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Cloven operadic categories: An approach to operadic categories with cardinalities in finite unordered sets

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We introduce and study operadic categories with cardinalities in finite sets and establish conditions under which their associated theories of operads and algebras are equivalent to the standard framework developed in [7]. Our approach is particularly natural in applications to the operadic category of graphs and the related category of modular operads and their clones.

Contents

Introduction

Recall that the traditional operads [1, 11] in a symmetric monoidal category $\mathtt{V}$, which is typically the category of sets or graded vector spaces, are collections $\mathcal{P} = \{\mathcal{P}(n) \,|\, n \geq 1\}$ of objects of $\mathtt{V}$ such that

(i) each $\mathcal{P}(n)$ is a module over the symmetric group Σ_n, and

(ii) there are composition laws

$$\mu_\rho : \mathcal{P}(k)\otimes\mathcal{P}(n_1)\otimes\cdots\otimes\mathcal{P}(n_k) \longrightarrow \mathcal{P}(m),\ m = n_1 + \cdots + n_k,\ k \geq 1,\ n_1,\ldots,n_k \geq 1, \tag{1a}$$

satisfying appropriate associativity and equivariance axioms; we assumed for simplicity that there is no $\mathcal{P}(0)$. The meaning of the index ρ at μ_ρ is explained below. We call the above form of definition the *skeletal* definition of (classical) operads, the natural numbers featured in (1a) are the *arities* of the corresponding components. An equivalent, *non-skeletal* definition with arities in finite sets was given in [10, Section II.1.7]. Operads in this setup appear as collections $\boldsymbol{\mathcal{P}} = \{\,\boldsymbol{\mathcal{P}}(X) \,|\, X \text{ a finite set }\}$, with two kinds of data:

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(i) a natural morphism $\varphi_* : \boldsymbol{\mathcal{P}}(X') \to \boldsymbol{\mathcal{P}}(X'')$ specified for each isomorphism $\varphi : X' \to X''$ of finite sets, and

(ii) composition laws

$$\mu_g : \boldsymbol{\mathcal{P}}(Y) \otimes \bigotimes_{y \in Y} \boldsymbol{\mathcal{P}}(g^{-1}(y)) \longrightarrow \boldsymbol{\mathcal{P}}(X), \tag{1b}$$

given for each epimorphism $g : X \to Y$ of finite sets,

satisfying again appropriate axioms. In (1b), $g^{-1}(y)$ is the set-theoretic preimage of $y \in Y$, and the display involves the "unordered tensor product" [10, Definition I.1.58] of the preimages indexed by the *unordered* set Y.

Notice that the operations in (1a) are parametrized by maps $\rho : \{1, \ldots, m\} \to \{1, \ldots, k\}$ so that n_i appears as the cardinality of the preimage $\rho^{-1}(i)$, $i \in \{1, \ldots, k\}$. We emphasize that ρ can be an *arbitrary*, not necessarily order-preserving map. Therefore both (1a) and (1b) are parametrized by maps of finite sets. The sets in (1a) belong to the small category of finite ordinals. Their order gives a preferred order of the preimages $\rho^{-1}(i)$, which circumvents the use of the unordered tensor product. On the other hand, (1b) refers to the big category of all finite sets, but the formula is "canonical," since it does not refer to any particular choice of the skeletal subcategory of finite sets, realized by finite ordinals in (1a).

Let us briefly mention non-Σ (non-symmetric) operads. In the traditional presentation, such an operad $\underline{\mathcal{P}}$ is a collection of objects of $\mathtt{V}$, no group actions assumed, with ρ in (1a) order-preserving. In the non-skeletal setup, it is a collection $\underline{\boldsymbol{\mathcal{P}}}$ indexed by finite ordered sets, with composition laws (1b) parametrized by *order-preserving* maps. But to keep both definitions equivalent, we still need natural actions $\varphi_* : \underline{\boldsymbol{\mathcal{P}}}(X') \to \underline{\boldsymbol{\mathcal{P}}}(X'')$ of order-preserving maps which do not appear in the skeletal version.

The preference for the skeletal version for classical operads is quite understandable. However, the picture changes for cyclic operads and their clones, such as modular operads. Here the skeletal version requires the choice of a skeletal subcategory of the category of finite sets with the cyclic group action, and a "nice" linear formula as in (1a) moreover requires non-canonical choices of linear orders of cyclically ordered sets. The resulting formulas are so clumsy as to be practically useless, cf. [8, Definition 6.23]. On the other hand, the non-skeletal version, written out in [4, Definition A.1] or [8, Definition 6.16], is sensible and intuitively clear.

So both approaches have their merits, depending on the context. The applications we had had in mind, together with the motivating example of Batanin's k-trees, lead us in [7] to the definition of operadic categories over finite ordinals. Our aim is to allow *all* finite sets and to determine when both approaches lead to equivalent notions of operads and their algebras. A typical example for the later approach is the category of graphs for which, in contrast to the skeletal version, no orders of vertices and (half)edges are required. We believe that extending the concept of operadic categories to allow the standard, "naïve" approach to graphs will make the theory more amenable for applications.

An anonymous referee suggested the following example of a situation in which one is naturally led to consider operads with arities in arbitrary finite sets. Let $\mathcal{P} = \{\mathcal{P}(n) \mid n \geq 1\}$ be the standard classical operad as in (1a), with the structure expressed in the equivalent form of $\circ_i$-operations

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1), \ \ m, n \geq 1, 1 \leq i \leq m. \tag{2a}$$

The frequently used bar construction $B(\mathcal{P})$ of $\mathcal{P}$ is formed by rooted trees with decorated vertices. The decoration is such that a vertex v with n incoming edges is decorated by the component $\mathcal{P}(n)$ of $\mathcal{P}$. The differential is given by the sum of edge contractions. However, since the incoming edges of v are unordered, it is not possible to determine which structure map (2a) of $\mathcal{P}$ corresponds to a given edge contraction. A standard way to overcome this difficulty is to allow the operad to have arities in arbitrary finite sets, and the structure operations of the form

$$\circ_e : \mathcal{P}(X) \otimes \mathcal{P}(Y) \longrightarrow \mathcal{P}((X \setminus \{e\}) \cup Y), \ \ e \in X, \ \ X \cap Y = \emptyset. \tag{2b}$$

Then the vertex v is decorated by the component $\mathcal{P}(X)$ of this extended operad, with X the (finite) set of incoming edges of v. The structure operation corresponding to the contraction of an incoming edge e of v is precisely $\circ_e$ in (2b). The applicability of this procedure relies on the equivalence between the 'standard' definition of operads and operads with arities in finite sets, cf. [10, Section II.1.7] again. Such an extension is standard when working with operads and their generalization. The goal of the present article is to lift this procedure to the level of operadic categories governing the given type of operads.

Terminology and notation. We will call operadic categories over finite ordinals introduced in [7] *thin*, those over arbitrary finite sets treated in this article *thick*. In other words, thin stands for skeletal and thick stands for non-skeletal. Whenever it makes sense to do so, the symbols for objects related to thick operadic categories are typed in **bold**. For instance, the cardinality functor of a thin operadic category $\mathtt{O}$ will be written as $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$, its thick version as $|\text{-}| : \mathbf{O} \to \mathbf{Fin}$.

Operadic categories and their operads in [7] were unital by definition. However, in our subsequent work [6] we realized that for some applications is is better to consider unitality as an additional, independent, property. We follow this convention in the present article.

Results and novelties. Theorem 28 establishes, and its proof explicitly describes, a natural equivalence between the categories of thin and thick operadic categories, respectively, such that the categories of operads and their algebras of the corresponding operadic categories are naturally equivalent. For such equivalences to exist, we must restrict our focus to operadic categories which are *cloven*. This property, introduced in Definitions 6 and 9, requires the existence of a *cleavage* – a functorial choice of certain lifts. The categories of graphs analyzed in detail in the last section provide convincing and easy to work with examples of thick operadic categories.

Regarding the applicability of our approach, most operadic categories that a working mathematician may encounter are either ordered, meaning by definition that the cardinality functor factorizes through the category of finite ordinals and order-preserving maps, or cloven. Ordered operadic categories can be treated by a simpler version of our theory. The only "non-synthetic" examples of operadic categories that are neither cloven nor ordered we know of are the operadic category of Batanin's k-trees and the operadic category of vines, cf. non-Examples 12 and 13. Our approach does not produce thick counterparts of these categories with equivalent categories of operads and their algebras, and we doubt that they exist.

Let us see what happens if we replace finite sets by finite *ordered* sets. For the lack of better terminology we will call an operadic category *semi-ordered*, if a factorization of the cardinality functor through the category of ordered sets and their, not-necessarily order-preserving, morphisms is given. Obviously, a thin operadic category is always semi-ordered. A semi-ordered operadic category is ordered, if the cardinality functor factorizes through the category of ordered sets and *order-preserving* maps. A suitably modified notion of a cleavage makes sense for ordered and semi-ordered operadic categories; thin operadic categories are always cloven in this modified sense.

In Theorem 43 we establish an equivalence between the category of thin operadic categories and the category of thick semi-ordered cloven operadic categories, which further restricts to an equivalence of the subcategories of ordered operadic categories. This equivalence has the expected property that the corresponding operadic categories have equivalent categories of operads and their algebras. We are aware that this setup is not very interesting, as it does not achieve our main goal of avoiding the use of ordered sets. We include it only for the sake of completeness.

Plan of the paper. In Sections 1–5, we introduce thick operadic categories, the concept of cleavage, the related notions of operads and their algebras, and prove their basic properties. Section 6 contains the main results of the paper. It introduces two functors – the extension and the restriction – between the category of cloven thin and the category of cloven thick operadic categories which together form a pair of mutually inverse equivalences. We show that the corresponding categories of operads and their algebras are equivalent. Section 7 is devoted to a detailed study of the thick version of the category of graphs. In the Appendix we briefly sketch the (semi)ordered version of our theory.

Requirements. The standard, "thin" operadic categories, their operads and algebras were introduced in [7]. We will assume familiarity with this apparatus. Some knowledge of the classical operads and PROPs may ease reading, cf. [9] or the monograph [10]. The present article is a loose continuation of the series [3, 4, 6] of papers devoted to general properties of operadic categories, so acquaintance with those sources is also welcome.

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1 Thick operadic categories

Operadic categories were introduced in [7] and further studied in several subsequent papers. Their salient feature is that each object has its cardinality, which is a finite ordinal. We shall refer to these categories as *thin* operadic categories. The aim of this section is to introduce a "thick" version with cardinalities in arbitrary finite sets.

Finite sets. In the following text, **Fin** will denote the category of (all) finite sets. For a map $\phi : S \to T$ in **Fin**, $\phi^{-1}(t)$ will be the set-theoretic preimage of $t \in T$. By $\mathtt{Fin}$ we denote the skeletal category of finite sets. Its objects are finite ordinals $\underline{n} := (1, \ldots, n)$, $n \geq 0$, with the convention that $\underline{0}$ is the empty set. The morphisms in $\mathtt{Fin}$ are all, not necessarily order-preserving, maps. For $f : \underline{m} \to \underline{n}$ and $i \in \underline{n}$ we denote by $f^{-1}(i)$ the pullback of f along the map $\underline{1} \to \underline{n}$ which picks up $i \in \underline{n}$, as in

$$\begin{tikzcd} f^{-1}(i) \arrow[r, "I_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{m} \arrow[d, "f"] \\ \underline{1} \arrow[r, "1 \longmapsto i"] & \underline{n}. \end{tikzcd} \tag{3}$$

The set $f^{-1}(i)$ is unique by the skeletality of $\mathtt{Fin}$. We will need at some places in Section 6 to emphasize that $f^{-1}(i)$ is a pullback, not an inverse image, we will thus write more specifically $f^{-1}_{pb}(i)$ instead.

We are going to define operadic categories with arbitrary finite sets as cardinalities. They are straightforward modifications of "thin" operadic categories as defined in [7, Section 1] which assumed cardinalities in $\mathtt{Fin}$. We will however leave out all references to chosen local terminal objects, since the related issue of unitality has a different flavor in the non-skeletal case, cf. the remarks following the definition. The same definition appeared also in [5].

Definition 1. A *thick (strict, nonunital) operadic category* is a category **O** equipped with a "cardinality" functor $|\text{-}| : \mathbf{O} \to \mathbf{Fin}$ with the following properties. For every $\boldsymbol{f} : \boldsymbol{T} \to \boldsymbol{S}$ in **O** and every element $s \in |\boldsymbol{S}|$ there is given an object $\boldsymbol{f}^{-1}(s) \in \mathbf{O}$, *the sth fiber* of $\boldsymbol{f}$, such that

$$|\boldsymbol{f}^{-1}(s)| = |\boldsymbol{f}|^{-1}(s). \tag{4a}$$

We moreover require that

(i) for any commutative diagram

$$\begin{tikzcd} \boldsymbol{T} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{h}"'] & & \boldsymbol{S} \arrow[dl, "\boldsymbol{g}"] \\ & \boldsymbol{R} & \end{tikzcd} \tag{4b}$$

in **O** and every $r \in |\boldsymbol{R}|$, one is given a morphism

$$\boldsymbol{f}_r : \boldsymbol{h}^{-1}(r) \to \boldsymbol{g}^{-1}(r)$$

in **O** such that $|\boldsymbol{f}_r| : |\boldsymbol{h}^{-1}(r)| \to |\boldsymbol{g}^{-1}(r)|$ equals the map $|\boldsymbol{h}|^{-1}(r) \to |\boldsymbol{g}|^{-1}(r)$ of the preimages induced by

$$\begin{tikzcd} |\boldsymbol{T}| \arrow[rr, "|\boldsymbol{f}|"] \arrow[dr, "|\boldsymbol{h}|"'] & & |\boldsymbol{S}| \arrow[dl, "|\boldsymbol{g}|"] \\ & |\boldsymbol{R}| & \end{tikzcd} .$$

This assignment must moreover assemble to a functor $\mathrm{Fib}_r : \mathbf{O}/\boldsymbol{R} \to \mathbf{O}$.

