplicial sets $\mathcal{Q}_{(X,Y)}$ and $\mathcal{Q}'_{(X,Y)}$ are subsimplicial sets of the full simplex $\Delta_{M_{XY}}$.

Consider then the contravariant functor $\mathcal{M}^{(2)}(\mathcal{Q}, \mathcal{Q}') : (S, M) \times (S', M') \to \mathrm{Vect}$ that maps $(X, Y) \mapsto \mathcal{M}^{(2)}(X, Y)$ to the vector space of real-valued (measurable) functions on the simplicial set of probabilities $\mathcal{Q}_{(X,Y)} \times \mathcal{Q}'_{(X,Y)}$. For $X \in \mathrm{Obj}(S), Y \in \mathrm{Obj}(S')$, the semigroup $\mathcal{S}_{(X,Y)}$ acts on $\mathcal{M}^{(2)}(X, Y)$ by

$$((X', Y') \cdot f)(P, Q) = \sum_{(x',y') \in M_{X'Y'}} P(x', y')^{\alpha} Q(x', y')^{1-\alpha} f((P, Q)|_{(X',Y')=(x',y')}), \tag{8.9}$$

for $(X', Y') \in \mathcal{S}_X$ and $(P, Q) \in \mathcal{Q}_{(X,Y)} \times \mathcal{Q}'_{(X,Y)}$, and with $\{(X', Y') = (x', y')\} = \pi^{-1}(x', y')$ under the surjection $\pi : M_{(X',Y')} \to M_{(X,Y)}$ determined by the morphism $\pi : (X', Y') \to (X, Y)$ (which exists by the definition of the semigroup $\mathcal{S}_{(X,Y)}$). This gives $\mathcal{M}^{(2)}(\mathcal{Q}, \mathcal{Q}')$ a structure of $\mathcal{A}$-module, which we denote by $\mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}')$. It is then shown in §3.7 of [107] that the Kullback–Leibler divergence (8.8) is a 1-cocycle in the resulting cochain complex $(C^{\bullet}(\mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}')), \delta)$.

8.4 Cohomological integrated information

Consider the setting as in the previous subsection, with $\mathcal{Q} : (S, M) \times (S', M') \to \Delta$ a given probability functor and $\mathcal{Q}'_{\lambda} : (S, M) \times (S', M') \to \Delta$ a probability functor with the property that, for all (X, Y) with $X \in \mathrm{Obj}(S)$ and $Y \in \mathrm{Obj}(S')$, the simplicial set $\mathcal{Q}'_{\lambda,(X,Y)}$ is contained in the subspace $\Omega_{\lambda,(X,Y)} \subset \Delta_{M_{XY}}$

$$\Omega_{\lambda,(X,Y)} = \{Q(X, Y) \in \Delta_{M_{XY}} \,|\, Q(Y_i|X) = Q(Y_i|X_i)\,\} \tag{8.10}$$

as in (8.1), for a partition λ of $S = \sqcup_{i=1}^{N} S_i$ and $S' = \sqcup_{i=1}^{N} S'_i$ so that $X_i \in \mathrm{Obj}(S_i)$ and $Y_i \in \mathrm{Obj}(S'_i)$.

Given $P(X, Y) \in \mathcal{Q}_{(X,Y)}$, let $Q^*_{\alpha}(X, Y) \in \mathcal{Q}'_{\lambda,(X,Y)}$ be obtained by taking

$$Q^*_{\alpha,\lambda}(X, Y) := \mathrm{argmin}_{Q \in \mathcal{Q}'_{\lambda,(X,Y)}} \mathrm{KL}_{\alpha}(P(X, Y)||Q(X, Y))\,. \tag{8.11}$$

as in (8.3) and

$$Q^*_{\alpha}(X, Y) := \mathrm{argmin}_{\lambda}\, \mathrm{KL}_{\alpha}(P(X, Y)||Q^*_{\alpha,\lambda}(X, Y))\,. \tag{8.12}$$

In the case where $\alpha = 1$, the minimizer $Q^*_{1,\lambda}(X, Y)$ can be determined as recalled above, through the orthogonal projection method of information geometry for the divergence $D(P||Q) = \mathrm{KL}(P||Q)$. The case of $\alpha \neq 1$ can also be treated similarly, using a divergence $D_f(P||Q)$ with $f(x) = \frac{1}{\alpha-1}(x^{\alpha-1} - 1)$, with the general formalism for the information geometry orthogonal projection theorem recalled in §8.1 above (see §3.4 of [2]).

The following result is then a direct consequence of the result of §3.7 of [107] recalled in the previous subsection.

Proposition 8.2 *The minimizer (8.12) determines a probability functor*

$$\mathcal{Q}^*_{\alpha} : (S, M) \times (S', M') \to \Delta$$

$$\mathcal{Q}^*_{\alpha,(X,Y)} := \{(P, Q^*_{\alpha}) \in \mathcal{Q}_{(X,Y)} \times \mathcal{Q}'_{(X,Y)} \,|\, Q^*_{\alpha} = \mathrm{argmin}_{\lambda, Q \in \mathcal{Q}'_{\lambda,(X,Y)}} \mathrm{KL}_{\alpha}(P||Q)\},$$

and a contravariant functor $\mathcal{M}^{(2)}(\mathcal{Q}, \mathcal{Q}^*_{\alpha}) : (S, M) \times (S', M') \to \mathrm{Vect}$ that maps (X, Y) to the vector space of real-valued (measurable) functions on $\mathcal{Q}^*_{\alpha,(X,Y)}$. The action (8.9) restricted to $(P, Q^*_{\alpha}) \in \mathcal{Q}^*_{\alpha,(X,Y)}$ gives $\mathcal{M}^{(2)}(\mathcal{Q}, \mathcal{Q}^*_{\alpha})$ the structure of an $\mathcal{A}$-module $\mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}^*_{\alpha})$, hence we obtain a cochain complex $(C^{\bullet}(\mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}^*_{\alpha})), \delta)$.

Definition 8.3 *The cohomological integrated information*

$$\mathrm{IIH}^*(\mathcal{Q}) := \mathrm{IIH}^*((S, M) \times (S', M'), \mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}^*_{\alpha}))$$

is the cohomology of the cochain complex $(C^{\bullet}(\mathcal{M}^{(2)}_{\alpha}(\mathcal{Q}, \mathcal{Q}^*_{\alpha})), \delta)$ obtained as in Proposition 8.2.

In particular, the usual geometric integrated information of (8.7) is identified with an element of the cohomological integrated information, which corresponds to the 1-cocycle given by the Kullback–Leibler divergence, in the case $\alpha = 1$. One interprets then the rest of the cohomological integrated information as measures of the difference between $P(X,Y)$ and its best approximation $Q^*_\alpha(X,Y) \in \mathcal{Q}^*_{(X,Y)}$ when measured using the higher cocycles. These can be seen as relative versions of the higher mutual information functionals of cohomological information, in the same way as the Kullback–Leibler divergence can be seen as a relative version, for a pair of measures, of the Shannon entropy.

8.5 Categorical Hopfield dynamics and integrated information

We show here that our formulation of Hopfield dynamics allows for a way of keeping track of the behavior of integrated information along solutions of the dynamics, namely of the change in integrated information that occurs in the subsequent steps of the dynamics.