(ii) In the situation of (4b), for any $s \in |\boldsymbol{S}|$, one has the equality

$$\boldsymbol{f}^{-1}(s) = \boldsymbol{f}^{-1}_{|\boldsymbol{g}|(s)}(s). \tag{4c}$$

(iii) Let

$$\begin{tikzcd}
& \boldsymbol{S} \arrow[dr, "\boldsymbol{a}"] \arrow[dd, "\boldsymbol{g}"] & \\
\boldsymbol{T} \arrow[ur, "\boldsymbol{f}"] \arrow[rr, "\boldsymbol{b}"] \arrow[dr, "\boldsymbol{h}"'] & & \boldsymbol{Q} \arrow[dl, "\boldsymbol{c}"] \\
& \boldsymbol{R} &
\end{tikzcd}$$

be a commutative diagram in $\mathbf{O}$, and let $q \in |\boldsymbol{Q}|, r := |\boldsymbol{c}|(q) \in |\boldsymbol{R}|$. Then

$$\begin{tikzcd}
\boldsymbol{h}^{-1}(r) \arrow[rr, "\boldsymbol{f}_r"] \arrow[dr, "\boldsymbol{b}_r"'] & & g^{-1}(r) \arrow[dl, "\boldsymbol{a}_r"] \\
& \boldsymbol{c}^{-1}(r) &
\end{tikzcd}$$

commutes by (i), so it induces a morphism $(\boldsymbol{f}_r)_q : \boldsymbol{b}_r^{-1}(q) \to \boldsymbol{a}_r^{-1}(q)$. We have by (4c)

$$\boldsymbol{a}^{-1}(q) = \boldsymbol{a}_r^{-1}(q) \text{ and } \boldsymbol{b}^{-1}(q) = \boldsymbol{b}_r^{-1}(q).$$

We then require that $\boldsymbol{f}_q = (\boldsymbol{f}_r)_q$.

We will also assume that the set $\pi_0(\mathbf{O})$ of connected components is small with respect to a sufficiently big ambient universe.

Equation (4a) implies that the fibers $\boldsymbol{f}^{-1}(s)$, $s \in \boldsymbol{S}$, of a morphism $\boldsymbol{f} : \boldsymbol{T} \to \boldsymbol{S}$ in $\mathbf{O}$ are mutually disjoint and the cardinality $|\boldsymbol{T}| \in \mathbf{Fin}$ of $\boldsymbol{T}$ equals their disjoint union

$$|\boldsymbol{T}| = \bigsqcup_{s \in |\boldsymbol{S}|} |\boldsymbol{f}^{-1}(s)|. \tag{5}$$

Let us recall the concept of unitality for "thin" operadic categories $\mathtt{O}$ introduced in [7, Section 1] and further refined in [6, Section 2].

Definition 2. Suppose that a family

$$\big\{U_c \in \mathtt{O} \mid c \in \pi_0(\mathtt{O})\big\} \tag{6}$$

of local terminal objects of a thin operadic category $\mathtt{O}$ is specified, with U_c belonging to the connected component c, and such that $|U_c| = \underline{1}$ for each c.

Then $\mathtt{O}$ is *left unital* if all fibers of the identity automorphism $\mathbb{1} : S \to S$ belong to the set (6) of the chosen local terminal objects, for each $S \in \mathtt{O}$. The category $\mathtt{O}$ is *right unital* if the fiber functor $\mathrm{Fib}_1 : \mathtt{O}/U_c \to \mathtt{O}$ is the domain functor for each $c \in \pi_0(\mathtt{O})$. Finally, $\mathtt{O}$ is *unital* if it is both left and right unital.

Definition 3 formally makes sense also in the thick case. While the right unitality occurs quite often, a thick operadic category $\mathbf{O}$ can be left unital only if $|\boldsymbol{T}| = \{1\}$ for all $\boldsymbol{T} \in \mathbf{O}$. Indeed, the cardinalities of the fibers of $\mathbb{1} : \boldsymbol{T} \to \boldsymbol{T}$ are $\{t\}$, $t \in |\boldsymbol{T}|$, by (4a), but we required that all the chosen local terminal objects have cardinalities $\{1\}$. It is more natural to assume only that the fibers of identity automorphisms are *isomorphic* to the chosen local terminal objects, as done in Definition 14, Section 3.

Example 3. The category $\mathbf{Fin}$ is a thick operadic category, with the cardinality given by the identity functor $\mathbf{Fin} \to \mathbf{Fin}$. It is right unital, with the family (6) consisting of a single object $\{1\}$. More sophisticated examples of thick unital operadic categories are provided by various categories of graphs analyzed in Section 7.

2 Cloven operadic categories

Given a functor $F : \mathtt{E} \to \mathtt{B}$ and objects $S \in \mathtt{E}$, $X \in \mathtt{B}$, we express that $F(S) = X$ by $S \rightsquigarrow X$. This notation will typically be used when F is a cardinality functor of an operadic category. The definition below has been inspired by the notion of a normal splitting cleavage of a Grothendieck fibration. However, we will require no fibration property of F, and the lifts will be required only for isomorphisms.

Definition 4. A *cleavage* for $F : \mathtt{E} \to \mathtt{B}$ is a choice, for each isomorphism $\sigma : X \to Y$ in $\mathtt{B}$ and an object $S \in \mathtt{E}$ with $F(S) = X$, of an object $T \in \mathtt{E}$ and of a *lift* $\widetilde{\sigma} : S \to T$ such that $F(\widetilde{\sigma}) = \sigma$, i.e.

$$\begin{tikzcd} S \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & T \arrow[d, squiggly] \\ X \arrow[r, "\sigma", "\cong"'] & Y. \end{tikzcd} \tag{7}$$

We moreover require the functoriality of the lifts, that is $\widetilde{\sigma''\sigma''} = \widetilde{\sigma''}\widetilde{\sigma'}$ in

$$\begin{tikzcd} S \arrow[r, dashed, "\widetilde{\sigma'}"] \arrow[rr, dashed, bend left, "\widetilde{\sigma''\sigma''}"] \arrow[d, squiggly] & T \arrow[r, dashed, "\widetilde{\sigma''}"] \arrow[d, squiggly] & R \arrow[d, squiggly] \\ X \arrow[r, "\sigma'", "\cong"'] & Y \arrow[r, "\sigma''", "\cong"'] & Z. \end{tikzcd}$$

and that the identities are lifted to the identities. We then say that the functor F is *cloven*.

Remark 5. Since the lifts are required only for isomorphism, their functoriality implies that the chosen lift in (7) is uniquely determined either by its domain, or by its target. We will call the diagrams as in (7) the *lifting squares*.

Definition 6. A thick operadic category $\mathbf{O}$ is *cloven* if the cardinality functor $|\text{-}| : \mathbf{O} \to \mathbf{Fin}$ is cloven by a cleavage compatible with the fiber functor in the following sense. Every choice of isomorphisms $\rho : X \to Y$ and $\sigma : A \to B$ in $\mathbf{Fin}$ together with objects $\boldsymbol{S}, \boldsymbol{T} \in \mathbf{O}$ such that $|\boldsymbol{S}| = X$, $|\boldsymbol{T}| = A$, connected by a morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$, determine the remaining objects in the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & \widetilde{\boldsymbol{S}} \arrow[rr, dashed, "\widetilde{\boldsymbol{f}}"] \arrow[dd, squiggly] & & \widetilde{\boldsymbol{T}} \arrow[dd, squiggly] \\ \boldsymbol{S} \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[rr, "\boldsymbol{f}"] \arrow[dd, squiggly] & & \boldsymbol{T} \arrow[ur, dashed, "\widetilde{\sigma}"'] \arrow[dd, squiggly] & \\ & Y \arrow[rr, dashed, "|\widetilde{\boldsymbol{f}}|"] & & B \\ X \arrow[ur, "\rho", "\cong"'] \arrow[rr, "|\boldsymbol{f}|"] & & A \arrow[ur, "\sigma", "\cong"'] & \end{tikzcd} \tag{8}$$

with commuting upper and bottom squares, and whose left and right faces are lifting squares.

Namely, $\widetilde{\rho} : \boldsymbol{S} \to \widetilde{\boldsymbol{S}}$ is the lift of ρ, $\widetilde{\sigma} : \boldsymbol{T} \to \widetilde{\boldsymbol{T}}$ the lift of σ and $\widetilde{\boldsymbol{f}} := \widetilde{\sigma}\boldsymbol{f}\widetilde{\rho}^{-1}$. For each $a \in A$ and $b := \sigma(a) \in B$ we have the induced isomorphism $\rho_a : |\boldsymbol{f}|^{-1}(a) \to |\widetilde{\boldsymbol{f}}|^{-1}(b)$ between the set-theoretic preimages. We require the existence of the lifting square

$$\begin{tikzcd} \boldsymbol{f}^{-1}(a) \arrow[r, dashed, "\widetilde{\rho_a}"] \arrow[d, squiggly] & \widetilde{\boldsymbol{f}}^{-1}(b) \arrow[d, squiggly] \\ |\boldsymbol{f}|^{-1}(a) \arrow[r, "\rho_a", "\cong"'] & |\widetilde{\boldsymbol{f}}|^{-1}(b) \end{tikzcd} \tag{9}$$

where $\widetilde{\boldsymbol{f}}^{-1}(b)$ is the fiber of $\widetilde{\boldsymbol{f}}$ over $b \in B$.

Non-Example 7. The thick operadic category $\mathbf{1}$ with one object $\bullet$ of cardinality $\{1\}$ is not cloven – there is no lift of the isomorphism σ in the diagram

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & ? \arrow[d, squiggly] \\ \{1\} \arrow[r, "\sigma", "\cong"'] & \{x\} \end{tikzcd}$$

if $x \neq 1$. All examples of non-cloven thick operadic categories we know are of this or similar "synthetic" type.

Example 8. The thick operadic category **Fin**, with the cleavage given by $\widetilde{\sigma} := \sigma$ for any isomorphism in **Fin**, is cloven. This follows from the simple fact that the upper and bottom squares in (8) are the same, so the same are also the upper and bottom horizontal maps in (9). Typical examples of thick cloven operadic categories are provided by the categories of (unlabeled, non-oriented) graphs presented in Section 7.

Definition 9. A standard, "thin" operadic category $\mathtt{O}$ as in [7, Section 1] is *cloven* if the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$ is cloven compatibly with the fiber functor in the sense analogous to that in Definition 6. Explicitly, every automorphisms (permutations) $\rho : \underline{m} \to \underline{m}$, $\sigma : \underline{n} \to \underline{n}$ together with objects $S, T \in \mathtt{O}$ with $|S| = \underline{m}$, $|T| = \underline{n}$, that are connected by a morphism $f : S \to T$, determine the diagram

$$\begin{tikzcd}
& \widetilde{S} \arrow[rr, dashed, "\widetilde{f}"] \arrow[dd, rightsquigarrow] & & \widetilde{T} \arrow[dd, rightsquigarrow] \\
S \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[rr, "f"] \arrow[dd, rightsquigarrow] & & T \arrow[ur, dashed, "\widetilde{\sigma}"'] \arrow[dd, rightsquigarrow] & \\
& \underline{m} \arrow[rr, dashed, "|\widetilde{f}|"] & & \underline{n} \\
\underline{m} \arrow[ur, "\rho", "\cong"'] \arrow[rr, "|f|"] & & \underline{n} \arrow[ur, "\sigma", "\cong"'] &
\end{tikzcd} \tag{10}$$

analogous to (8). We have for, each $i \in \underline{n}$ and $j := \sigma(i)$, the canonical induced isomorphism $\rho_i : |f|^{-1}(i) \to |\widetilde{f}|^{-1}(j)$ between the pullbacks and require the existence of the lifting diagram

$$\begin{tikzcd}
f^{-1}(i) \arrow[r, dashed, "\widetilde{\rho_i}"] \arrow[d, rightsquigarrow] & \widetilde{f}^{-1}(j) \arrow[d, rightsquigarrow] \\
|f|^{-1}(i) \arrow[r, "\rho_i", "\cong"'] & |\widetilde{f}|^{-1}(j)\ .
\end{tikzcd} \tag{11}$$

Example 10. The skeletal category $\mathtt{Fin}$ is cloven, as is the category $\mathtt{Gr}$ of graphs and its modifications listed in diagram (37) of [4]. Operadic categories of graphs are addressed in detail in Section 7 of this article.

Non-Example 11. Recall that a standard, thin operadic category $\mathtt{O}$ is *ordered* if the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{Fin}$ factorizes through the subcategory $\Delta \subset \mathtt{Fin}$ of finite ordinals and their order-preserving maps. Non-unary ordered thin operadic categories, such as Δ itself, serve as generic examples of non-cloven thin operadic categories. Indeed, take $T \in \mathtt{O}$ such that $|T| = \underline{n}$, $n \geq 2$, and an automorphism $\sigma : \underline{n} \to \underline{n}$ which does not preserve order. Its lift $\widetilde{\sigma}$ of must satisfy $|\widetilde{\sigma}| = \sigma$, which is not possible, since σ does not preserve the order of $\underline{n}$. Below we give two examples of non-ordered thin operadic categories which are not cloven.

Non-Example 12. We claim that Batanin's thin operadic category Ω_k of k-trees [7, §1.1] is cloven only if $k = 0$, and ordered only if $k \leq 1$. Indeed, the category Ω_0 is the terminal unary operadic category $\mathtt{1}$, therefore (trivially) ordered and cloven. The category Ω_1 equals Δ, which is ordered, therefore not cloven by non-Example 11.

Let us analyze Ω_2. Its objects are morphisms of $\Delta \subset \mathtt{Fin}$, i.e. order-preserving maps $T : \underline{m} \to \underline{n}$. Morphisms $F : T' \to T''$ of objects of Ω_2 are commutative diagrams in the category of sets

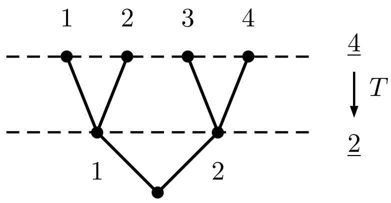
$$\begin{tikzcd}
\underline{m}' \arrow[r, "\omega"] \arrow[d, "T'"'] & \underline{m}'' \arrow[d, "T''"'] \\
\underline{n}' \arrow[r, "\varsigma"] & \underline{n}''
\end{tikzcd} \tag{12}$$

such that

(i) ς is order-preserving and

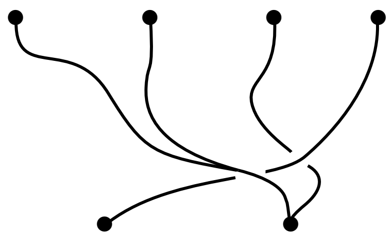
(ii) for each $i \in \underline{n}'$, the restriction of ω to the preimage $T'^{-1}(i)$ is order-preserving.

The cardinality of $T : \underline{m} \to \underline{n} \in \Omega_2$ is $\underline{m} \in \mathtt{Fin}$, and the morphism F represented by (12) is mapped by the cardinality functor to $\omega : \underline{m}' \to \underline{m}''$. The objects of Ω_2 can be viewed as trees with two levels of vertices, whence the name. For instance, the 2-tree $T : \underline{4} \to \underline{2}$ with $T(1) = T(2) := 1$, $T(3) = T(4) := 2$ of cardinality $\underline{4}$ is depicted as



It is easy to see that the automorphism $\sigma : \underline{4} \to \underline{4}$ given by $\sigma(1) := 2$, $\sigma(2) := 1$, $\sigma(3) := 4$ and $\sigma(4) := 3$ does not admit a lift, so Ω_2 is not cloven. To show that Ω_2 is not ordered, consider a 2-tree $S : \underline{2} \to \underline{1}$ and a morphism $F : T \to S$ given by the morphism $\omega : \underline{4} \to \underline{2}$ of leaves with $\omega(1) := 1$, $\omega(2) := 2$, $\omega(3) := 1$ and $\omega(2) := 2$. The map $|F| = \omega$ is not order-preserving. The categories Ω_k, $k \geq 3$, can be discussed similarly.

Non-Example 13. The thin operadic category $\mathtt{Vin}$ of vines has the same objects as $\mathtt{Fin}$ but morphisms $\underline{m} \to \underline{n}$ are isotopy classes of merging descending strings in $\mathbb{R}^3$ as in



cf. [3, Example 1.4]. The cardinality $|\text{-}| : \mathtt{Vin} \to \mathtt{Fin}$ is the identity on objects and maps $i \in \underline{m}$ to the element of $\underline{n}$ connected with i by a string. It follows immediately from the definition that the subcategory of automorphism of objects of cardinality $\underline{n}$ equals the braided group Br_n on n elements. A cleavage thus must be a functorial section of the projection $\mathrm{Br}_n \twoheadrightarrow \Sigma_n$ of Br_n to the symmetric group. Sections exist, but the functoriality cannot be achieved if $n \geq 2$. This example was suggested to me by M. Batanin.

We have seen that there are two issues that can prevent the existence of a cleavage. Either some automorphisms have no lifts as in non-Examples 11 and 12, or the functoriality of lifts cannot be achieved, as in non-Example 13.

3 Cloven unitality

This section introduces the concept of unitality for thick cloven operadic categories. The following definition is to be compared with Definition 2, which recalls the unitality for standard, thin operadic categories.

Definition 14. Assume that a family

$$\big\{\boldsymbol{U}_c \in \mathbf{O} \mid c \in \pi_0(\mathbf{O})\big\} \tag{13}$$

of local terminal objects of a thick operadic category $\mathbf{O}$ is given, with $\boldsymbol{U}_c$ belonging to the connected component c, and such that $|\boldsymbol{U}_c| = \{1\}$ for each c.

The category $\mathbf{O}$ is *left unital* if, for each $\boldsymbol{T} \in \mathbf{O}$ with $|\boldsymbol{T}| = \{1, \ldots, n\}$, the fiber $\mathbb{1}_{\boldsymbol{T}}^{-1}(1)$ of the identity $\mathbb{1}_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}$ equals $\boldsymbol{U}_c$ with some $c \in \pi_0(\mathbf{O})$. The category $\mathbf{O}$ is *right unital* if the fiber functor $\mathrm{Fib}_1 : \mathbf{O}/\boldsymbol{U}_c \to \mathbf{O}$ is the domain functor for each $c \in \pi_0(\mathbf{O})$. The category $\mathbf{O}$ is *unital* if it is both left and right unital.

Let $\mathbf{O}$ be a thick cloven operadic category equipped with a family (13). For each singleton $\{x\}$ define $\boldsymbol{U}_c^{\{x\}}$ as the target of the lift of the unique isomorphism $\{1\} \to \{x\}$ in the lifting square

$$\begin{tikzcd} \boldsymbol{U}_c \arrow[r, dashed] \arrow[d, squiggly] & \boldsymbol{U}_c^{\{x\}} \arrow[d, squiggly] \\ \{1\} \arrow[r, "\cong"] & \{x\}. \end{tikzcd} \tag{14}$$

Since $\boldsymbol{U}_c^{\{x\}}$ is isomorphic to $\boldsymbol{U}_c$, it is local terminal too. We thus have, for each $c \in \pi_0(\mathbf{O})$, an action of the big groupoid $\mathfrak{G}$ of one-point sets on the local terminal objects in the connected component c. Notice that $\boldsymbol{U}_c^{\{1\}} = \boldsymbol{U}_c$

Proposition 15. *Let $\mathbf{O}$ be a thick cloven operadic category. If $\mathbf{O}$ is left unital, the fiber functor $Fib_{\{x\}} : \mathbf{O}/\boldsymbol{U}_c^{\{x\}} \to \mathbf{O}$ is the domain functor for each $c \in \pi_0(\mathbf{O})$ and a singleton $\{x\}$. If $\mathbf{O}$ is right unital, the fiber $\mathbb{1}_{\boldsymbol{S}}^{-1}(x)$ of the identity $\mathbb{1}_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{S}$ over $x \in |\boldsymbol{S}|$ equals $\boldsymbol{U}_{c_x}^{\{x\}}$ with some $c_x \in \pi_0(\mathbf{O})$, for each $\boldsymbol{S} \in \mathbf{O}$.*

More generally, if $\mathbf{O}$ is right unital, $\sigma : X \to Y$ be an isomorphism of finite sets and $\widetilde{\sigma} : \boldsymbol{S} \to \widetilde{\boldsymbol{S}}$ the lift in the lifting square

$$\begin{tikzcd} \boldsymbol{S} \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & \widetilde{\boldsymbol{S}} \arrow[d, squiggly] \\ X \arrow[r, "\sigma", "\cong"'] & Y, \end{tikzcd}$$

then, for each $x \in X$ and $y := \sigma(x)$, one has $\widetilde{\sigma}^{-1}(y) = \mathbb{1}_{\boldsymbol{S}}^{-1}(x) = \boldsymbol{U}_{c_x}^{\{x\}}$ with some $c_x \in \pi_0(\mathbf{O})$.

Proof. Suppose that $\mathbf{O}$ is left unital and let $! : \boldsymbol{S} \to \boldsymbol{U}_c^{\{x\}}$ resp. $! : \boldsymbol{S} \to \boldsymbol{U}_c$ be the unique maps to the appropriate local terminal objects. They determine the upper square of the diagram

$$\begin{tikzcd} & \boldsymbol{S} \arrow[rr, dashed, "!"] \arrow[dd, squiggly] & & \boldsymbol{U}_c^{\{x\}} \arrow[dd, squiggly] \\ \boldsymbol{S} \arrow[ur, dashed, "\mathbb{1}_{\boldsymbol{S}}"] \arrow[rr, "!"] \arrow[dd, squiggly] & & \boldsymbol{U}_c \arrow[ur, dashed] \arrow[dd, squiggly] & \\ & X \arrow[rr, dashed, "!"] & & \{x\} \\ X \arrow[ur, "\mathbb{1}_X"] \arrow[rr, "!"] & & \{1\} \arrow[ur, "!"] & \end{tikzcd} \tag{15}$$

of the obvious maps and their lifts. The left face of (15) is a lifting square since cleavage preserves identities, and the right face is a lifting square by the definition of $\boldsymbol{U}_c^{\{x\}}$. The associated lifting square (9) is

$$\begin{tikzcd} !^{-1}(1) \arrow[r, dashed, "\widetilde{\mathbb{1}_X}"] \arrow[d, squiggly] & !^{-1}(x) \arrow[d, squiggly] \\ X \arrow[r, "\mathbb{1}_X"] & X. \end{tikzcd}$$

Since the lift of the identity automorphism is the identity automorphism, $!^{-1}(1) = !^{-1}(x)$. Moreover, $!^{-1}(1) = \boldsymbol{S}$ by the left unitality. This proves the first part of the proposition.

Assume that $\mathbf{O}$ is right unital. Consider an arbitrary isomorphism of finite sets $\omega : A \to X$.

The lift of the inverse of ω gives the left square of the diagram

$$\begin{tikzcd}
& \boldsymbol{S} \arrow[rr, dashed, "\widetilde{\sigma}"] \arrow[dd, rightsquigarrow] & & \widetilde{\boldsymbol{S}} \arrow[dd, rightsquigarrow] \\
\boldsymbol{T} \arrow[ur, dashed, "\widetilde{\omega}"] \arrow[rr, "\mathbb{1}_{\boldsymbol{T}}"] \arrow[dd, rightsquigarrow] & & \boldsymbol{T} \arrow[ur, dashed, "\widetilde{\sigma\omega}"] \arrow[dd, rightsquigarrow] & \\
& X \arrow[rr, "\sigma"] & & Y \\
A \arrow[ur, "\omega"] \arrow[rr, "\mathbb{1}_A"] & & A\,. \arrow[ur, "\sigma\omega"] &
\end{tikzcd} \tag{16}$$

Given $a \in A$, $x := \omega(a)$ and $y := \sigma(x)$, diagram (16) induces the lifting square

$$\begin{tikzcd}
\mathbb{1}_{\boldsymbol{T}}^{-1}(a) \arrow[r, dashed] \arrow[d, rightsquigarrow] & \widetilde{\sigma}^{-1}(y) \arrow[d, rightsquigarrow] \\
\{a\} \arrow[r, "\cong"] & \{y\}.
\end{tikzcd} \tag{17}$$

Choosing $A := \{1, \ldots, n\}$ and $\omega : \{1, \ldots, n\} \to X$ an arbitrary map such that $\omega(1) = x$, diagram (17) with $a := 1$ gives

$$\begin{tikzcd}
\boldsymbol{U}_c := \mathbb{1}_{\boldsymbol{T}}^{-1}(1) \arrow[r, dashed] \arrow[d, rightsquigarrow] & \widetilde{\sigma}^{-1}(y) \arrow[d, rightsquigarrow] \\
\{1\} \arrow[r, "\cong"] & \{x\},
\end{tikzcd} \tag{18a}$$

so $\widetilde{\sigma}^{-1}(y) = \boldsymbol{U}_c^{\{x\}}$ by (14). Likewise, diagram (16) with $\boldsymbol{T} = \boldsymbol{S}$, $A = X$ and $\omega = \mathbb{1}_X$ induces, for $a = x \in X$ and $y := \sigma(x)$,

$$\begin{tikzcd}
\mathbb{1}_{\widetilde{\boldsymbol{S}}}^{-1}(x) \arrow[r, dashed] \arrow[d, rightsquigarrow] & \widetilde{\sigma}^{-1}(y) \arrow[d, rightsquigarrow] \\
\{x\} \arrow[r, "\mathbb{1}"] & \{x\}.
\end{tikzcd} \tag{18b}$$

Diagrams 18a and (18b) together imply the rest of the proposition. □

Recall an important class of morphisms of a standard thin unital operadic category, formed by *quasibijections*, which are morphisms whose all fibers are the chosen local terminal objects [3, Section 2]. A similar definition makes sense also in the thick case, with the role of the chosen local terminal objects played by $\boldsymbol{U}_c^{\{x\}}$ for $c \in \pi_0(\mathbf{O})$ and singletons $\{x\}$. Proposition 15 has an obvious

Corollary 16. *Lifts of isomorphisms in a cloven unital operadic category are quasibijections.*

4 Operads

Operadic categories which we have discussed so far form the initial stage of the triad

$$\boxed{\text{operadic categories}} \Longrightarrow \boxed{\text{operads}} \Longrightarrow \boxed{\text{algebras}} \tag{19}$$

in which "$A \Longrightarrow B$" must be read as "A governs B." In this section we discuss the next one.

We start by notation that will simplify future exposition. Let $\mathbf{O}$ be a cloven thick operadic category and $\boldsymbol{\mathcal{E}} = \{\boldsymbol{\mathcal{E}}(\boldsymbol{T})\}_{\boldsymbol{T} \in \mathbf{O}}$ a collection of objects of a symmetric monoidal category $\mathtt{V}$. We say that $\boldsymbol{\mathcal{E}}$ is a *cloven module* if the lifts $\widetilde{\sigma} : \boldsymbol{S} \to \boldsymbol{T}$ of isomorphisms $\sigma : X \to Y$ in the lifting square

$$\begin{tikzcd}
\boldsymbol{S} \arrow[r, dashed, "\widetilde{\sigma}", "\cong"'] \arrow[d, rightsquigarrow] & \boldsymbol{T} \arrow[d, rightsquigarrow] \\
X \arrow[r, "\sigma", "\cong"'] & Y
\end{tikzcd} \tag{20}$$

act contravariantly and functorially on $\mathcal{E}$ via isomorphisms $\mathcal{E}(\sigma) : \mathcal{E}(\boldsymbol{T}) \xrightarrow{\cong} \mathcal{E}(\boldsymbol{S})$.

For a morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ with fibers $\{\boldsymbol{F}_a\}_{a\in|\boldsymbol{T}|}$, we denote by $\mathcal{E}(\boldsymbol{f})$ the "unordered tensor product" $\bigotimes_{a\in|\boldsymbol{T}|} \mathcal{E}(\boldsymbol{F}_a)$ given by the colimit in [10, Definition I.1.58]. In the situation of diagram (8) we have a natural isomorphism

$$\mathcal{E}(\rho) := \bigotimes_{a\in A} \mathcal{E}(\rho_a) : \mathcal{E}(\widetilde{\boldsymbol{f}}) \xrightarrow{\cong} \mathcal{E}(\boldsymbol{f}), \tag{21}$$

where $\rho_a : |\boldsymbol{f}|^{-1}(a) \to |\widetilde{\boldsymbol{f}}|^{-1}(\sigma(a))$ is the induced isomorphism between the set-theoretic preimages. Regarding notation, we hope that there will be no confusion between $\mathcal{E}(\sigma) : \mathcal{E}(\boldsymbol{T}) \to \mathcal{E}(\boldsymbol{S})$ and the morphism in (21); the meaning will always be clear from the context.

Definition 17. Let $\mathbf{O}$ be a thick cloven operadic category as in Definition 6. A (nonunital) *operad* is a cloven module $\mathcal{P} = \{\mathcal{P}(\boldsymbol{T})\}_{\boldsymbol{T}\in\mathbf{O}}$ with the composition law

$$\boldsymbol{\mu_f} : \mathcal{P}(\boldsymbol{T}) \otimes \mathcal{P}(\boldsymbol{f}) \longrightarrow \mathcal{P}(\boldsymbol{S}) \tag{22}$$

specified for each morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ and subject to the following properties.