We consider then again the setting we described in §6. For a given network G, consider a categorical Hopfield dynamics as in (6.5) (or (6.8) or (6.9)) on the category $\Sigma^{\rm eq}_{\mathcal{C}}(G)$, with a given initial condition $\Phi_0 \in \Sigma^{\rm eq}_{\mathcal{C}}(G)$ and with a functor $T \in \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E)$ that determines the dynamics, as in §6. As shown in §6, the assignment $\Phi_n \mapsto \Phi_{n+1}$ given by the dynamics is an endofunctor of $\Sigma^{\rm eq}_{\mathcal{C}}(G)$.

Proposition 8.4 *The Hopfield dynamics (6.5) determines a functor*

$$\mathcal{T}_n : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \Sigma^{\rm eq}_{\mathcal{C}}(G)^2$$

mapping the initial condition Φ_0 to the pair of summing functors (Φ_n, Φ_{n+1}). Let $\mathcal{I} : \mathcal{C} \to \mathcal{IS}$ be a functor compatible with coproducts. Composition with the functor $C^\bullet(\mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha))$ and passing to cohomology determines a functor

$$\mathrm{IIH}^\bullet_n : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \Sigma_{\rm GrVect}(G) \subset \mathrm{Func}(P(G), \mathrm{GrVect})$$

$$\mathrm{IIH}^\bullet_n(\Phi_0) = \mathrm{IIH}^\bullet((S,M)^{G'} \times (S',M')^{G'}, \mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha)) \tag{8.13}$$

that assigns to an initial condition Φ_0 the cohomological integrated information of the network G in the n-th step of the Hopfield evolution.

Proof. The functoriality of the assignment $\Phi_0 \mapsto (\Phi_n, \Phi_{n+1})$ follows from Lemma 6.3. We then consider the composition $\mathcal{I}^2 \circ \mathcal{T}_n$, with $\mathcal{I}^2 : \mathcal{C}^2 \to \mathcal{IS}^2$. This is a functor $\mathcal{I}^2 \circ \mathcal{T}_n : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \mathrm{Func}(P(G), \mathcal{IS}^2)$ that maps Φ_0 to the functor $G' \mapsto (S,M)^{G'}_n \times (S,M)^{G'}_{n+1} \in \mathrm{Obj}(\mathcal{IS}^2)$ where $(S,M)^{G'}_n = \mathcal{I}(\Phi_n(G))$ and $(S,M)^{G'}_{n+1} = \mathcal{I}(\Phi_{n+1}(G'))$. As in Corollary 5.19 we can then compose with the functor $\mathcal{K} = C^\bullet(\mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha))$ and obtain a functor $\mathcal{K} \circ \mathcal{I}^2 \circ \mathcal{T}_n : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \mathrm{Func}(P(G), \mathrm{Ch}(\mathbb{R}))$

$$G' \mapsto (C^\bullet((S,M)^{G'}_n \times (S,M)^{G'}_{n+1}, \mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha)), \delta).$$

Further passing to cohomology gives $\mathrm{IIH} \circ \mathcal{K} \circ \mathcal{I}^2 \circ \mathcal{T}_n : \Sigma^{\rm eq}_{\mathcal{C}}(G) \to \mathrm{Func}(P(G), \mathrm{GrVect})$

$$G' \mapsto \mathrm{IIH}^\bullet((S,M)^{G'}_n \times (S,M)^{G'}_{n+1}, \mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha)).$$

We refer to the functor obtained in this way as $\mathrm{IIH}^\bullet_n(\Phi_0)$. □

8.6 Integrated information and Gamma networks

We now consider how to adapt the formalism of information cohomology to deal with data of networks. This in particular will provide us with a notion of "random graphs" that is more general than the usual models such as the Erdős–Rényi graphs discussed in Proposition 7.3, based on finite information structures and probability functors as in [106] (see §5.4.1 above). We proceed as in the case of finite information structures of [106].

Definition 8.5 *A graph information structure consists of a pair (S, M) of a thin category S, defined as in [106], consisting of random variables X with morphisms describing a "coarsening" relation (see our summary of [106] in §5.4.1) and a functor*

$$M : S \to \mathcal{G} = \mathrm{Func}(\mathbf{2}, \mathcal{F}) .$$

Probability functors on graph information structures are functors $\mathcal{Q} : (S \times \mathbf{2}, M) \to \Delta$ that assign to a pair of random variables X_E, X_V simplicial sets $\mathcal{Q}_{X_E}$, $\mathcal{Q}_{X_V}$ of probabilities over the vertex sets M_{X_E}, M_{X_V}, with source and target morphisms.

Remark 8.6 In the definition above, we view the functor M equivalently as an object

$$M \in \mathrm{Func}(\mathbf{2} \times S, \mathcal{F}).$$

To a pair X_E, X_V of random variables in S, the functor M assigns sets given by their ranges M_{X_E}, M_{X_V} endowed with source and target maps $s, t : M_{X_E} \to M_{X_V}$. These data determine a directed random graph G_X with these sets as vertices and edges. Thus, we can identify each pair $(S \times \mathbf{2}, M)$ with a category $\mathcal{G}_{(S,M)}$ of random graphs $G_X \in \mathrm{Obj}(\mathcal{G}_{(S,M)})$. A probability functor $\mathcal{Q}$ can be seen as a functor $\mathcal{Q} : \mathcal{G}_{(S,M)} \to \Delta$, from a category of random graphs to simplicial sets.

The same construction above can be adapted to the case where the category of finite sets $\mathcal{F}$ is replaced by pointed finite sets $\mathcal{F}_*$ and the functors $\mathcal{Q}$ take values in Δ_*. Proceeding as in Lemma 7.5, we can then consider Gamma networks obtained in the following way.

Lemma 8.7 *Let $\mathcal{C}$ be a category of resources, with an associated Gamma-space $\Gamma_{\mathcal{C}} : \Delta_* \to \Delta_*$. Given a probability functor $\mathcal{Q} : \mathcal{G}_{(S,M)} \to \Delta_*$, we obtain an associated Gamma network*

$$\mathcal{E}^{\mathcal{Q}}_{\mathcal{C}} = \Gamma_{\mathcal{C}} \circ \mathcal{Q} : \mathcal{G}_{(S,M)} \to \Delta_* \qquad \mathcal{E}^{\mathcal{Q}}_{\mathcal{C}}(G_X) = \Gamma_{\mathcal{C}}(\mathcal{Q}_{G_X}) . \tag{8.14}$$

Note that we can view $\mathcal{E}^{\mathcal{Q}}_{\mathcal{C}}$ itself as a new probability functor, assigning to $G_X \in \mathcal{G}_{(S,M)}$ the simplicial set $\Gamma_{\mathcal{C}}(\mathcal{Q}_{G_X})$. Thus, we can view the Gamma-space $\Gamma_{\mathcal{C}}$ as an endofunctor of the category of probability functors $\mathcal{Q} : \mathcal{G}_{(S,M)} \to \Delta_*$.

Consider then the case of pairs of random variables (X, Y), as in our discussion of Kullback–Leibler divergence and integrated information in §8.3 and §8.4.

Proposition 8.8 *The joint distribution of a pair of random variables (X, Y) in $(S \times \mathbf{2}, M) \times (S' \times \mathbf{2}, M')$ determines a subgraph $G_{(X,Y)}$ of the Kronecker product $G_X \times G_Y$. A probability functor $\mathcal{Q} : \mathcal{G}_{(S,M)\times(S,M')} \to \Delta$ has an associated cohomological integrated information $\mathrm{IIH}^*(\mathcal{Q})$ as in Definition 8.3 that measures the amount of information in the associated simplicial set $\mathcal{Q}_{(X,Y)}$ of probabilities that is not reducible to a decomposition into independent subsystems.*

Proof. We consider information structures $(S \times \mathbf{2}, M)$ and $(S' \times \mathbf{2}, M')$, which correspond, respectively, to categories of random graphs $\mathcal{G}_{(S,M)}$ and $\mathcal{G}_{(S,M')}$. A pair of *independent* random variables $(X, Y) \in (S \times \mathbf{2}, M) \times (S' \times \mathbf{2}, M')$ will correspond to the Kronecker product of the random graphs $G_X \times G_Y$. For a more general pair (X, Y), the joint distribution will determine a subgraph $G_{(X,Y)} \subset G_X \times G_Y$. We consider probability functors $\mathcal{Q} : (S \times \mathbf{2}, M) \times (S' \times \mathbf{2}, M') \to \Delta$ that assign to a pair of random variables (X, Y) the simplicial set $\mathcal{Q}_{G_{(X,Y)}}$, which is a subsimplicial set in the full simplex $\Delta_{X,Y}$ on the set $V_{G_X} \times V_{G_Y}$. As in §8.4, we can then consider those functors $\mathcal{Q}'_\lambda : (S \times \mathbf{2}, M) \times (S' \times \mathbf{2}, M') \to \Delta$ with the property that the simplicial set $\mathcal{Q}'_{\lambda, G_{(X,Y)}}$ is contained in the subspace

$$\Omega_{\lambda,(X,Y)} = \{ Q(X, Y) \in \Delta_{X,Y} \,|\, Q(Y_i|X) = Q(Y_i|X_i) \} ,$$

for a partition λ of $S = \sqcup_i S_i$ and $S' = \sqcup_i S'_i$ with $X_i \in \mathrm{Obj}(S_i)$ and $Y_i \in \mathrm{Obj}(S'_i)$. We can then proceed as in §8.4 and minimize the Kullback–Leibler divergence as in (8.11) and (8.12). The resulting minimizer determines a probability functor

$$\mathcal{Q}^*_\alpha : \mathcal{G}_{(S,M)\times(S,M')} \to \Delta , \tag{8.15}$$

with respect to which one can compute the cohomological integrated information as the cohomology of the cochain complex $(C^\bullet(\mathcal{M}^{(2)}_\alpha(\mathcal{Q}, \mathcal{Q}^*_\alpha)), \delta)$. One obtains in this way a cohomological integrated information $\mathrm{IIH}^*(\mathcal{Q})$ as in Definition 8.3. Assume that the random variables X, Y describe the activated subnetwork, in response to an external stimulus, at time t and at time $t+1$. Then the integrated information $\mathrm{IIH}^*(\mathcal{Q})$ described above captures the amount of information in the associated simplicial set $\mathcal{Q}_{(X,Y)}$ of probabilities that is not reducible to a decomposition into independent subsystems, in which the variables Y_i of a subsystem at time $t+1$ are only correlated to the variables X_i of the same subsystem at time t. □

Consider now probability functors $\mathcal{Q} : \mathcal{G}_{(S,M)\times(S,M')} \to \Delta$ as in Proposition 8.8 and the composition $\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}} = \Gamma_{\mathcal{C}} \circ \mathcal{Q}$ with a Gamma-space of a category $\mathcal{C}$ of resources as in 8.7. The chain rule for the Kullback–Leibler divergence (for $\alpha = 1$) then allows us to compare the integrated information of $\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}}$ and $\mathcal{Q}$, hence to measure the effect of $\Gamma_{\mathcal{C}}$ on integrated information.

Proposition 8.9 *For the functor $\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}} : \mathcal{G}_{(S,M)\times(S,M')} \to \Delta$, the Kullback–Leibler divergence*

$$KL(P(X,Y)||Q^*(X,Y))$$

for $\alpha = 1$, with $P \in (\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}}(G_{(X,Y)}))_n$ and $Q^* \in \mathcal{Q}^*_{\alpha, G_{(X,Y)}}$, for $\mathcal{Q}^* : \mathcal{G}_{(S,M)\times(S,M')} \to \Delta$ the KL-minimizer, is of the form

$$KL(P(X,Y)||Q^*(X,Y)) = KL(P'(X,Y)||Q^*(X,Y)) + S(P''),$$

where S is the Shannon entropy, and $P(X,Y) = P'(X,Y)\, P''$ with $P'(X,Y) \in \mathcal{Q}_{G_{(X,Y)}}$ and P'' is a probability in the simplicial sets $\{\Gamma_{\mathcal{C}}([n])\}_{n\in\mathbb{N}}$.

Proof. The image $\mathcal{Q}_{G_{(X,Y)}}$ is some simplicial set K with K_n the set of n-simplexes in the n-th skeleton. Thus we can write a probability $P(X,Y) \in \mathcal{Q}_{G_{(X,Y)}}$ as $\{P_\sigma(X,Y)\}_{\sigma\in K_n}$ with each $P_\sigma(X,Y)$ a probability in an n-simplex σ. With the same notation, using the fact that $\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}}(G_{(X,Y)})$ is a simplicial set obtained as the coend of the $(\mathcal{Q}_{G_{(X,Y)}})_n \wedge \Gamma_{\mathcal{C}}([n])$, we can write a probability $P(X,Y) \in \mathcal{E}_{\mathcal{C}}^{\mathcal{Q}}(G_{(X,Y)})$ as a collection

$$\{P_{\sigma,\tau}(X,Y)\,|\,\sigma \in (\mathcal{Q}_{G_{(X,Y)}})_n\,,\ \tau \in \Gamma_{\mathcal{C}}([n])_m\}\,,$$

with $\Gamma_{\mathcal{C}}([n])_m$ the set of m-simplexes in the skeleton. Moreover, since the simplicial sets $\Gamma_{\mathcal{C}}([n])$ are independent of the random variables (X,Y), we can further write these as products of independent probabilities

$$\{P'_\sigma(X,Y)\, P''_\tau\,|\,\sigma \in (\mathcal{Q}_{G_{(X,Y)}})_n\,,\ \tau \in \Gamma_{\mathcal{C}}([n])_m\}\,.$$

The chain rule for the Kullback–Leibler divergence then gives

$$KL(P_{\sigma,\tau}(X,Y)||Q_{\sigma,\tau}(X,Y)) = \sum P'_\sigma(X,Y)\, P''_\tau \log Q_{\sigma,\tau}(X,Y)$$

$$-\sum P'_\sigma(X,Y)\, \log P'_\sigma(X,Y) - \sum P''_\tau \log P''_\tau$$

$$= \sum_\tau P''_\tau\, KL(P'(X,Y)||Q_\tau(X,Y)) + S(P'')\,,$$

where $P'(X,Y) = \{P'_\sigma(X,Y)\}$ and $Q_\tau(X,Y) = \{Q_{\sigma,\tau}(X,Y)\}$. Convexity of the Kullback–Leibler divergence gives

$$\sum_\tau P''_\tau\, KL(P'(X,Y)||Q_\tau(X,Y)) \geq KL\left(P'(X,Y)||\sum_\tau P''_\tau Q_\tau(X,Y)\right),$$

and the minimizer $Q^*(X,Y)$ of $KL(P'(X,Y)||Q'(X,Y))$ over $Q'(X,Y) \in \Omega_{\lambda,X,Y}$ also minimizes $KL(P'(X,Y)||Q(X,Y))$ with respect to $Q(X,Y) \in \Omega_{\lambda,X,Y}$. □

We can interpret this result as saying that the integrated informations of $\mathcal{E}_{\mathcal{C}}^{\mathcal{Q}}$ and of $\mathcal{Q}$ differ by the Shannon entropy of $\Gamma_{\mathcal{C}}$, where the latter is understood as the Shannon entropy functional from the simplicial sets $\{\Gamma_{\mathcal{C}}([n])\}_{n\in\mathbb{N}}$ to $\mathbb{R}$ (see the similar discussion of information-loss functionals on Gamma-spaces in [76]).