(i) Associativity. Let $\boldsymbol{T} \xrightarrow{\boldsymbol{f}} \boldsymbol{S} \xrightarrow{\boldsymbol{g}} \boldsymbol{R}$ be morphisms in $\mathbf{O}$, and $\boldsymbol{h} := \boldsymbol{g}\boldsymbol{f} : \boldsymbol{T} \to \boldsymbol{R}$ as in (4b). Then the following diagram of composition laws of $\mathcal{P}$ combined with the canonical isomorphisms of iterated products in $\mathtt{V}$ commutes:

$$\begin{tikzcd}
\bigotimes_{r\in|\boldsymbol{R}|} \mathcal{P}(\boldsymbol{R})\otimes\mathcal{P}(\boldsymbol{g})\otimes\mathcal{P}(\boldsymbol{f}_r) \arrow[rr, bend left=10, "\mathbb{1}\,\otimes \bigotimes_r \boldsymbol{\mu}_{\boldsymbol{f}_r}"] \arrow[d, "\boldsymbol{\mu_g}\otimes\mathbb{1}"'] & & \mathcal{P}(\boldsymbol{R})\otimes\mathcal{P}(\boldsymbol{h}) \arrow[dl, bend left=10, "\boldsymbol{\mu_h}"'] \\
\mathcal{P}(\boldsymbol{S})\otimes \bigotimes_{r\in|\boldsymbol{R}|} \mathcal{P}(\boldsymbol{f}_r) \cong \mathcal{P}(\boldsymbol{S})\otimes\mathcal{P}(\boldsymbol{f}) \arrow[r, "\boldsymbol{\mu_f}"] & \mathcal{P}(\boldsymbol{T}) &
\end{tikzcd} \tag{23a}$$

The isomorphism $\bigotimes_{r\in|\boldsymbol{R}|} \mathcal{P}(\boldsymbol{f}_r) \cong \mathcal{P}(\boldsymbol{f})$ used in the bottom line holds by (4c).

(ii) Compatibility with the action. In the situation of (8), we require the commutativity of

$$\begin{tikzcd}
\mathcal{P}(\boldsymbol{T}) \otimes \mathcal{P}(\boldsymbol{f}) \arrow[r, "\boldsymbol{\mu_f}"] & \mathcal{P}(\boldsymbol{S}) \arrow[d, "\mathcal{P}(\rho)"] \\
\mathcal{P}(\widetilde{\boldsymbol{T}})\otimes\mathcal{P}(\widetilde{\boldsymbol{f}}) \arrow[u, "\mathcal{P}(\sigma)\,\otimes\,\mathcal{P}(\rho)"] \arrow[r, "\boldsymbol{\mu}_{\widetilde{\boldsymbol{f}}}"] & \mathcal{P}(\widetilde{\boldsymbol{S}}).
\end{tikzcd} \tag{23b}$$

The following definition assumes that $\mathbf{O}$ is a cloven thick unital operadic category with collection (13) of chosen local terminal objects. Each family of morphisms

$$\{\eta_c : \mathbb{1}_{\mathtt{V}} \longrightarrow \mathcal{P}(\boldsymbol{U}_c) \mid c \in \pi_0(\mathbf{O})\}, \tag{24a}$$

where $\mathbb{1}_{\mathtt{V}}$ is the monoidal unit of $\mathtt{V}$, generates the morphisms

$$\boldsymbol{\eta}_c^{\{x\}} : \mathbb{1}_{\mathtt{V}} \xrightarrow{\eta_c} \mathcal{P}(\boldsymbol{U}_c) \xrightarrow{\cong} \mathcal{P}(\boldsymbol{U}_c^{\{x\}}) \tag{24b}$$

specified for each $c \in \pi_0(\mathbf{O})$ and a singleton $\{x\}$; the isomorphism $\mathcal{P}(\boldsymbol{U}_c) \xrightarrow{\cong} \mathcal{P}(\boldsymbol{U}_c^{\{x\}})$ is given by the cloven module action of $\boldsymbol{U}_c \to \boldsymbol{U}_c^x$ in (14). Notice that $\boldsymbol{\eta}_c^{\{x\}} = \eta_c$.

Definition 18. An operad $\mathcal{P}$ as in Definition 17 is *unital* if it is equipped with morphisms (24a) and the following conditions are satisfied for each $\boldsymbol{T} \in \mathbf{O}$.

(i) The diagram

$$\begin{tikzcd}
\mathcal{P}(\boldsymbol{U}_c)\otimes\mathcal{P}(\boldsymbol{T}) \arrow[r, "\boldsymbol{\mu}_!"] & \mathcal{P}(\boldsymbol{T}) \arrow[d, equal] \\
\mathbb{1}_{\mathtt{V}} \otimes\mathcal{P}(\boldsymbol{T}) \arrow[u, "\eta_c\otimes\mathbb{1}"] \arrow[r, "\cong"] & \mathcal{P}(\boldsymbol{T}),
\end{tikzcd}$$

in which $\boldsymbol{\mu}_!$ is the operation associated to the unique morphism $! : \boldsymbol{T} \to \boldsymbol{U}_c$, commutes.

(ii) The diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{T}) \otimes \bigotimes_{c_x} \mathcal{P}(\boldsymbol{U}_{c_x}^{\{x\}}) \arrow[r, "\boldsymbol{\mu}_{\mathbb{1}}"] & \mathcal{P}(\boldsymbol{T}) \arrow[d, equal] \\ \mathcal{P}(\boldsymbol{T}) \otimes \mathbb{I}_{\mathtt{V}}^{\otimes |\boldsymbol{T}|} \arrow[u, "\mathbb{1} \otimes \bigotimes_{c_x} \boldsymbol{\eta}_{c_x}^{\{x\}}"] \arrow[r, "\cong"] & \mathcal{P}(\boldsymbol{T}), \end{tikzcd} \tag{25}$$

where $\boldsymbol{U}_{c_x}^{\{x\}}$ is the fiber of $\mathbb{1}_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}$ over x, and x runs over the set $|\boldsymbol{T}|$, commutes.

It follows from the compatibility with the cloven module action required in (ii) of Definition 17 that it suffices to verify the properties (i) and (ii) only for $\boldsymbol{T} \in \mathbf{O}$ with $|\boldsymbol{T}| = \{1, \ldots, n\}$, $n \geq 1$. The compatibility also implies that a stronger version of the diagram in (i) of Definition 18 holds, namely

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{U}_c^{\{x\}}) \otimes \mathcal{P}(\boldsymbol{T}) \arrow[r, "\boldsymbol{\mu}_!"] & \mathcal{P}(\boldsymbol{T}) \arrow[d, equal] \\ \mathbb{I}_{\mathtt{V}} \otimes \mathcal{P}(\boldsymbol{T}) \arrow[u, "\boldsymbol{\eta}_c^{\{x\}} \otimes \mathbb{1}"] \arrow[r, "\cong"] & \mathcal{P}(\boldsymbol{T}), \end{tikzcd} \tag{26}$$

where $! : \boldsymbol{T} \to \boldsymbol{U}_c^{\{x\}}$ is the unique morphism to a local terminal object. To verify the commutativity of (26), notice that (ii) of Definition 17 applied to

$$\begin{tikzcd} & \boldsymbol{T} \arrow[rr, dashed, "!"] \arrow[dd, squiggly] & & \boldsymbol{U}_c^{\{x\}} \arrow[dd, squiggly] \\ \boldsymbol{T} \arrow[ur, dashed, "\mathbb{1}_{\boldsymbol{T}}"] \arrow[rr, "!"] \arrow[dd, squiggly] & & \boldsymbol{U}_c \arrow[ur, dashed, "!"] \arrow[dd, squiggly] & \\ & |\boldsymbol{T}| \arrow[rr, dashed, "!"] & & \{x\}. \\ |\boldsymbol{T}| \arrow[ur, "\mathbb{1}_{|\boldsymbol{T}|}"] \arrow[rr, "!"] & & \{1\} \arrow[ur, "!"] & \end{tikzcd}$$

implies the commutativity of the square in

$$\begin{tikzcd} & \mathcal{P}(\boldsymbol{U}_c) \otimes \mathcal{P}(\boldsymbol{T}) \arrow[r, "\boldsymbol{\mu}_!"] & \mathcal{P}(\boldsymbol{T}) \arrow[dd, equal] \\ \mathbb{I}_{\mathtt{V}} \otimes \mathcal{P}(\boldsymbol{T}) \arrow[ur, bend left, "\eta_c \otimes \mathbb{1}"] \arrow[dr, bend right, "\boldsymbol{\eta}_c^{\{x\}} \otimes \mathbb{1}"'] & & \\ & \mathcal{P}(\boldsymbol{U}_c^{\{x\}}) \otimes \mathcal{P}(\boldsymbol{T}) \arrow[uu, "\mathcal{P}(!) \otimes \mathbb{1}"'] \arrow[r, "\boldsymbol{\mu}_!"] & \mathcal{P}(\boldsymbol{T}), \end{tikzcd}$$

while the triangle is commutative by the definition of $\boldsymbol{\eta}_c^{\{x\}}$.

It can be proven that the cloven module action of a unital operad $\mathcal{P}$ over a thick unital operadic category $\mathbf{O}$ is related with family (24b) via the diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{T}) \otimes \bigotimes_{x \in X} \mathcal{P}(\boldsymbol{U}_{c_x}^{\{x\}}) \arrow[r, "\boldsymbol{\mu}_{\widetilde{\sigma}}"] & \mathcal{P}(\boldsymbol{S}) \\ \mathcal{P}(\boldsymbol{T}) \otimes \bigotimes_{x \in X} \mathbb{I}_{\mathtt{V}} \arrow[u, "\mathbb{1} \otimes \bigotimes \boldsymbol{\eta}_{c_x}^{\{x\}}"] \arrow[r, "\cong"] & \mathcal{P}(\boldsymbol{T}), \arrow[u, "\mathcal{P}(\sigma)"'] \end{tikzcd} \tag{27}$$

where $\widetilde{\sigma} : \boldsymbol{S} \to \boldsymbol{T}$ is as in (20), $\boldsymbol{U}_{c_x}^{\{x\}}$ is the fiber of $\mathbb{1}_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{S}$ over x, which equals the fiber $\widetilde{\sigma}^{-1}(y)$ over $y := \sigma(x)$ by the second part of Proposition 15, and where x runs over X.

Morphisms (24b) were generated from the unit maps (24a) using the cloven module structure of $\mathcal{P}$. The following proposition shows that, vice versa, the module structure of $\mathcal{P}$ is determined by a family

$$\boldsymbol{\eta}_c^{\{x\}} : \mathbb{I}_{\mathtt{V}} \longrightarrow \mathcal{P}(\boldsymbol{U}_c^{\{x\}}), \ c \in \pi_0(\mathbf{O}), \ \{x\} \text{ a sindleton,} \tag{28}$$

of morphisms as in (24b), suitably compatible with the structure operations of $\mathcal{P}$.

Proposition 19. *Let* $\mathbf{O}$ *be a thick unital operadic category and* $\boldsymbol{\mathcal{P}} = \{\boldsymbol{\mathcal{P}}(\boldsymbol{T})\}_{\boldsymbol{T}\in\mathbf{O}}$ *a collection with composition laws* (22) *which are associative as required in (i) of Definition 17. Assume moreover that* $\boldsymbol{\mathcal{P}}$ *comes with a family* (28) *satisfying (25) and (26). Diagrams (27) then define a cloven module action that satisfies (ii) of Definition (17).*

As we will not need the above proposition, we omit its technical, but straightforward, proof.

Remark 20. If we take the hypotheses of Proposition 19 as the definition of unitality, unital operads over thick operadic categories are automatically equipped with a cloven module action that satisfies (ii) of Definition 18.

The thin analog of a cloven module over a thin operadic category $\mathtt{O}$ obvious. It is a collection $\mathcal{E} = \{\mathcal{E}(T)\}_{T\in\mathtt{O}}$ such that the lifts $\widetilde{\sigma} : S \to T$ in the lifting square

$$\begin{tikzcd} S \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & T \arrow[d, squiggly] \\ \underline{n} \arrow[r, "\sigma", "\cong"'] & \underline{n} \end{tikzcd} \tag{29}$$

act contravariantly and functorially on $\mathcal{E}$ via isomorphisms $\mathcal{E}(\sigma) : \mathcal{E}(T) \xrightarrow{\cong} \mathcal{E}(S)$. An analog of the notation in (21) is also clear. We are ready for

Definition 21. A (nonunital) *operad* over a thin cloven operadic category $\mathtt{O}$ is a cloven module $\mathcal{P} = \{\mathcal{P}(T)\}_{T\in\mathtt{O}}$ with the composition law

$$\mu_f : \mathcal{P}(T) \otimes \mathcal{P}(f) \longrightarrow \mathcal{P}(S)$$

specified for each morphism $f : S \to T$. We moreover demand the following two properties.

(i) Associativity. The diagram in (i) of [7, Definition 1.11], which was the blueprint for its thick version (23a), commutes for each chain $T \xrightarrow{f} S \xrightarrow{g} R$ of maps in $\mathtt{O}$.

(ii) Compatibility with the action. In the situation of (10), we require the commutativity of

$$\begin{tikzcd} \mathcal{P}(T) \otimes \mathcal{P}(f) \arrow[r, "\mu_f"] & \mathcal{P}(S) \arrow[d, "\mathcal{P}(\rho)"] \\ \mathcal{P}(\widetilde{T}) \otimes \boldsymbol{\mathcal{P}}(\widetilde{f}) \arrow[u, "\mathcal{P}(\sigma) \otimes \mathcal{P}(\rho)"] \arrow[r, "\mu_{\widetilde{f}}"] & \mathcal{P}(\widetilde{S}). \end{tikzcd}$$

Suppose that $\mathtt{O}$ is unital, and the operad $\mathcal{P}$ is unital in the standard sense, that is, there are unit maps $\eta_c : \mathbb{1}_{\mathtt{V}} \to \mathcal{P}(U_c)$, $c \in \pi_0(\mathtt{O})$, satisfying items (ii) and (iii) of [7, Definition 1.11]. It can be easily shown that the cloven module action is related with the unit maps of $\mathcal{P}$ by the diagram

$$\begin{tikzcd} \mathcal{P}(T) \otimes \bigotimes_{i\in|S|} \mathcal{P}(U_{c_i}) \arrow[r, "\mu_{\widetilde{\sigma}}"] & \mathcal{P}(S) \\ \mathcal{P}(T) \otimes \bigotimes_{i\in|S|} \mathbb{1}_{\mathtt{V}} \arrow[u, "\mathbb{1} \otimes \bigotimes \eta_{c_i}"] \arrow[r, "\cong"] & \mathcal{P}(T), \arrow[u, "\mathcal{P}(\sigma)"'] \end{tikzcd} \tag{30}$$

where $\widetilde{\sigma} : S \to T$ is as in (29) and U_{c_i} is the fiber of $\mathbb{1}_S : S \to S$ over i. Diagram 30 can be used as a definition of the cloven module action. The following proposition with straightforward proof has to be compared with Remark 20.

Proposition 22. *Unital* $\mathtt{O}$*-operads in the sense of the original definition [7, Definition 1.11] are automatically* $\mathtt{O}$*-operads in Definition 21.*

5 Algebras

This section is devoted to the lowest, terminal level of the triad (19). In the "standard" definition of operads [7, Definition 1.20], algebras appear as collections indexed by the connected components of the background operadic category. They are subject to an associativity axiom, which uses the notions of the source and the target of an object. We therefore open this section with a thick version of these notions.