8.7 Homotopy types, spectra, information and cohomology

In this final subsection we outline some connections between some of the threads developed in the previous parts of the paper. In particular, we return to the theme of homotopy types. We start from the viewpoint that neural codes generate homotopy types, in the form of the nerve simplicial set of an open covering associated to a (convex) code, as in [29], [73], and that activated subnetworks of a given network also generate homotopy types in the form of the associated clique complexes. We have discussed in §5.5 and §5.7 how both of these constructions of simplicial sets can be incorporated into the general framework of information structures discussed in §5.4. We have also discussed in §7.4 and §7.5 how Gamma networks, especially those obtained as composition $\Gamma_{\mathcal{C}} \circ \mathcal{Q}$ of a classical Gamma-space $\Gamma_{\mathcal{C}}$ with a functor $\mathcal{Q} : \mathcal{G} \to \Delta$ from a category of (random) graphs to simplicial sets, transform these homotopy types into new homotopy types that incorporate topological structure arising from the category of resources $\mathcal{C}$. This has the effect of combining the simplicial sets $\mathcal{Q}_X$ obtained from information structures with those obtained via the spectra associated to Gamma-spaces, into a single object. For example, when the input simplicial set is the clique complex of the activated part of the network, or the nerve complex of a neural code, the output through the Gamma network can be thought of as a total measure of topological complexity associated to the system and its subsystems *together* with the associated category of resources. Thus, non-trivial homotopy types coming from these clique complexes $K(G)$ (or from nerves of covering complexes) is reflected in the non-trivial topology of their "representation" under the Gamma-space associated to the category $\mathcal{C}$, in the non-trivial homotopy type of the simplicial sets $\Gamma_{\mathcal{C}}(\Sigma^n(K(G)))$, in which the homotopy structure of $K(G)$ is combined with the homotopy structure of the spectrum determined by the Gamma-space $\Gamma_{\mathcal{C}}$, which reflects the contribution of the additional structure that the network carries, determined by the category $\mathcal{C}$ of resources, see Proposition 7.2.

There are two forms of (co)homology one can associate to this object, as a measurement of its topological structure: the information cohomology that we discussed in §5.4 and §8 and the generalized cohomology determined by the homotopy-theoretic spectra discussed in §7.4. Again we can consider possible combinations of these two kinds of (co)homological structures that capture both the informational and the structural sides of the topology of the system.

The main property of homotopy-theoretic spectra is the fact that they determine generalized cohomology theories. Given a spectrum $\mathbb{S}$, the associated generalized cohomology theory is defined by

$$H^k(A, \mathbb{S}) := \pi_k(\Sigma(A) \wedge \mathbb{S}),$$

for simplicial sets A, with $\Sigma(A)$ the suspension spectrum.

In §7.4 we considered the spectra $\Sigma(K(G)) \wedge \Gamma_{\mathcal{C}}$, where we write here $\Gamma_{\mathcal{C}}$ for the spectrum $X_n = \Gamma_{\mathcal{C}}(S^n)$ determined by the Gamma-space, together with the map $\Sigma(K(G)) \wedge \Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{C}}(\Sigma(K(G)))$ as in Proposition 7.2. These determine the generalized cohomology $H^\bullet(K(G), \Gamma_{\mathcal{C}})$.

We have seen in Proposition 5.25 that the simplicial set $K(G)$ given by the clique complex of the network G can be realized as a special case of our more general construction of simplicial sets $\mathcal{Q}_X$ associated to a probability functor $\mathcal{Q}$ and random variables X in the finite information structure functorially associated to a pair (G, Φ) of a network and a summing functor $\Phi \in \Sigma_{\mathcal{C}}(V_G)$.

Thus, it is also natural to consider the spectrum $\Sigma(K(G)) \wedge \Gamma_{\mathcal{C}}$ as a special case of a sheaf of spectra $X \mapsto \Sigma(\mathcal{Q}_X) \wedge \Gamma_{\mathcal{C}}$ and the associated generalized cohomologies $H^\bullet(\mathcal{Q}_X, \Gamma_{\mathcal{C}})$.

In §5.4 and §8 we have considered information cohomologies $H^\bullet(C^\bullet(\mathcal{F}_\alpha(\mathcal{Q}_X), \delta))$. We can also extend these by considering the more general information cohomology groups

$$H^\bullet(C^\bullet(\mathcal{F}_\alpha(\Sigma^k(\mathcal{Q}_X) \wedge \Gamma_{\mathcal{C}}(S^m)), \delta)).$$

While information cohomology itself is not a generalized cohomology theory (it cannot be expected in general to satisfy the Steenrod axioms), one can ask the question of whether a generalized cohomology theory modeled on the case of the $H^\bullet(\mathcal{Q}_X, \Gamma_{\mathcal{C}})$ described above can be constructed that incorporates information measures such as Shannon entropy, Kullback–Leibler divergence, integrated information, in the way that the information cohomology does (see the discussion in §8.6). For some more interpretation of this encoding of homotopy types via Gamma networks see [77].

Appendix

A Probabilistic and persistent Gamma-spaces

Throughout the paper we have worked with "classical" Gamma-spaces, namely functors $\Gamma_{\mathcal{C}} : \mathcal{F}_* \to \Delta_*$ from finite pointed sets to finite simplicial sets, as well as with some generalizations, that we referred to as Gamma-networks. Much of what has been formulated in those terms can be adapted easily to two further variants of the notion of Gamma-space: a *probabilistic* version of Gamma-spaces already considered in [76], which we recall here in §A.2, and a *persistent* version that we introduce in §A.4.

We decided to present these two variants separately as an appendix, rather than blending them into the main text, to maintain clarity of exposition. It should be kept in mind though, that both incorporating probabilistic structures and introducing filtrations that account for the change of topological structure over time, are important features for viable applications to neuronal networks. Since adapting the results of the paper to probabilistic and persistent Gamma-spaces does not present technical obstacles, we will not give a detailed account here, beyond briefly presenting these two notions in §A.2 and A.4, and their combined form in §A.5. In §A.6 and A.7 we discuss briefly some motivations for introducing these variants. We also include in this Appendix a brief account of possible variants of the nerve construction.

Having a persistent version of Gamma-spaces and spectra is useful when one needs to keep into account dependence on some scale parameter (or more generally some parameter in an ordered set, such as time) and keep track of when the topological structures considered undergo changes with respect to that parameter (for instance when homotopy and homology groups acquire or lose new generators). Having a probabilistic version allows for considering probabilistic superpositions of objects and morphisms in the categories involved, for example when assignment of resources involves a random rather than a simply a deterministic choice.

A.1 Simplicial topology enriched with probabilities

As in the earlier sections, we write Δ for the category of simplicial sets (Δ_* for pointed simplicial sets), namely the category of functors $S : \triangle^{\rm op} \to$ Sets from the simplex category $\triangle$ to sets (respectively, pointed sets). Enrichments of simplicial structures with probabilities have been variously considered, for instance in [22], [76], [26].

In the general setting of [21], [22], [23], [85], also used in [26], one constructs a category of probability distributions, where objects are triples (Ω, Σ, P) of a set, a σ-algebra, and a probability distribution, and with morphisms given by "transition measures".

More precisely, consider pairs (Ω, Σ) with Ω a set and $\Sigma \subset \mathcal{P}(\Omega)$ a collection of subsets satisfying

(a) $\Omega \in \Sigma$.

(b) If $X, Y \in \Sigma$, then $X \setminus Y \in \Sigma$.

(c) The union of all elements of any countable subcollection of Σ belongs to Σ.

Let $(S, +, 0)$ be a commutative semigroup with zero. An S-valued (finitely additive) measure on (Ω, Σ) is a map $\mu : \Sigma \to S$ such that $\mu(\emptyset) = 0$ and $\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y)$.

A (σ-additive) *probability distribution* P on (Ω, Σ) is a $(\mathbb{R}_+, +, 0)$-valued measure such that $P(\Omega) = 1$ and for any countable subfamily $\{X_i\} \subset \Sigma$ with empty pairwise intersections we have $P(\cup_{i=1}^\infty X_i) = \sum_{i=1}^\infty P(X_i)$.

A *category of probability distributions* is obtained as follows. Denote by $Cap(\Omega, \Sigma)$ the set of probability distributions on the σ-algebra (Ω, Σ). Given two such sets $Cap(\Omega_1, \Sigma_1)$ and $Cap(\Omega_2, \Sigma_2)$, call "a transition measure" Π between them a function $\Pi\{*|\omega\}$ upon $\Sigma_2 \times \Omega_1$ such that for any fixed $X \in \Sigma_2$, $\Pi\{X|\omega_1\}$ is a Σ_1-measurable function on Ω_1, and for any fixed $\omega_1 \in \Omega_1$, $\Pi\{X|\omega_1\}$ is a probability distribution on Σ_2. A transition measure Π determines a map $Cap(\Omega_1, \Sigma_1) \to Cap(\Omega_2, \Sigma_2)$ given by

$$\Pi P_1(X_2) := \int_{\Omega_1} \Pi\{X_2|\omega_1\} P_1\{d\omega_1\}.$$

One can then take morphisms between objects $(\Omega_1, \Sigma_1, P_1)$ and $(\Omega_2, \Sigma_2, P_2)$ to be transition measures $\Pi : Cap(\Omega_1, \Sigma_1) \to Cap(\Omega_2, \Sigma_2)$ such that $P_2 = \Pi P_1$. Note that this definition of the category of probability distributions differs slightly from [22], [26]: it has been adapted for compatibility with the setting of [76].

A.1.1 Probability distributions on finite sets

If Ω is a finite set, then the collection of all subsets $X \subseteq \Omega$ is a σ-algebra, and probability distributions on it are in the bijective correspondence with maps $P : \Omega \to [0,1]$ such that $\sum_{x\in\Omega} P(x) = 1$. The transition measures $\Pi\{*|\omega\}$ are simply stochastic matrices with obvious properties.

Thus, the category described above reduces to the category $\mathcal{FP}$ used in [76] with objects the pairs (X, P) of a finite set and a probability distribution and morphisms $S : (X, P) \to (Y, Q)$ given by stochastic matrices: $S_{yx} \geq 0$ and $\sum_{y\in Y} S_{yx} = 1$ for all $x \in X$, satisfying $Q = SP$. This category $\mathcal{FP}$ is the undercategory $\mathbb{I}/\mathrm{FinStoch}$ of the category FinStoch of stochastic maps of [6] (see Remark 2.2 of [76]), just as the more general category of probability distributions described above is the undercategory of the one of [22].

In other words, such distributions can be considered as probabilistic enrichment of the simplex Δ_X whose vertices are coordinate points in $\mathbb{R}^X$. If we consider the category of *pointed finite sets* (X, x), morphisms are $(X, x) \to (Y, y)$ in which $X \to Y$ are maps sending x to y. A probabilistic enrichment of such category based on the transition measures $\Pi\{*|\omega\}$ is described as a wreath product of the category of pointed sets with the category of probability distributions, cf. [76], Sec. 2. This is the basic example for a more general construction of probabilistic categories obtained as wreath products of a category $\mathcal{C}$ (with sum and zero object) and the category of finite probability distributions $\mathcal{FP}$.

A.2 Probabilistic Gamma-spaces

A version of Gamma-spaces based on the category $\square_*$ of cubical sets with connections rather than the category Δ_* of simplicial sets was introduced in [76]. Versions of Gamma-spaces that incorporate probabilistic data, using the category $\mathcal{C} = \mathcal{FP} = \mathbb{I}/\mathrm{FinStoch}$, were also introduced in [76].

In the setting of probabilistic Gamma-space of [76] one constructs functors $\Gamma_{\mathcal{PC}} : \mathcal{PF}_* \to \mathcal{P}\square_*$, where $\mathcal{PC}$ is a probabilistic category of resources (an explicit example is discussed in §A.3). The category $P(X)$ is replaced by the category $P(\Lambda X)$ with $\Lambda X = \sum_i \lambda_i(X_i, x_i)$ an object in the probabilistic category $\mathcal{PF}_*$ of pointed sets. This category $P(\Lambda X)$ has objects the subsystems $\Lambda A = \sum_i \lambda_i(A_i, x_i)$ where $A_i \subseteq X_i$ is a pointed subset and morphisms that are the identity on Λ and (deterministic) pointed inclusions on the sets. A summing functor $\Phi_{\Lambda X} : P(\Lambda X) \to \mathcal{PC}$ has the form $\Phi_{\Lambda X}(\Lambda A) = \sum_i \lambda_i \Phi_{X_i}(A_i)$ where the $\Phi_{X_i} : P(X_i) \to \mathcal{PC}$ are summing functors. Thus, when we interpret an object ΛX of $\mathcal{PF}_*$ as a probabilistic assignment of sets X_i (which here we think of as certain systems of neurons), we think of a summing functor $\Phi_{\Lambda X}$ as the corresponding probabilistic assignment of (non-deterministic) transition systems to each X_i and all its subsets A_i in a consistent way. The choice of working with cubical rather than simplicial sets does not alter the homotopy type of the resulting construction.

It is often convenient to work with cubical sets because of the fact that transition systems and higher dimensional automata have a natural formulation in terms of cubical sets [36]. The values in $\mathcal{P}\square_*$ of the functor $\Gamma_{\mathcal{PC}} : \mathcal{PF}_* \to \mathcal{P}\square_*$, simply keep track of the probabilities $\Lambda = (\lambda_i)$ assigned to the subsystems X_i of an object ΛX of $\mathcal{PF}_*$ by considering the object in $\mathcal{P}\square_*$ given by the same combination of cubical nerves of the categories of summing functors of the subsystems, $\sum_i \lambda_i \mathcal{N}_{\mathrm{cube}}(\Sigma_{\mathcal{PC}}(X_i, x_i))$.

A.3 Probabilistic transition systems

As an example of a relevant probabilistic category to consider in this setting, we describe more explicitly the probabilistic category of transition systems, where the probabilistic category $\mathcal{PC}$ can be constructed as in [76], by taking a wreath product $\mathcal{FP} \wr \mathcal{C}$ of the category $\mathcal{C}$ of transition systems described above with a category $\mathcal{FP}$ of finite probabilites.