We define the *source* $\mathbf{s}(\boldsymbol{T})$ of an object $\boldsymbol{T} \in \mathbf{O}$ to be the list of the connected components of the fibers of the identity $\mathbb{1}_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}$. The *target* $\mathbf{t}(\boldsymbol{T}) \in \pi_0(\mathbf{O})$ is the connected component of $\boldsymbol{T}$.

Lemma 23. *Let $\boldsymbol{S}, \boldsymbol{T}$ be objects of $\mathbf{O}$. The lift $\widetilde{\sigma} : \boldsymbol{S} \to \boldsymbol{T}$ in the lifting square (20) acts functorially on the sources by an isomorphism $\mathbf{s}(\sigma) : \mathbf{s}(\boldsymbol{S}) \to \mathbf{s}(\boldsymbol{T})$.*

Proof. Embed $\widetilde{\sigma} : \boldsymbol{S} \to \boldsymbol{T}$ to the diagram

$$
\begin{tikzcd}
& \boldsymbol{T} \arrow[rr, dashed, "\mathbb{1}_{\boldsymbol{T}}"] \arrow[dd, squiggly] & & \boldsymbol{T} \arrow[dd, squiggly] \\
\boldsymbol{S} \arrow[ur, dashed, "\widetilde{\sigma}"] \arrow[rr, "\mathbb{1}_{\boldsymbol{S}}"] \arrow[dd, squiggly] & & \boldsymbol{S} \arrow[ur, dashed, "\widetilde{\sigma}"] \arrow[dd, squiggly] & \\
& Y \arrow[rr, dashed, "\mathbb{1}_Y"] & & Y. \\
X \arrow[ur, "\sigma", "\cong"'] \arrow[rr, "\mathbb{1}_X"] & & X \arrow[ur, "\sigma", "\cong"'] &
\end{tikzcd}
$$

The associated lifting squares (9) provide the morphism $\widetilde{\sigma_x} : \mathbb{1}_{\boldsymbol{S}}^{-1}(x) \to \mathbb{1}_{\boldsymbol{T}}^{-1}(\sigma(x))$, for each $x \in X$. Therefore the fiber of $\mathbb{1}_{\boldsymbol{S}}$ over x belongs to the same component of $\mathbf{O}$ as the fiber of $\mathbb{1}_{\boldsymbol{T}}$ over $\sigma(x)$. The assignment

$$
\mathbf{s}(\sigma) : \text{the component of } \ \mathbb{1}_{\boldsymbol{S}}^{-1}(x) \longmapsto \text{the component of } \ \mathbb{1}_{\boldsymbol{T}}^{-1}(\sigma(x)), \ x \in X, \tag{31}
$$

defines the requisite action $\mathbf{s}(\sigma) : \mathbf{s}(\boldsymbol{S}) \to \mathbf{s}(\boldsymbol{T})$. $\square$

Given a collection $\mathbf{A} = \{\mathbf{A}_c \in \mathtt{V} \mid c \in \pi_0(\mathbf{O})\}$ indexed by the set of connected components of $\mathbf{O}$, we denote for $\boldsymbol{T} \in \mathbf{O}$

$$
\mathbf{A}_{\mathbf{s}(\boldsymbol{T})} := \bigotimes_{c \in \mathbf{s}(\boldsymbol{T})} \mathbf{A}_c.
$$

Definition 24. An *algebra* for an operad $\mathcal{P}$ is a collection $\mathbf{A} = \{\mathbf{A}_c\}_{c \in \pi_0(\mathbf{O})}$ of objects of $\mathtt{V}$, along with the actions

$$
\boldsymbol{\alpha}_{\boldsymbol{T}} : \mathcal{P}(\boldsymbol{T}) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{T})} \longrightarrow \mathbf{A}_{\mathbf{t}(\boldsymbol{T})}, \ \boldsymbol{T} \in \mathbf{O}.
$$

We assume the following three properties.

(i) Associativity. For each morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ with fibers $\boldsymbol{F}_a$, $a \in |\boldsymbol{T}|$, we require the commutativity of

$$
\begin{tikzcd}
\mathcal{P}(\boldsymbol{T}) \otimes \bigotimes_a \big(\mathcal{P}(\boldsymbol{F}_a) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{F}_a)}\big) \arrow[r, "\mathbb{1} \otimes \bigotimes_a \boldsymbol{\alpha}_{\boldsymbol{F}_a}"] \arrow[d, "\cong"'] & \mathcal{P}(\boldsymbol{T}) \otimes \bigotimes_a \mathbf{A}_{\mathbf{t}(\boldsymbol{F}_a)} \arrow[r, "\boxed{1}", "\cong"'] & \mathcal{P}(T) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{T})} \arrow[d, "\boldsymbol{\alpha}_{\boldsymbol{T}}"] \\
\mathcal{P}(\boldsymbol{T}) \otimes \mathcal{P}(\boldsymbol{f}) \otimes \bigotimes_a \mathbf{A}_{\mathbf{s}(\boldsymbol{F}_a)} \arrow[d, equal, "\boldsymbol{\mu}_{\boldsymbol{f}} \otimes \mathbb{1}^{\otimes |\boldsymbol{T}|}"'] & & \mathbf{A}_{\mathbf{t}(\boldsymbol{T})} \arrow[d, equal, "\boxed{2}"] \\
\mathcal{P}(\boldsymbol{S}) \otimes \bigotimes_a \mathbf{A}_{\mathbf{s}(\boldsymbol{F}_a)} \arrow[r, "\boxed{3}", "\cong"'] & \mathcal{P}(\boldsymbol{S}) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{S})} \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{S}}"] & \mathbf{A}_{\mathbf{t}(\boldsymbol{S})}.
\end{tikzcd}
\tag{32a}
$$

In the above diagram, $\boxed{1}$ uses the isomorphism between the connected components of the fibers $\boldsymbol{F}_a$ and the source of $\boldsymbol{T}$ implied by the existence of the induced maps $\boldsymbol{f}_a : \boldsymbol{F}_a \to \mathbb{1}_{\boldsymbol{T}}^{-1}(a)$, $a \in |\boldsymbol{T}|$, guaranteed by (i) of Definition 1 applied to

$$
\begin{tikzcd}
\boldsymbol{S} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{f}"'] & & \boldsymbol{T} \arrow[dl, "\mathbb{1}_{\boldsymbol{T}}"] \\
& \boldsymbol{T}. &
\end{tikzcd}
$$

Since $\boldsymbol{S}$ and $\boldsymbol{T}$ are connected by $\boldsymbol{f}$, they belong to the same component of $\mathbf{O}$, thus $\mathbf{t}(\boldsymbol{S}) = \mathbf{t}(\boldsymbol{T})$, which explains $\boxed{2}$. By (ii) of Definition 1 applied to

$$\begin{tikzcd} \boldsymbol{S} \arrow[rr, "\mathbb{1}_{\boldsymbol{S}}"] \arrow[dr, "\boldsymbol{f}"'] & & \boldsymbol{S} \arrow[dl, "\boldsymbol{f}"] \\ & \boldsymbol{T}, & \end{tikzcd}$$

the set of fibers of $\mathbb{1}_{\boldsymbol{F}a}$, $a \in |\boldsymbol{T}|$, is the same as the set of fibers of $\mathbb{1}_{\boldsymbol{S}}$, which explains $\boxed{3}$.

(ii) Compatibility with the cloven action. Given the lifting square (20), the diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{S}) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{S})} \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{S}}"] & \mathbf{A}_{\mathbf{t}(\boldsymbol{S})} \arrow[d, equal] \\ \mathcal{P}(\boldsymbol{T}) \otimes \mathbf{A}_{\mathbf{s}(\boldsymbol{T})} \arrow[u, "\mathcal{P}(\sigma)\otimes \mathbf{A}_{\mathbf{s}(\sigma)}", "\cong"'] \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{T}}"] & \mathbf{A}_{\mathbf{t}(\boldsymbol{T})}, \end{tikzcd} \tag{32b}$$

where $\mathbf{A}_{\mathbf{s}(\sigma)} : \mathbf{A}_{\mathbf{s}(\boldsymbol{S})} \xrightarrow{\cong} \mathbf{A}_{\mathbf{s}(\boldsymbol{T})}$ is the isomorphism given by the permutation of the factors according to $\mathbf{s}(\sigma)$, commutes.

(iii) If the operad $\mathcal{P}$ is unital, then we moreover assume that, for each $c \in \pi_0(\mathbf{O})$, the diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{U}_c) \otimes \mathbf{A}_c \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{U}_c}"] & \mathbf{A}_c \arrow[d, equal] \\ \mathbb{1}_{\mathtt{V}} \otimes \mathbf{A}_c \arrow[u, "\eta_c \otimes \mathbb{1}"] \arrow[r, "\cong"] & \mathbf{A}_c \end{tikzcd} \tag{32c}$$

commutes.

In the following proposition, $\mathcal{P}$ will be an unital operad over a thick cloven unital operadic category $\mathbf{O}$, and η_c will be as in (24a).

Proposition 25. *If $\mathcal{P}$ is unital, then (ii) of Definition 24 holds automatically, and the following generalization*

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{U}_c^{\{x\}}) \otimes \mathbf{A}_c \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{U}_c^{\{x\}}}"] & \mathbf{A}_c \arrow[d, equal] \\ \mathbb{1}_{\mathtt{V}} \otimes \mathbf{A}_c \arrow[u, "\boldsymbol{\eta}_c^{\{x\}} \otimes \mathbb{1}"] \arrow[r, "\cong"] & \mathbf{A}_c \end{tikzcd}$$

of (32c), where $\boldsymbol{\eta}_c^{\{x\}}$ is as in (24b), commutes.

Proof. The first part is a consequence of the relation between the cloven module action on $\mathcal{P}$ and its unital structure expressed by the commutativity of (27), The second part follows from the definition (24b) of $\boldsymbol{\eta}_c^{\{x\}}$ and the commutativity of the diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{U}_c^{\{x\}}) \otimes \mathbf{A}_c \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{U}_c^{\{x\}}}"] & \mathbf{A}_c \arrow[d, equal] \\ \mathcal{P}(\boldsymbol{U}_c) \otimes \mathbf{A}_c \arrow[u, "\cong"] \arrow[r, "\boldsymbol{\alpha}_{\boldsymbol{U}_c}"] & \mathbf{A}_c \end{tikzcd}$$

implied by (ii) of Definition 24. □

Let us briefly address algebras in the context of a thin cloven operadic background category $\mathtt{O}$. For $T \in \mathtt{O}$ denote by $s(T)$ the list of connected components of the fibers of $\mathbb{1}_T : T \to T$, and by $t(T)$ the connected component of T. As in Lemma 23 we notice that the lifts $\widetilde{\sigma}$ in (29) act functorially by isomorphisms $s(\sigma) : s(S) \to s(T)$ of the sources.

Definition 26. An *algebra* for an $\mathtt{O}$-operad $\mathcal{P}$ is a collection $A = \{A_c\}_{c \in \pi_0(\mathtt{O})}$, equipped with the actions

$$\alpha_T : \mathcal{P}(T) \otimes A_{s(T)} \longrightarrow A_{t(T)},\ T \in \mathtt{O},$$

which satisfy the following three axioms.

(i) Associativity. A thin analog of diagram (32a), which actually appeared in Definition 48 of [2], commutes for each morphism $S \to T$.

(ii) Compatibility with the cloven action. An obvious thin analog of diagram (32b), namely

$$\begin{tikzcd}
\mathcal{P}(S) \otimes A_{s(S)} \arrow[r, "\alpha_S"] & A_{t(S)} \arrow[d, equal] \\
\mathcal{P}(T) \otimes A_{s(T)} \arrow[u, "\mathcal{P}(\sigma) \otimes A_{s(\sigma)}", "\cong"'] \arrow[r, "\alpha_T"] & A_{t(T)},
\end{tikzcd}$$

commutes for each lifting square (29).

(iii) If $\mathcal{P}$ is unital, we assume that the diagram

$$\begin{tikzcd}
\mathcal{P}(U_c) \otimes A_c \arrow[r, "\alpha_{U_c}"] & A_c \arrow[d, equal] \\
\mathbb{1}_{\mathtt{V}} \otimes A_c \arrow[u, "\eta_c \otimes \mathbb{1}"] \arrow[r, "\cong"] & A_c
\end{tikzcd}$$

commutes for each $c \in \pi_0(\mathtt{O})$,.

As in the thick case, the compatibility (ii) is automatic if $\mathcal{P}$ is unital. A $\mathcal{P}$-algebra is then the same as a morphism from $\mathcal{P}$ to a suitable colored endomorphism operad, so our definition agrees with Definition 1.12 of [7].

6 The equivalences

Operadic functors were introduced, in the context of the standard, thin operadic categories, in [7, page 1635]. Among other properties, operadic functors were required to commute with the cardinality functors, to send fiber to fibers, and to send the chosen local terminal objects to the chosen local terminal objects in the unital case. A modification to the cloven case is given in

Definition 27. An operadic functor $F : \mathtt{O}' \to \mathtt{O}''$ between thin cloven operadic categories is *cloven* if the images

$$\begin{tikzcd}
F(S) \arrow[r, dashed, "F(\widetilde{\sigma})"] \arrow[d, squiggly] & F(T) \arrow[d, squiggly] \\
\underline{n} \arrow[r, "\sigma", "\cong"'] & \underline{n}
\end{tikzcd} \tag{33a}$$

of the lifting squares

$$\begin{tikzcd}
S \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & T \arrow[d, squiggly] \\
\underline{n} \arrow[r, "\sigma", "\cong"'] & \underline{n}
\end{tikzcd} \tag{33b}$$

in $\mathtt{O}'$ are lifting squares in $\mathtt{O}''$. Operadic functors between thick operadic categories and their cloven versions are defined analogously.

Notice that, since F commutes with the cardinality functor, $|F(\widetilde{\sigma})| = |\widetilde{\sigma}| = \sigma$, the diagram in (33a) is indeed the F-image of (33b).

Let us denote by $\mathtt{COpCat}$ the category of thin cloven operadic categories and their cloven operadic functors, and by **COpCat** its thick version. The aim of this section is to establish the central result of this article, namely

Theorem 28. *There exists a natural correspondence* $\mathbf{E} : \mathtt{O} \leftrightarrow \mathbb{O} : \mathrm{R}$ *(extension and restriction) which extends to an equivalence of the categories* $\mathtt{COpCat}$ *and* $\mathbb{COpCat}$*. The categories of operads and their algebras over the corresponding operadic categories are naturally equivalent.*

The thin operadic category $\mathtt{O}$ *is unital if and only if the corresponding thick operadic category* $\mathbb{O}$ *is unital. If this is the case, then the above equivalence restricts to an equivalence of the categories of unital operads and their algebras.*

We denote, as before, by $\mathbb{Fin}$ the category of finite sets and by $\mathtt{Fin}$ its skeletal subcategory of finite ordered sets $\underline{n} := \{1, \ldots, n\}$, $n \geq 0$. The blackboard $\mathbb{O}$ will serve as the generic name of a thick cloven operadic category, the typewriter $\mathtt{O}$ the generic name of a thin cloven one.

The rest of this section is devoted to the proof of Theorem 28. First we define the restriction and the extension. Then, in Proposition 29 we prove that they are inverse to each other and in Proposition 32 we investigate the unitality. Proposition 34 establishes the equivalence of the associated categories of operads and Proposition 36 addresses algebras.

The restriction. The "thin" operadic category $\mathrm{R}\,\mathbb{O}$ is the full subcategory of $\mathbb{O}$ consisting of objects whose cardinalities are finite ordered sets $\underline{n} = \{1, \ldots, n\} \in \mathtt{Fin}$. Let $\boldsymbol{f} : S \to T$ be a morphism in $\mathbb{O}$, $|S| = \underline{m}$, $|T| = \underline{n}$, and $f : S \to T$ be the same morphism considered as a morphism in $\mathrm{R}\,\mathbb{O}$.

Given $i \in \underline{n}$, the ith fiber $f^{-1}(i)$ of f in $\mathrm{R}\,\mathbb{O}$ is constructed from the ith fiber $\boldsymbol{f}^{-1}(i)$ of f in $\mathbb{O}$ as follows. The cardinality $|\boldsymbol{f}^{-1}(i)|$ is a subset of $\underline{m}$. Let $\sigma_i : |\boldsymbol{f}^{-1}(i)| \xrightarrow{\cong} \underline{k_i}$ be the unique order-preserving isomorphism to some finite ordinal $\underline{k_i} \in \mathtt{Fin}$. Then the fiber $f^{-1}(i)$ is defined as the target of the lift $\widetilde{\sigma}_i$ of σ_i in the lifting square

$$\begin{tikzcd} \boldsymbol{f}^{-1}(i) \arrow[r, dashed, "\widetilde{\sigma}_i"] \arrow[d, squiggly] & f^{-1}(i) \arrow[d, squiggly] \\ |\boldsymbol{f}^{-1}(i)| \arrow[r, "\sigma_i", "\cong"'] & \underline{k_i}\,. \end{tikzcd} \tag{34}$$

Its cardinality is $\underline{k_i}$.

The extension. The objects of the extension $\mathbf{E}\,\mathtt{O}$ are equivalence classes of pairs consisting of an isomorphism $\sigma : X \xrightarrow{\cong} \underline{n}$ and an object $A \in \mathtt{O}$ with $|A| = \underline{n}$, written as $(X \xrightarrow{\sigma} \underline{n} \leftsquigarrow A)$ or as

$$\begin{tikzcd} & A \arrow[d, squiggly] \\ X \arrow[r, "\sigma", "\cong"'] & \underline{n}, \end{tikzcd} \tag{35}$$

modulo the relation $(X \xrightarrow{\sigma'} \underline{n} \leftsquigarrow A') \sim (X \xrightarrow{\sigma''} \underline{n} \leftsquigarrow A'')$ if there exists an automorphism $\phi : \underline{n} \to \underline{n}$ such that A'' is the target of the lift $\widetilde{\phi}$ of ϕ in the lifting square of

$$\begin{tikzcd} & A' \arrow[r, dashed, "\widetilde{\phi}"] \arrow[d, squiggly] & A'' \arrow[d, squiggly] & \\ X \arrow[r, "\sigma'", "\cong"'] & \underline{n} \arrow[r, "\phi", "\cong"'] & \underline{n} & X \arrow[l, "\sigma''"', "\cong"] \end{tikzcd} \tag{36}$$

and $\sigma'' = \phi\sigma'$. The cardinality of such an equivalence class is X by definition.

The morphisms of $\mathbf{E}\,\mathtt{O}$ are equivalence classes too; a morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ from the equivalence class $\boldsymbol{S}$ of $(X \xrightarrow{\xi} \underline{m} \leftsquigarrow S)$ to the equivalence class $\boldsymbol{T}$ of $(Y \xrightarrow{\eta} \underline{n} \leftsquigarrow T)$ is the equivalence class of a morphism $f : S \to T$ in

$$\begin{tikzcd} & S \arrow[r, "f"] \arrow[d, squiggly] & T \arrow[d, squiggly] & \\ X \arrow[r, "\xi", "\cong"'] & \underline{m} \arrow[r, "|f|"] & \underline{n} & Y. \arrow[l, "\eta"', "\cong"] \end{tikzcd} \tag{37}$$

The cardinality functor is given by $|\boldsymbol{f}| := \eta^{-1}|f|\xi : X \to Y$. The equivalence, by definition, identifies $f : S \to T$ with $\widetilde{f} := \widetilde{\sigma} f \widetilde{\rho}^{-1} : \widetilde{S} \to \widetilde{T}$ in the diagram

$$
\begin{tikzcd}
 & & & \widetilde{S} \arrow[rr, "\widetilde{f}"] \arrow[dd, rightsquigarrow] & & \widetilde{T} \arrow[dd, rightsquigarrow] & \\
 & & S \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[rr, "f"] \arrow[dd, rightsquigarrow] & & T \arrow[ur, dashed, "\widetilde{\sigma}"] \arrow[dd, rightsquigarrow] & & \\
 & & & \underline{m} \arrow[rr, "|\widetilde{f}|"] & & \underline{n} & \\
X \arrow[rr, "\xi", "\cong"'] & & \underline{m} \arrow[ur, "\rho", "\cong"'] \arrow[rr, "|f|"] & & \underline{n} \arrow[ur, "\sigma", "\cong"'] & & Y \arrow[ll, "\eta"', "\cong"]
\end{tikzcd}
\tag{38}
$$

in which $\rho : \underline{m} \to \underline{m}$ and $\sigma : \underline{n} \to \underline{n}$ are isomorphisms in $\mathtt{Fin}$, and $\widetilde{\rho} : S \to \widetilde{S}$, $\widetilde{\sigma} : T \to \widetilde{T}$ their lifts. Notice that $(X \xrightarrow{\rho\xi} \underline{m} \leftsquigarrow \widetilde{S})$ is another representative of $\boldsymbol{S}$ and $(Y \xrightarrow{\sigma\eta} \underline{n} \leftsquigarrow \widetilde{T})$ another representative of $\boldsymbol{T}$, so $\widetilde{f}$ indeed represents a morphism $\boldsymbol{S} \to \boldsymbol{T}$.

The diagram below defines the categorical composite of a morphism $\boldsymbol{S} \to \boldsymbol{T}$ represented by the left part of the diagram, followed by a morphism $\boldsymbol{T} \to \boldsymbol{R}$ represented by the right part,

$$
\begin{tikzcd}
 & S \arrow[r, "f"] \arrow[d, rightsquigarrow] & T' \arrow[rr, "\widetilde{\eta'\eta''^{-1}}"] \arrow[d, rightsquigarrow] & & T'' \arrow[r, "g"] \arrow[d, rightsquigarrow] & R \arrow[d, rightsquigarrow] & \\
X \arrow[r, "\xi", "\cong"'] & \underline{m} \arrow[r, "|f|"] & \underline{n} & Y \arrow[l, "\eta'"', "\cong"] \arrow[r, "\eta'", "\cong"'] & \underline{n} \arrow[r, "|g|"] & \underline{k} & Z. \arrow[l, "\omega"', "\cong"]
\end{tikzcd}
$$

In this diagram, the corner $(X \xrightarrow{\xi} \underline{m} \leftsquigarrow S)$ represents $\boldsymbol{S}$, $(Y \xrightarrow{\eta'} \underline{n} \leftsquigarrow T')$ resp. $(Y \xrightarrow{\eta''} \underline{n} \leftsquigarrow T'')$ are two equivalent representatives of $\boldsymbol{T}$, and $(Z \xrightarrow{\omega} \underline{k} \leftsquigarrow R)$ represents $\boldsymbol{R}$. The middle square is a lifting square in $\mathtt{O}$.

Let us describe the fiber $\boldsymbol{f}^{-1}(y)$ over $y \in Y$ of a morphism $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ represented by $f : S \to T$ in (37). Denote $i := \eta(y) \in \underline{n}$ and let $f^{-1}(i)$ be the the ith fiber of f in $\mathtt{O}$. In this situation we have the corner

$$
\begin{tikzcd}
 & & f^{-1}(i) \arrow[d, rightsquigarrow] \\
(|f|\xi)^{-1}(i) \arrow[r, "\xi", "\cong"'] & |f|^{-1}(i) \arrow[r, "can", "\cong"'] & |f|^{-1}_{pb}(i)
\end{tikzcd}
\tag{39}
$$

where $|f|^{-1}_{pb}(i)$ is the ith fiber of $|f|$ in $\mathtt{Fin}$, i.e. a pullback, and can is the unique canonical order-preserving isomorphism between the set-theoretic preimage $|f|^{-1}(i) \subset \underline{m}$ and the ordinal $|f|^{-1}_{pb}(i) \in \mathtt{Fin}$. Then $\boldsymbol{f}^{-1}(y) \in \mathbf{E}\,\mathtt{O}$ is the equivalence class of the corner (39).

Let us check that our definition of $\boldsymbol{f}^{-1}(y)$ is independent of the choice of a representative of $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$. Consider therefore a representative $\widetilde{f} : \widetilde{S} \to \widetilde{T}$ as in (38). In this situation we have the diagram

$$
\begin{tikzcd}
 & & f^{-1}(i) \arrow[r, dashed, "\widetilde{\rho_{pb}}"] \arrow[d, rightsquigarrow] \arrow[dl, phantom, "\boxed{\text{left}}"] \arrow[dr, phantom, "\boxed{\text{lifting square}}"] & \widetilde{f}^{-1}(j) \arrow[d, rightsquigarrow] \arrow[dr, phantom, "\boxed{\text{right}}"] & & \\
(|f|\xi)^{-1}(i) \arrow[r, "\xi", "\cong"'] & |f|^{-1}(i) \arrow[r, "can", "\cong"'] & |f|^{-1}_{pb}(i) \arrow[r, "\rho_{pb}"] & |\widetilde{f}|^{-1}_{pb}(j) & |\widetilde{f}|^{-1}(j) \arrow[l, "can"', "\cong"] & (|\widetilde{f}|\rho\xi)^{-1}(j) \arrow[l, "\eta"', "\cong"]
\end{tikzcd}
\tag{40}
$$

in which $j := \sigma^{-1}(i)$, ρ_{pb} is the canonical map between the pullbacks induced by ρ, $\widetilde{\rho_{pb}}$ its lift, and the meaning of other objects is similar as in (39). Here we very crucialy needed (11) to see that the target of $\widetilde{\rho_{pb}}$ equals the fiber $\widetilde{f}^{-1}(j)$.

The left corner in (40) is a representative of $\boldsymbol{f}^{-1}(y)$ calculated using $f : S \to T$, the right corner a representative calculated using $\widetilde{f} : \widetilde{S} \to \widetilde{T}$. Since clearly $(|f|\xi)^{-1}(i) = (|\widetilde{f}|\rho\xi)^{-1}(j)$ and since the composite

$$(|f|\xi)^{-1}(i) \xrightarrow{\xi} |f|^{-1}(i) \xrightarrow{\text{can}} |f|^{-1}_{pb}(i) \xrightarrow{\rho_{\text{pb}}} |\widetilde{f}|^{-1}_{pb}(j)$$

equals the composite

$$(|\widetilde{f}|\rho\xi)^{-1}(j) \xrightarrow{\eta} |\widetilde{f}|^{-1}(j) \xrightarrow{\text{can}} |\widetilde{f}|^{-1}_{pb}(j),$$

the left and the right corners of (40) represent the same object of $\mathbf{E}\,\mathtt{O}$, cf. the diagram in (36). We leave the painstaking verification that the above constructions produce cloven operadic categories as claimed to the reader.

Proposition 29. *There are canonical natural isomorphisms* $\mathrm{R}\,\mathbf{E}\,\mathtt{O} \cong \mathtt{O}$ *and* $\mathbf{E}\,\mathrm{R}\,\mathbf{O} \cong \mathbf{O}$ *of cloven operadic categories.*

Proof. The pair of mutually inverse functors $J : \mathrm{R}\,\mathbf{E}\,\mathtt{O} \longleftrightarrow \mathtt{O} : I$ is given as follows. The objects of $\mathrm{R}\,\mathbf{E}\,\mathtt{O}$ are the equivalence classes of the corners

$$\begin{tikzcd} & A \arrow[d, squiggly] \\ \underline{n} \arrow[r, "\sigma", "\cong"'] & \underline{n}\,, \end{tikzcd} \quad A \in \mathtt{O}.$$

The functor J assigns to the equivalence class of the above corner the domain $\widetilde{A} \in \mathtt{O}$ of the lift $\widetilde{\sigma}$ in the lifting diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & A \arrow[d, squiggly] \\ \underline{n} \arrow[r, "\sigma", "\cong"'] & \underline{n}\,. \end{tikzcd}$$

The inverse I of J sends $T \in \mathtt{O}$, $|T| = \underline{n}$, to the equivalence class of the corner

$$\begin{tikzcd} & T \arrow[d, squiggly] \\ \underline{n} \arrow[r, "\text{identity}"] & \underline{n}\,. \end{tikzcd}$$

Let us describe the pair of mutually inverse functors $F : \mathbf{E}\,\mathrm{R}\,\mathbf{O} \longleftrightarrow \mathbf{O} : G$. The objects of $\mathbf{E}\,\mathrm{R}\,\mathbf{O}$ are the equivalence classes of corners

$$\begin{tikzcd} & \boldsymbol{S} \arrow[d, squiggly] \\ X \arrow[r, "\varkappa", "\cong"'] & \underline{n}, \end{tikzcd} \quad \boldsymbol{S} \in \mathbf{O}.$$

The functor F assigns to the related equivalence class the domain $\widetilde{\boldsymbol{S}} \in \mathbf{O}$ of the lift $\widetilde{\varkappa}$ in the lifting square

$$\begin{tikzcd} \widetilde{\boldsymbol{S}} \arrow[r, dashed, "\widetilde{\varkappa}"] \arrow[d, squiggly] & \boldsymbol{S} \arrow[d, squiggly] \\ X \arrow[r, "\varkappa", "\cong"'] & \underline{n}\,. \end{tikzcd}$$