The resulting categories has objects that are formal convex combinations of objects of $\mathcal{C}$ (finite sets of objects of $\mathcal{C}$ with a probability distribution) and morphisms consists of a stochastic matrix that relates the probabilities on the objects, together with a set of morphisms in $\mathcal{C}$ with assigned probabilities, with a compatibility between the probability distribution of this set of morphisms and the stochastic matrix. More precisely, the resulting category has objects given by finite combinations $\Lambda\tau := \sum_k \lambda_k (S_k, \iota_k, \mathcal{L}_k, \mathcal{T}_k)$ and morphisms given by a stochastic map S with $S\Lambda = \Lambda'$ and morphisms $F : \Lambda\tau \to \Lambda'\tau'$ with $F = F_{ab,r} = (\sigma_{ab,r}, \lambda_{ab,r})$ with probabilities μ^r_{ab} with $\sum_r \mu^r_{ab} = S_{ab}$. The objects of this category can be seen as non-deterministic automata with states set $S = \cup_k S_k$ which are a combination of subsystems S_k that are activated with probabilities λ_k. A morphism in this category consists of a stochastic map affecting the probabilities of the subsystems and non-deterministic maps $(\sigma_{ab,r}, \lambda_{ab,r})$ of the states and labeling systems and transitions, applied with probabilities μ^r_{ab}.

A.4 Persistent Gamma-spaces and persistent spectra

We develop here a new formalism that extends and combines the constructions of [74] and [76] of Gamma-spaces enriched with probabilistic data and of persistent topology.

We are interested here in a variant of Segal's construction of Gamma-spaces and associated spectra, which we will then also combine with probabilistic data as in [76], and which allows us to also incorporate persistent topology structures, following the point of view we adopted in [74], and the categorical setting for persistence described in [18].

A.4.1 Thin categories and persistence diagrams

A persistence diagram in a category $\mathcal{C}$ indexed by a thin category $(S, \leq)$ (as in Definition 5.12) is a functor

$$P : (S, \leq) \to \mathcal{C}\,. \tag{A.1}$$

In particular a *pointed simplicial persistence diagram* is a functor $P : (S, \leq) \to \Delta_*$ to the category Δ_* of pointed simplicial sets. We write

$$\mathcal{C}^{(S,\leq)} := \mathrm{Func}((S, \leq), \mathcal{C}) \tag{A.2}$$

for the category of persistence diagrams, with objects given by functors as in (A.1) and morphisms given by natural transformations of these functors. This is the categorical viewpoint on persistent topology developed in [18] and used in [74].

A.4.2 Persistent Gamma-spaces

We can accommodate the notion of persistent topology in the setting of Gamma-spaces in the following way.

Definition A.1 *Let $\Gamma\mathcal{S}$ denote the category of Gamma-spaces*

$$\Gamma\mathcal{S} := \mathrm{Func}(\mathcal{F}_*, \Delta_*)\,. \tag{A.3}$$

We define persistent Gamma-spaces *to be persistence diagrams in the category of Gamma-spaces*

$$\Gamma\mathcal{S}^{(S,\leq)} := \mathrm{Func}((S, \leq), \mathrm{Func}(\mathcal{F}_*, \Delta_*))\,, \tag{A.4}$$

which we can equivalently view as functors from the category $\mathcal{F}_*$ to the category of pointed simplicial persistence diagrams, or as functors from $\mathcal{F}_* \times (S, \leq)$ to Δ_*

$$\Gamma\mathcal{S}^{(S,\leq)} \simeq \mathrm{Func}(\mathcal{F}_*, \mathrm{Func}((S, \leq), \Delta_*)) \simeq \mathrm{Func}(\mathcal{F}_* \times (S, \leq), \Delta_*).$$

Correspondingly, we introduce a notion of *persistent spectra*. Let $\Sigma\mathbb{S}$ denote the category of symmetric spectra (see [97]). This category has objects given by sequences $X = \{X_n\}_{n\in\mathbb{N}}$ of pointed simplicial sets with a basepoint-preserving left action of the symmetric group Σ_n on X_n and with structure maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ such that the composition $S^k \wedge X_n \to X_{n+k}$ is equivariant with respect to the action of $\Sigma_k \times \Sigma_n$. It has morphisms $f \in \mathrm{Mor}_{\Sigma\mathbb{S}}(X,Y)$ given by a collection $f = \{f_n\}$ of morphisms $f : X_n \to Y_n$ in Δ_* that are Σ_n-equivariant and compatible with the structure maps through the commutative diagrams

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{id \wedge f_n} & S^1 \wedge Y_n \\ \downarrow \sigma_n^X & & \downarrow \sigma_n^Y \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}\,. \end{array}$$

Definition A.2 *Persistent spectra are persistence diagrams in the category of spectra*

$$\Sigma\mathbb{S}^{(S,\leq)} = \mathrm{Func}((S,\leq), \Sigma\mathbb{S}).$$

Lemma A.3 *A persistent Gamma-space determines a persistent spectrum.*

Proof. By applying the Segal construction [96] pointwise we see that a persistent Gamma-space gives rise to an associated persistent spectrum, by first upgrading a persistent Gamma-space $\Gamma : \mathcal{F}_* \to \mathrm{Func}((S,\leq),\Delta_*)$ to a functor

$$\Gamma : \Delta_* \to \mathrm{Func}((S,\leq),\Delta_*)$$

seen equivalently as a persistent endofunctor of the category of simplicial sets

$$\Gamma : (S,\leq) \to \mathrm{Func}(\Delta_*,\Delta_*).$$

The associated persistent spectrum is given by the functor determined by assigning $X(s)_n := F(s)(S^n)$. □

The Segal construction [96] of Gamma-spaces and spectra associated to categories with coproduct and zero object can be extended to the persistent setting as follows.

Proposition A.4 *Let $\mathcal{C}$ be a category with coproduct and zero object and let $\mathcal{C}^{(S,\leq)}$ be the category of persistence diagrams in $\mathcal{C}$ indexed by a thin category $(S,\leq)$. The category $\mathcal{C}^{(S,\leq)}$ determines a persistent Gamma-space $\Gamma_{\mathcal{C}^{(S,\leq)}} : \mathcal{F}_* \to \Delta_*^{(S,\leq)}$ and an associated persistent spectrum $\mathcal{S}_{\mathcal{C}^{(S,\leq)}}$ in the category $\Sigma\mathbb{S}^{(S,\leq)}$.*

Proof. If $\mathcal{C}$ is a category with coproduct $\oplus$ and a zero object 0, then the category of persistence diagrams $\mathcal{C}^{(S,\leq)} = \mathrm{Func}((S,\leq),\mathcal{C})$ endowed with the pointwise coproduct has zero object given by the functor $F_0(s) = 0$ for all $s \in S$ and $F_0(s \leq s') = \mathrm{id}_0$.

The nerve construction is given by a functor $\mathcal{N} : \mathrm{Cat} \to \Delta$ from the category of small categories to simplicial sets defined on a small category $\mathcal{D}$ by $\mathcal{N}(\mathcal{D})_n := \mathrm{Obj}(\mathrm{Func}([n],\mathcal{D}))$ where $[n] = \{0 < 1 < \cdots < n\}$ the ordered set seen as a thin category with a unique morphism $i \to j$ for $i \leq j$.

Consider then the categories $\mathrm{Func}((S,\leq),\mathcal{D})$ and $\mathrm{Func}([n],\mathrm{Func}((S,\leq),\mathcal{D}))$, which we can identify with $\mathrm{Func}((S,\leq),\mathrm{Func}([n],\mathcal{D})) = \mathrm{Func}([n],\mathcal{D})^{(S,\leq)}$. The set of objects of $\mathrm{Func}([n],\mathcal{D})^{(S,\leq)}$ consists of objects of $\mathrm{Func}([n],\mathcal{D})$ parameterized by elements $s \in S$, hence the nerve $\mathcal{N}(\mathrm{Func}((S,\leq),\mathcal{D}))$ determines a pointed simplicial persistence diagram in $\Delta_*^{(S,\leq)}$.

Given a category $\mathcal{C}$ with coproduct and zero object, and the associated category of persistence diagrams $\mathcal{C}^{(S,\leq)}$, for each finite pointed set $X \in \mathrm{Obj}(\mathcal{F}_*)$ we can consider the category $\Sigma_{\mathcal{C}^{(S,\leq)}}(X)$ of summing functors

$$\Phi_X : P(X) \to \mathcal{C}^{(S,\leq)}$$

from the category of pointed subsets of X with inclusions such that $\Phi_X(*) = F_0$. The base point of X is sent to the zero object of $\mathcal{C}^{(S,\leq)}$ and $\Phi_X(A \cup A') = \Phi_X(A) \oplus \Phi_X(A')$, with $\oplus$ the coproduct of $\mathcal{C}^{(S,\leq)}$, whenever $A \cap A' = \{*\}$. Any such summing functor $\Phi_X : P(X) \to \mathcal{C}^{(S,\leq)}$ determines a functor $\psi_X : (S,\leq) \to \Sigma_{\mathcal{C}}(X)$, where $\Sigma_{\mathcal{C}}(X)$ is the category of summing functors $\Phi_X : P(X) \to \mathcal{C}$, by $\Phi_X(A)(s) = \psi_X(s)(A)$. Thus, we can identify $\Sigma_{\mathcal{C}^{(S,\leq)}}(X)$ with $\Sigma_{\mathcal{C}}(X)^{(S,\leq)}$, and applying the nerve construction $\mathcal{N}(\Sigma_{\mathcal{C}}(X)^{(S,\leq)})$ as above we obtain a pointed simplicial persistence diagram in $\Delta_*^{(S,\leq)}$. Thus, using the Segal construction of [96], we can associate to a category of persistence diagrams $\mathcal{C}^{(S,\leq)}$ over a category $\mathcal{C}$ with coproduct and zero object a persistent Gamma-space

$$\Gamma_{\mathcal{C}^{(S,\leq)}} : \mathcal{F}_* \to \Delta_*^{(S,\leq)}$$

and an associated persistent spectrum $\mathcal{S}_{\mathcal{C}^{(S,\leq)}}$. □

A.5 Probabilistic persistent Gamma-spaces

Consider as in [76] the category $\mathcal{FP} = \mathbb{I}/\text{FinStoch}$ of finite probabilities with stochastic maps as morphisms (see [6] and Remark 2.2 of [76]). Given a category $\mathcal{C}$ with categorical sum and zero object, consider the wreath product $\mathcal{PC} := \mathcal{FP} \wr \mathcal{C}$, which has objects given by formal convex linear combinations $\Lambda C := \sum_i \lambda_i C_i$ of objects $C_i \in \text{Obj}(\mathcal{C})$ with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$, and morphisms $\phi : \Lambda C \to \Lambda' C'$ given by pairs $\phi = (S, F)$ of a stochastic map with $S\Lambda = \Lambda'$ (a morphisms of $\mathcal{FP}$) and a finite collection $F = \{F_{ab,r}\}$ of morphisms $F_{ab,r} \in \text{Mor}_{\mathcal{C}}(C_b, C'_a)$ with assigned probabilities μ^r_{ab} satisfying $\sum_r \mu^r_{ab} = S_{ab}$. As shown in §2 of [76] the category $\mathcal{PC}$ constructed in this way has a zero object and a coproduct of the form $\Lambda C \oplus \Lambda' C' = \sum_{ij} \lambda_i \lambda'_j C_i \oplus_{\mathcal{C}} C'_j$.

The morphisms in the category $\mathcal{PC}$ can be interpreted as a collection of morphisms $F_{ab,r}$ in $\mathcal{C}$ that are chosen and applied with probability μ^r_{ab} where $\sum_r \mu^r_{ab}$ is the stochastic matrix that relates the probabilities Λ and Λ' in $\mathcal{FP}$ associated to the objects ΛC and $\Lambda' C'$ of $\mathcal{PC}$. Thus, both the objects and the morphisms of $\mathcal{C}$ are made non-deterministic by passing to $\mathcal{PC}$, in a way that preserves the property that the category has coproduct and zero object. Thus, the Segal construction applied to $\mathcal{C}$ can be extended to the probabilistic category $\mathcal{PC}$.

Moreover, it is shown in [76] that the Segal construction itself can be made probabilistic, by considering a version of Gamma-spaces based on the category $\Box_*$ of pointed cubical sets with connections (which are homotopy equivalent to the usual Gamma-spaces valued in pointed simplicial sets) and then defining stochastic Gamma-spaces as functors

$$\Gamma : \mathcal{PF}_* \to \mathcal{P}\Box_* \tag{A.5}$$

where $\mathcal{PF}_*$ and $\mathcal{P}\Box_*$ are the probabilistic categories associated to pointed sets and to pointed cubical sets with connections. It is shown in §5 of [76] that to any category $\mathcal{C}$ with zero object and categorical sum, one can associate, using the Segal construction, a probabilistic Gamma-space $\Gamma_{\mathcal{PC}} : \mathcal{PF}_* \to \mathcal{P}\Box_*$ which is the functor determined by $\Gamma_{\mathcal{PC}}(\Lambda X) = \sum_i \lambda_i \mathcal{N}_{\text{cube}}(\Sigma_{\mathcal{PC}}(X_i))$, seen as an object in $\mathcal{P}\Box_*$. The cubical nerve $\mathcal{N}_{\text{cube}}(\Sigma_{\mathcal{PC}}(X))$ is homotopy equivalent to the simplicial nerve $\mathcal{N}(\Sigma_{\mathcal{PC}}(X))$ [3], meant as the equivalence of the respective realizations (see §4 of [3]).

The cubical nerve of a category $\mathcal{D}$ is obtained by considering functors $\text{Func}(\mathcal{I}^n, \mathcal{D})$, where the objects of $\mathcal{I}^n$ are the vertices of the n-cube (sequences $(s_1, \ldots, s_n)$ with digits $s_i \in \{0, 1\}$) and morphisms generated by the edges of the cube. The maps $\mathcal{N}_{\text{cube}}(\mathcal{D})_n \to \mathcal{N}_{\text{cube}}(\mathcal{D})_m$ are induced by precomposition of the functors $\mathcal{I}^n \to \mathcal{D}$ with the morphisms $\mathcal{I}^m \to \mathcal{I}^n$ of the cubical category (box category).

By combining this construction with the construction of persistent Gamma-spaces we introduced in §A.4, we obtain the following.

Proposition A.5 *Let $\mathcal{C}$ be a category with zero object and categorical sum, with $\mathcal{PC} = \mathcal{FP} \wr \mathcal{C}$ the associated probabilistic category, and let $(S, \leq)$ be a thin category. The Segal construction determines a probabilistic persistent spectrum*

$$\Gamma_{\mathcal{PC}^{(S,\leq)}} : \mathcal{PF}_* \to \mathcal{P}\Box_*^{(S,\leq)} . \tag{A.6}$$

The construction described in Proposition 7.2 of [74] can be seen as a special case of the construction obtained here above.

A.6 Modeling constraints and the role of persistence

As observed in §3.2 of [89], a diagram in a thin category $(S, \leq)$ is just a selection of a subset $A \subseteq S$. A cone on A with vertex x is a lower bound x for A, since it consists of an arrow from x to each element $a \in A$, while a cocone is an upper bound. Limits and colimits then correspond to greatest lower bounds and least upper bounds for subsets $A \subseteq S$. Thus, functors that are compatible with limits and colimits can be viewed as describing constrained optimization settings where certain maximization or minimization conditions are imposed.