To describe the value $G\boldsymbol{S} \in \mathbf{E}\,\mathrm{R}\,\mathbf{O}$ of the inverse of F at some $\boldsymbol{S} \in \mathbf{O}$, $|\boldsymbol{S}| = X$, choose an arbitrary isomorphism $\omega : X \to \underline{n}$ and take the lift of ω as in

$$\begin{tikzcd} \boldsymbol{S} \arrow[r, dashed, "\widetilde{\omega}"] \arrow[d, squiggly] & \widetilde{\boldsymbol{S}} \arrow[d, squiggly] \\ X \arrow[r, "\omega", "\cong"'] & \underline{n}\,. \end{tikzcd} \tag{41}$$

Then $G\boldsymbol{S}$ is the equivalence class of the corner

$$\begin{tikzcd} & \widetilde{\boldsymbol{S}} \arrow[d, squiggly] \\ X \arrow[r, "\omega", "\cong"'] & \underline{n}\,. \end{tikzcd}$$

It is simple to check that the above construction leads to well-defined pairs of mutually inverse functors that preserve the fibers. □

Corollary 30. *Each thick cloven operadic category is isomorphic to the extension of a thin cloven operadic category. Reciprocally, each thin cloven operadic category is isomorphic to the restriction of a thick cloven operadic category.*

Before we prove, in Proposition 32 below, that the correspondence of $\mathbf{E} : \mathtt{O} \leftrightarrow \mathbf{O} : \mathrm{R}$ preserves the unitality, we formulate

Lemma 31. *Let* $\mathbf{O}$ *be a thick cloven operadic category and* $\mathtt{O}$ *the corresponding thin one. Then there is a natural isomorphism* $\pi_0(\mathtt{O}) \cong \pi_0(\mathbf{O})$ *of the sets of connected components.*

Proof. We may assume that $\mathbf{O} = \mathbf{E}\,\mathtt{O}$, by Corollary 30, so we have the inclusion $\mathtt{O} \hookrightarrow \mathbf{O}$ of categories. Since each object of $\mathbf{O}$ is isomorphic to some object of $\mathtt{O}$, cf. diagram (41), the inclusion induces an epimorphism $\pi_0(\mathtt{O}) \twoheadrightarrow \pi_0(\mathbf{O})$. To show that is an isomorphism, we need to prove that two objects S and T, connected by a zigzag of maps in $\mathbf{O}$, are connected by a zigzag in $\mathtt{O}$.

This statement follows from the definition of morphisms in $\mathbf{E}\,\mathtt{O}$, but we prove it directly. Consider e.g. the simple zigzag

$$\begin{tikzcd} S \arrow[r, "\boldsymbol{f}"] \arrow[d, squiggly] & \boldsymbol{R} \arrow[d, squiggly] & T \arrow[l, "\boldsymbol{g}"'] \arrow[d, squiggly] \\ \underline{m} \arrow[r, "|\boldsymbol{f}|"] & X & \underline{n} \arrow[l, "|\boldsymbol{g}|"'] \end{tikzcd} \tag{42}$$

with $S, T \in \mathtt{O}$ and $\boldsymbol{R} \in \mathbf{O}$. Choose an isomorphism $\omega : X \to \underline{k}$ and let R be the target of the lift $\widetilde{\omega}$ in the lifting square

$$\begin{tikzcd} \boldsymbol{R} \arrow[r, dashed, "\widetilde{\omega}"] \arrow[d, squiggly] & R \arrow[d, squiggly] \\ X \arrow[r, "\omega", "\cong"'] & \underline{k}. \end{tikzcd}$$

Then we replace the zigzag in (42) in $\mathbf{O}$ by the zigzag $S \xrightarrow{f} R \xleftarrow{g} T$ in $\mathtt{O}$, where $f = \widetilde{\omega}\boldsymbol{f}$ and $g = \widetilde{\omega}\boldsymbol{g}$. More complex zigzags can be treated similarly. □

Proposition 32. *Let* $\mathtt{O}$ *be a thin cloven operadic category and* $\mathbf{O}$ *the corresponding thick one. Then the category* $\mathtt{O}$ *is unital if and only if* $\mathbf{O}$ *is unital.*

Proof. By Lemma 31, the sets of connected components of the respective categories are isomorphic. Referring to Corollary 30 we may assume that $\mathtt{O} = \mathrm{R}\,\mathbf{O}$. The correspondence between the chosen local terminal objects in $\mathtt{O}$ and the chosen local terminal objects $\boldsymbol{U}_c$ in $\mathbf{O}$ is described by the lifting square

$$\begin{tikzcd} U_c \arrow[r, dashed, "\mathbb{1}"] \arrow[d, squiggly] & \boldsymbol{U}_c \arrow[d, squiggly] \\ \underline{1} \arrow[r, "\mathbb{1}"] & \{1\} \end{tikzcd}$$

in $\mathbf{O}$ which actually says that $U_c = \boldsymbol{U}_c$. □

Example 33. As expected, the category $\mathtt{Fin}$ of finite ordinals and the category **Fin** of finite sets correspond to each other in the correspondence of Theorem 28. The thick counterpart of the terminal thin unary operadic category $\mathtt{1}$ is the big groupoid $\mathfrak{G}$ of one-point sets.

Operads. In Proposition 34 below, $\mathtt{O}$ will be a thin cloven operadic category and $\mathbf{O}$ the corresponding thick one, in the correspondence of Theorem 28. We will also assume that the symmetric monoidal category $\mathtt{V}$ where our operads live is cocomplete. The notion of categories of operads is the expected one – their objects are operads and their morphisms are morphisms of the underlying collections which commute with the composition laws, and units in the unital case.

Proposition 34. *The categories of* $\mathtt{O}$*-operads and* $\mathbf{O}$*-operads are naturally equivalent. If* $\mathtt{O}$ *or, which is the same,* $\mathbf{O}$ *is unital, this equivalence restricts to the subcategories of unital operads.*

Proof. In view of Corollary 30 we may assume that $\mathtt{O} = \mathrm{R}\,\mathbf{O}$. Given an $\mathbf{O}$-operad $\boldsymbol{\mathcal{P}}$, the underlying collection of the corresponding $\mathrm{R}\,\mathbf{O}$-operad $\mathcal{P}$ is the restriction of the underlying collection of $\boldsymbol{\mathcal{P}}$ to objects of $\mathbf{O}$ whose arities are $\underline{n} \in \mathtt{Fin}$, $n \geq 0$. Let us define the composition laws of $\mathcal{P}$.

Consider a morphism $\boldsymbol{f} : S \to T$ of $\mathbf{O}$, with $|S| = \underline{m}$, $|T| = \underline{n}$, and denote by $f : S \to T$ the same morphism interpreted as a morphism in $\mathrm{R}\,\mathbf{O}$. Let $\boldsymbol{f}^{-1}(i)$ be the fiber of $\boldsymbol{f}$ over i in the category $\mathbf{O}$, and $f^{-1}(i)$ the ith fiber of f in $\mathtt{O}$, $i \in \underline{n}$. They are connected by the isomorphism $\widetilde{\sigma_i} : \boldsymbol{f}^{-1}(i) \to f^{-1}(i)$ in the lifting square (34). The cloven module structure of $\boldsymbol{\mathcal{P}}$ induces the isomorphism

$$\boldsymbol{\mathcal{P}}(\sigma) : \mathcal{P}(f) \cong \bigotimes_{i\in\underline{n}} f^{-1}(i) \xrightarrow{\bigotimes_{i\in\underline{n}} \boldsymbol{\mathcal{P}}(\sigma_i)} \bigotimes_{i\in\underline{n}} \boldsymbol{f}^{-1}(i) \cong \boldsymbol{\mathcal{P}}(\boldsymbol{f}).$$

The composition law $\mu_f : \mathcal{P}(T)\otimes\mathcal{P}(f) \longrightarrow \mathcal{P}(S)$ of $\mathcal{P}$ is then defined by the diagram

$$\begin{tikzcd}
\mathcal{P}(T)\otimes\mathcal{P}(f) \arrow[r, "\mu_f"] \arrow[d, "1\!\!1 \otimes \boldsymbol{\mathcal{P}}(\sigma)"'] & \mathcal{P}(S) \arrow[d, equal] \\
\boldsymbol{\mathcal{P}}(T)\otimes\boldsymbol{\mathcal{P}}(\boldsymbol{f}) \arrow[r, "\boldsymbol{\mu}_{\boldsymbol{f}}"] & \boldsymbol{\mathcal{P}}(S),
\end{tikzcd}$$

where $\boldsymbol{\mu}_{\boldsymbol{f}}$ in the bottom line is the composition law of $\boldsymbol{\mathcal{P}}$.

Let us explain how an $\mathrm{R}\,\mathbf{O}$-operad $\mathcal{P}$ determines the corresponding $\mathbf{O}$-operad $\boldsymbol{\mathcal{P}}$. Each $X \in \mathbf{Fin}$ determines the groupoid $\mathtt{G}(X)$, whose objects are isomorphisms $\sigma : X \xrightarrow{\cong} \underline{n}$ with some (necessarily unique) $\underline{n}$, and morphisms $\Omega : \sigma' \to \sigma''$ are commutative diagrams

$$\begin{tikzcd}
& X \arrow[dl, "\sigma'"'] \arrow[dr, "\sigma''"] & \\
\underline{n} \arrow[rr, "\omega"] & & \underline{n}.
\end{tikzcd} \tag{43}$$

of isomorphisms in $\mathbf{Fin}$. For each $\boldsymbol{S} \in \mathbf{O}$ with $|\boldsymbol{S}| = X$, $\mathcal{P}$ defines a contravariant functor $\mathcal{P}_{\boldsymbol{S}} : \mathtt{G}(X) \to \mathtt{V}$ as follows. Its value $\mathcal{P}(\sigma)$ on $\sigma : X \xrightarrow{\cong} \underline{n} \in \mathtt{G}(X)$ is $\mathcal{P}(S_\sigma)$, where S_σ is the right corner of the lifting square

$$\begin{tikzcd}
\boldsymbol{S} \arrow[r, dashed, "\widetilde{\sigma}"] \arrow[d, squiggly] & S_\sigma \arrow[d, squiggly] \\
X \arrow[r, "\sigma", "\cong"'] & \underline{n}
\end{tikzcd} \tag{44}$$

in $\mathbf{O}$. The value $\mathcal{P}_{\boldsymbol{S}}(\Omega)$ on the morphism Ω in (43) is given as follows. The commutative diagram in (43) determines the diagram of three lifting squares

$$\begin{tikzcd}
S_{\sigma'} \arrow[d, squiggly] \arrow[rr, dashed, bend left, "\widetilde{\omega}"] & \boldsymbol{S} \arrow[l, dashed, "\widetilde{\sigma'}"'] \arrow[r, dashed, "\widetilde{\sigma''}"] \arrow[d, squiggly] & S_{\sigma''} \arrow[d, squiggly] \\
\underline{n} \arrow[rr, bend right, "\omega", "\cong"'] & X \arrow[l, "\sigma'"', "\cong"] \arrow[r, "\sigma''", "\cong"'] & \underline{n}
\end{tikzcd}$$

with commutative upper and lower bases. The cloven module action on $\mathcal{P}$ induces the morphism $\mathcal{P}(\omega) : \mathcal{P}(S_{\sigma''}) \to \mathcal{P}(S_{\sigma'})$ which we interpret as $\mathcal{P}_{\boldsymbol{S}}(\Omega) : \mathcal{P}(\sigma'') \to \mathcal{P}(\sigma')$. This finishes the definition of the functor $\mathcal{P}_{\boldsymbol{S}}$. The piece $\boldsymbol{\mathcal{P}}(\boldsymbol{S})$ of the underlying collection of the $\mathbf{O}$-operad $\boldsymbol{\mathcal{P}}$ is the colimit

$$\boldsymbol{\mathcal{P}}(\boldsymbol{S}) := \operatorname{colim} \mathcal{P}_{\boldsymbol{S}},\ \ \boldsymbol{S} \in \mathbf{O}. \tag{45}$$

Since the indexing category $\mathtt{G}(X)$ is a groupoid, each isomorphism $\sigma : X \xrightarrow{\cong} \underline{n}$ determines a natural isomorphism

$$\pi_\sigma : \mathcal{P}(S_\sigma) \xrightarrow{\cong} \boldsymbol{\mathcal{P}}(\boldsymbol{S}). \tag{46}$$

Assume that $\boldsymbol{f} : \boldsymbol{S} \to \boldsymbol{T}$ is a morphism in $\mathbf{O}$, $|\boldsymbol{S}| = X$, $|\boldsymbol{T}| = Y$. To define the associated composition law $\boldsymbol{\mu_f} : \boldsymbol{\mathcal{P}}(\boldsymbol{T}) \otimes \boldsymbol{\mathcal{P}}(\boldsymbol{f}) \to \boldsymbol{\mathcal{P}}(\boldsymbol{S})$, choose two isomorphisms $\rho : X \xrightarrow{\cong} \underline{m}$, $\sigma : Y \xrightarrow{\cong} \underline{n}$, and construct the diagram

$$\begin{tikzcd}
& S_\rho \arrow[rr, dashed, "f_{\rho\sigma}"] \arrow[dd, rightsquigarrow] & & T_\sigma \arrow[dd, rightsquigarrow] \\
\boldsymbol{S} \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[rr, "\boldsymbol{f}"] \arrow[dd, rightsquigarrow] & & \boldsymbol{T} \arrow[ur, dashed, "\widetilde{\sigma}"'] \arrow[dd, rightsquigarrow] & \\
& \underline{m} \arrow[rr, dashed, "|f_{\rho\sigma}|"] & & \underline{n} \\
X \arrow[ur, "\rho", "\cong"'] \arrow[rr, "|\boldsymbol{f}|"] & & Y \arrow[ur, "\sigma", "\cong"'] &
\end{tikzcd} \tag{47}$$

in which $f_{\rho\sigma} := \widetilde{\sigma}\boldsymbol{f}\widetilde{\rho}^{-1}$ is a morphism in $\mathtt{O} = \mathrm{R}\,\mathbf{O}$ and the left and right faces are lifting squares. Given $y \in Y$ and $i := \sigma(y) \in \underline{n}$, the diagram (9) associated to (47) leads to the left lifting square of

$$\begin{tikzcd}
\boldsymbol{f}^{-1}(y) \arrow[r, "\widetilde{\rho_y}"] \arrow[d, rightsquigarrow] & \boldsymbol{f}_{\rho\sigma}^{-1}(i) \arrow[r, "\widetilde{\rho_i}"] \arrow[d, rightsquigarrow] & f_{\rho\sigma}^{-1}(i) \arrow[d, rightsquigarrow] \\
|\boldsymbol{f}|^{-1}(y) \arrow[r, "\rho_y"] & |\boldsymbol{f}_{\rho\sigma}|^{-1}(i) \arrow[r, "\rho_i"] & |f_{\rho\sigma}|^{-1}(i)\,.
\end{tikzcd}$$

The right lifting square expresses the relation between the fiber $\boldsymbol{f}_{\rho\sigma}^{-1}(i)$ of $f_{\rho\sigma}$ considered as a morphism in $\mathbf{O}$, and the fiber $f_{\rho\sigma}^{-1}(i)$ of $f_{\rho\sigma}$ in $\mathtt{O} = \mathrm{R}\,\mathbf{O}$, cf. (34). The isomorphisms

$$\pi_{\rho_i\sigma_y} : \mathcal{P}(f_{\rho\sigma}^{-1}(i)) \xrightarrow{\cong} \boldsymbol{\mathcal{P}}(\boldsymbol{f}^{-1}(y)),\ \ y \in Y,\ \ i := \sigma(y),$$

assemble to an isomorphism $\pi_{\rho\sigma} : \mathcal{P}(f_{\rho\sigma}) \cong \boldsymbol{\mathcal{P}}(\boldsymbol{f})$. We finally define the composition law $\boldsymbol{\mu_f}$ of $\boldsymbol{\mathcal{P}}$ via the diagram

$$\begin{tikzcd}
\boldsymbol{\mathcal{P}}(\boldsymbol{T}) \otimes \boldsymbol{\mathcal{P}}(\boldsymbol{f}) \arrow[r, "\boldsymbol{\mu_f}"] & \boldsymbol{\mathcal{P}}(\boldsymbol{S}) \\
\mathcal{P}(T_\sigma) \otimes \mathcal{P}(f_{\rho\sigma}) \arrow[r, "\mu_{f_{\rho\sigma}}"] \arrow[u, "\pi_\sigma \otimes \pi_{\rho\sigma}", "\cong"'] & \mathcal{P}(S_\rho) \arrow[u, "\cong", "\pi_\rho"']
\end{tikzcd}$$

in which $\mu_{f_{\rho\sigma}}$ is composition law of the $\mathtt{O}$-operad $\mathcal{P}$. It can be verified that this definition does not depend on the choices of the isomorphisms ρ and σ in (47).