Consider a thin category $(S, \leq)$ and a category $\mathcal{D}^{(S,\leq)} = \text{Func}((S, \leq), \mathcal{D})$ of persistence diagrams in $\mathcal{D}$ indexed by $(S, \leq)$. We can interpret the objects $D(s)$ and morphisms $D(s \leq s') : D(s) \to D(s')$ in $\mathcal{D}$ as families of objects in $\mathcal{D}$ subject to constraints encoded in $(S, \leq)$.

In our setting here we can consider a theory of resources formulated as in [27] and [40] (see §3.2 above). In particular, we assume given an abelian semigroup with partial ordering $(R, +, \succeq, 0)$ on the set R of isomorphism classes of objects in $\mathrm{Obj}(\mathcal{R})$, where $\mathcal{R}$ is a symmetric monoidal category describing resources.

We can use $(R, \succeq)$ as the indexing of persistence diagrams. Here we use the reverse ordering, since in $(R, +, \succeq, 0)$ the relation $A \succeq B$ means $\mathrm{Mor}_{\mathcal{R}}(A, B) \neq \emptyset$ hence resource A is convertible to resource B. Thus, in a category $\mathcal{D}^{(R,\succeq)}$ we have objects D_A, indexed by resources $A \in R$ with a morphism $D_A \to D_B$ whenever $A \succeq B$. The morphism describes the effect on the object D_A caused by converting the resource A to the resource B. The constraints here are encoded in the convertibility of resources.

Assuming that we are modeling the possible concurrent/distributed computing architectures or other resources associated to a population of neurons via a (probabilistic) Gamma-space $\Gamma_{\mathcal{PC}} : \mathcal{PF}_* \to \mathcal{P}\square_*$, where $\mathcal{PC}$ is the category of (non-deterministic) transition systems (or the probabilistic version of another category of resources), we can incorporate in this description the constraints given by the convertibility properties of computational, metabolic or informational resources by considering an associated persistent (probabilistic) Gamma-space

$$\Gamma_{\mathcal{PC}^{(R,\succeq)}} : \mathcal{PF}_* \to \mathcal{P}\square_*^{(R,\succeq)},$$

where $\mathcal{PC}^{(R,\succeq)}$ is the category of persistence diagrams of (non-deterministic) transition systems parameterized by the convertibility of resources $(R, \succeq)$ with the reverse ordering, as above.

Thus, for instance, we can view the category $\mathcal{PC}^{(R,\succeq)}$ as describing families of non-deterministic (computational) resources (transition systems) C_A associated to the available (metabolic) resources $A \in R$, with maps $C_A \to C_B$ that describe the change to a transition system affected by the conversion of resources $A \succeq B$. The persistent (probabilistic) Gamma-space carries this information about the dependence on resources and conversion of resources over into the construction of the resulting objects in $\mathcal{P}\square_*^{(R,\succeq)}$, which we now interpret as a family of (probabilistic) simplicial or cubical sets associated to the available resources in R with maps describing the effect of resource conversion. This provides a description of all the possible ways of assigning transition systems to a probabilistic set ΛX with assigned resources $A \in R$.

A.7 Persistence to model time and scale dependence

There is another use of the persistence structure in these models, which is in line with the more common use of persistent topology, namely as a way to keep track of the dependence on time and on scale.

In the previous subsection we have described how to use persistent (probabilistic) Gamma-spaces to model the dependence on constraints dictated by resources and convertibility of resources, where the latter are encoded in the structure of a preordered semigroup $(R, +, \succeq)$. A more common use of persistent topology is in capturing the dependence of a simplicial set on either a time variable or a scale factor. In this setting, the thin category of the form $(\mathcal{I}, \leq)$ where $\mathcal{I}$ is a subinterval of the real line $\mathbb{R}$, with its natural ordering $\leq$, where the variable $s \in \mathcal{I}$ represents either time or a scale variable. The scale dependence for example is used in the construction of the Vietoris–Rips simplicial complex associated to a set of data points. The time dependence is crucial for example in the analysis of the formation of non-trivial homology in the persistent topology of the simulations of the neural cortex in response to stimuli analyzed in [91].

The results of [91] show that nontrivial topological structures arise in the computational architecture of the response of the (simulated) neural cortex to stimuli. In simulations of the reconstructed microcircuitry, following a spatio-temporal stimulus to the network, during correlated activity, active cliques of increasingly high dimension are detected, with a large number of nontrivial homology generators forming and peaking at around 60–80 ms from the initial stimulus and then quickly disappearing. While different stimuli give rise to different patterns of activity, all have this common feature, where functional relations among increasingly high-dimensional cliques form and then disintegrate.

This kind of result motivates the introduction of a time scale for the birth and death of simplices and homology generators in various dimensions. A dependence on scale may also be similarly needed.

While one can try to incorporate both the time/scale dependence and the dependence on resources and their convertibility in a the persistent structure, it seems more natural to reserve persistence as a way to capture the time/scale dependence and incorporate the metabolic constraints and other resources constraints in a different way. This can be done by working directly with the symmetric monoidal category of resources $(\mathcal{R}, \otimes, \mathbb{I})$ instead of using the associated preordered semigroup $(R, +, \succeq)$.

A.8 Variants of the nerve construction

We can use Gamma-spaces and their probabilistic and persistent generalizations to associate in a functorial way to a given population of neurons endowed with certain probabilities of activation, described by an object ΛX in the category $\mathcal{PF}_*$, a (probabilistic) simplicial or cubical set $\Gamma_{\mathcal{PC}}(\Lambda X)$. The underlying category of summing functors describes all the consistent ways of assigning computational architectures (transition systems) or other kinds of resources, seen as elements in a category $\mathcal{PC}$, to all subsystems (subsets with probabilities) of ΛX.

The resulting object $\Gamma_{\mathcal{PC}}(\Lambda X)$ can itself be regarded as a computational architecture (a higher dimensional automaton) attached to ΛX. It encodes all the possible consistent assignments of transition systems to ΛX and to its constituent parts, and it inputs and outputs (probabilistic) simplicial data.

The nerve construction used in the notion of Gamma-spaces (and their probabilistic and persistent versions) can be adapted to accommodate other possible categorical models of concurrent/distributed computation in the resulting $\Gamma_{\mathcal{PC}}(\Lambda X)$. Indeed, the usual full and faithful nerve functor $\mathcal{N} : \mathrm{Cat} \to \mathrm{Func}(\Delta^{\mathrm{op}}, \mathrm{Set})$ provides a way of describing categories through simplicial sets. This nerve construction admits generalizations, as shown in [69], [108], obtained by considering certain classes of monads T on suitable categories (see [69] for the specific conditions on monads and categories), to which it is possible to associate a nerve functor

$$\mathcal{N}_T : \mathrm{Alg}(T) \to \mathrm{Func}(\Delta_T^{\mathrm{op}}, \mathrm{Set})$$

with Δ_T^{op} a category of T-simplicial sets, and $\mathrm{Alg}(T)$ the category of algebras over the monad T.

While we do not develop this viewpoint in the present paper, it is worth mentioning the fact that this can lead to other ways of adapting the formalism of Gamma-spaces to a probabilistic setting, where probabilities are interpreted from a monad viewpoint, see [44] (and also [39], [41], [43], [42], [44]) for recent developments of this approach to probability.

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