It remains to discuss the unitality, assuming as before that $\mathtt{O} = \mathrm{R}\,\mathbf{O}$. Let us invoke Lemma 31 and take $c \in \pi_0(\mathtt{O}) \cong \pi_0(\mathbf{O})$. The chosen local terminal object U_c of $\mathtt{O}$ is related to the corresponding chosen local terminal object of $\mathbf{O}$ by the trivial lifting square

$$\begin{tikzcd}
\boldsymbol{U}_c \arrow[r, dashed, "\mathbb{1}"] \arrow[d, rightsquigarrow] & U_c \arrow[d, rightsquigarrow] \\
\{1\} \arrow[r, "\mathbb{1}"] & \underline{1}
\end{tikzcd}$$

If an $\mathtt{O}$-operad $\mathcal{P}$ is the restriction of an $\mathbf{O}$-operad $\boldsymbol{\mathcal{P}}$ to $\mathtt{O}$, then $\mathcal{P}(U_c) = \boldsymbol{\mathcal{P}}(\boldsymbol{U}_c)$ since $\boldsymbol{U}_c \in \mathtt{O} = \mathrm{R}\,\mathbf{O}$. We define the unit maps for $\mathcal{P}$ using the unit maps $\eta_c : \mathbb{1}_{\mathtt{V}} \to \boldsymbol{\mathcal{P}}(\boldsymbol{U}_c)$ of $\boldsymbol{\mathcal{P}}$ as the composite

$$\mathbb{1}_{\mathtt{V}} \xrightarrow{\eta_c} \boldsymbol{\mathcal{P}}(\boldsymbol{U}_c) \xrightarrow{\mathbb{1}} \mathcal{P}(U_c).$$

Let, on the other hand, $\boldsymbol{\mathcal{P}}$ be the extension of $\mathcal{P}$. Then the unit maps of $\boldsymbol{\mathcal{P}}$ are defined using the unit maps $\eta_c : \mathbb{1}_{\mathtt{V}} \to \mathcal{P}(U_c)$ of $\mathcal{P}$ as the composite

$$\mathbb{1}_{\mathtt{V}} \xrightarrow{\eta_c} \mathcal{P}(U_c) \xrightarrow{\pi_{\mathbb{1}}} \boldsymbol{\mathcal{P}}(\boldsymbol{U}_c)$$

which uses the canonical isomorphism (46). We graciously leave to the reader the arduous task of verifying that the above constructions do what we claim they do. $\square$

Example 35. The necessity of the compatibility (23b) of the composition law (22) with the cloven action for Proposition 34 to hold can be seen in the following simple example. Let $\mathtt{1}$ be the terminal unary thin operadic category, cf. non-Example 7, and $\mathfrak{G}$ its thick counterpart introduced in Example 33. While the $\mathtt{1}$-operads are associative algebras, the $\mathfrak{G}$-operads without (23b) are families $\{\boldsymbol{\mathcal{P}}(x) \,|\, \{x\} \in \mathfrak{G}\}$ indexed by singletons, with left actions

$$\triangleright : \boldsymbol{\mathcal{P}}(x) \otimes \boldsymbol{\mathcal{P}}(y) \longrightarrow \boldsymbol{\mathcal{P}}(y)$$

such that the diagram

$$\begin{tikzcd} \boldsymbol{\mathcal{P}}(x)\otimes\boldsymbol{\mathcal{P}}(y)\otimes\boldsymbol{\mathcal{P}}(z) \arrow[r, "\mathbb{1}\otimes\triangleright"] \arrow[d, "\triangleright\otimes\mathbb{1}"'] & \boldsymbol{\mathcal{P}}(x)\otimes\boldsymbol{\mathcal{P}}(z) \arrow[d, "\triangleright"] \\ \boldsymbol{\mathcal{P}}(y)\otimes\boldsymbol{\mathcal{P}}(z) \arrow[r, "\triangleright"] & \boldsymbol{\mathcal{P}}(z) \end{tikzcd}$$

commutes for each singletons $\{x\}, \{y\}, \{z\} \in \mathfrak{G}$.

Algebras. In the following proposition, $\mathtt{O}$ will be a thin cloven operadic category, $\mathbf{O}$ the corresponding thick one, $\mathcal{P}$ an $\mathtt{O}$-operad and $\boldsymbol{\mathcal{P}}$ the corresponding $\mathbf{O}$-operad. The notion of categories of algebras is again the expected one.

Proposition 36. *The categories of $\boldsymbol{\mathcal{P}}$- and $\mathcal{P}$-algebras are naturally equivalent. If we declare collections with canonically isomorphic indexing sets to be* the same, *then this equivalence is actually an isomorphism of categories.*

Proof. By definition, the underlying collection of a $\mathcal{P}$-algebra A is of the form $\{A_c\}_{c\in\pi_0(\mathtt{O})}$, while the underlying collection of a $\boldsymbol{\mathcal{P}}$-algebra $\mathbf{A}$ is of the form $\{\mathbf{A}_c\}_{c\in\pi_0(\mathbf{O})}$. In view of Lemma 31 which we use to identify $\pi_0(\mathtt{O})$ with $\pi_0(\mathbf{O})$, the underlying collections of $\mathcal{P}$-algebras and $\boldsymbol{\mathcal{P}}$-algebras are therefore of the same form. The equivalence of the proposition acts as the identity on the underlying collections.

Replacing the operadic category $\mathtt{O}$ by an isomorphic one if necessary, we may assume, as in the proof of Proposition 34, that $\mathtt{O} = \mathrm{R}\,\mathbf{O}$, and refer to the explicit correspondence between $\mathbf{O}$-operads and $\mathtt{O}$-operads constructed there.

If $\mathbf{A} = \{\mathbf{A}_c\}_{c\in\pi_0(\mathbf{O})}$ is a $\boldsymbol{\mathcal{P}}$-algebra, the $\mathcal{P}$-action on the corresponding $\mathcal{P}$-algebra $A = \{A_c\}_{c\in\pi_0(\mathtt{O})}$ is the restriction of the $\boldsymbol{\mathcal{P}}$-action to the objects of $\mathrm{R}\,\mathbf{O}$. On the other hand, given a $\mathcal{P}$-algebra A, we have, in the notation of the second part of the proof of Proposition 34, cf. diagrams (43)–(46) in particular, the diagram

$$\begin{tikzcd} \mathcal{P}(S_{\sigma''})\otimes A_{s(S_{\sigma''})} \arrow[r, "\alpha_{S_{\sigma''}}"] \arrow[d, "\mathcal{P}(\omega)\otimes A_{s(\omega)}"'] & A_{t(S_{\sigma''})} \arrow[d, equal] \\ \mathcal{P}(S_{\sigma'})\otimes A_{s(S_{\sigma'})} \arrow[r, "\alpha_{S_{\sigma'}}"] & A_{t(S_{\sigma'})}. \end{tikzcd}$$

which is commutative by item (ii) of Definition 26. Passing to the colimit (45) gives the action

$$\boldsymbol{\alpha}_{\boldsymbol{S}} : \boldsymbol{\mathcal{P}}(\boldsymbol{S})\otimes A_{\mathbf{s}(\boldsymbol{S})} \to A_{\mathbf{s}(\boldsymbol{T})}.$$

It is simple to verify that the above constructions preserve unitality. $\square$

7 Graphs, modular operads, &c

The aim of this section is to introduce a thick version of the operadic category of graphs and show how they describe modular, resp. odd modular operads in the form of [4, Definitions A.1, A.2] when arities are finite sets rather than natural numbers. The presentation of the category of graphs here follows closely [3, Section 3]. We will ignore the genus grading of (odd) modular operads, as it does not bring anything conceptually new to the picture.

Definition 37. A *thick graph* Γ is a pair (g, σ) consisting of an order-preserving map

$$g : F \to V,\ V \neq \emptyset,$$

in the category **Fin** of finite sets together with an involution σ on F.

Elements of $\mathrm{Flg}(\Gamma) := F$ are the *flags* (also called *half-edges*) of Γ and elements of $\mathrm{Ver}(\Gamma) := V$ are its *vertices*. The elements of the set $\mathrm{Leg}(\Gamma) \subset F$ of fixed points of σ are called the *legs* (also called *hairs*) of Γ while nontrivial orbits $\mathrm{Edg}(\Gamma)$ of σ are its *edges*. The *endpoints* of an edge $e = \{h_1, h_2\} \in \mathrm{Edg}(\Gamma)$ are $g(h_1)$ and $g(h_2)$. We will use the notation (F, V) or (F, g, V) for a graph $\Gamma = (g, \sigma)$ if we want to specify its set of vertices and flags.

A thin version of graphs in Definition 37 was called in [3] *preordered* graphs, where "preordered" indicated the absence of the global orders of the legs. In our situation, nothing is ordered and this terminology would be misleading.

A *morphism* of graphs $\Phi : \Gamma' \to \Gamma''$ is a pair (ψ, ϕ) of morphisms of finite sets such that the diagram

$$\begin{tikzcd} F' \arrow[d, "g'"'] & F'' \arrow[l, hook', "\psi"'] \arrow[d, "g''"] \\ V' \arrow[r, two heads, "\phi"] & V'' \end{tikzcd} \tag{48}$$

commutes. We moreover require ϕ to be a surjection, and ψ equivariant with respect to the involutions, and bijection on fixed points. Thus ψ injectively maps flags to flags and bijectively legs to legs. The pair (ψ, ϕ) must satisfy the following condition: If $\phi(x) \neq \phi(y)$, $x, y \in V'$ and e' is an edge of Γ' with endpoints x and y, then there exists an edge e'' in Γ'' with endpoints $\phi(x)$ and $\phi(y)$ such that $e' = \psi(e'')$. Graphs and their morphisms form a category **prGR**.

The *fiber* $\Phi^{-1}(x)$ of a map $\Phi = (\psi, \phi) : \Gamma' \to \Gamma''$ in (48) over $x \in V''$ is a graph whose set of vertices is $\phi^{-1}(x)$ and whose set of flags is $(\phi g)^{-1}(x)$. The involution τ of $\Phi^{-1}(x)$ is defined by

$$\tau(h) := \begin{cases} h & \text{if } h \in \mathit{Im}(\psi), \\ \sigma(h) & \text{if } h \notin \mathit{Im}(\psi), \end{cases}$$

where σ is the involution of Γ.

For a finite set $X \in \mathbf{Fin}$, denote by C_X the graph $X \to \{1\}$ with the trivial involution on X. It can be viewed as a corolla with spikes indexed by X and the hub labelled by $\{1\}$. Let **GR** be the coproduct of the categories $\mathbf{prGR}/C_X$ for $X \in \mathbf{Fin}$. We are aware that we are skating on thin ice here, but we may always assume that we are working in the universe which is sufficiently big so that the coproduct makes sense. Moreover, we will always know in concrete situations what we are doing. Explicitly, an object of $\mathbf{prGR}/C_X$ is a morphism in **prGR** of the form

$$\begin{tikzcd} F \arrow[d, "g"'] & X \arrow[l, hook', "\psi"'] \arrow[d, "!"] \\ V \arrow[r, two heads, "!"] & \{1\}. \end{tikzcd} \tag{49}$$

The inclusion $\psi : X \hookrightarrow F$ induces an isomorphism between X and the set of legs of $\Gamma = (V, g, F)$. The set X thus appears as the set of labels of the legs of Γ, which we call the *global* labels of Γ. The morphisms of **GR** are the morphisms of **prGR** that keeps the global labels fixed.

Proposition 38. *The category* **GR** *is a thick operadic category.*

Proof. The cardinality functor assigns to a graph Γ its set $\mathrm{Ver}(\Gamma)$ of vertices. We describe the fiber structure and leave the verification of the axioms as an exercise. A morphisms $\Phi : \Gamma' \to \Gamma''$ in $\mathbf{GR}$ is, by definition, a diagram

$$\begin{tikzcd} F' & & F'' \arrow[ll, hook', "\psi"'] \\ & X \arrow[ul, hook] \arrow[ur, hook'] \arrow[d] & \\ & \{1\} & \\ V' \arrow[uuu, leftarrow, "g'"] \arrow[ur, two heads, "!"] \arrow[rr, "\phi"] & & V''. \arrow[uuu, leftarrow, "g''"'] \arrow[ul, two heads, "!"'] \end{tikzcd} \tag{50a}$$

It induces for each $x \in V''$ a commutative diagram

$$\begin{tikzcd} (\phi g')^{-1}(x) \arrow[d, "g'_x"'] & (g'')^{-1}(x) \arrow[l, hook', "\psi_x"'] \arrow[d, "g''_x"] \\ \phi^{-1}(x) \arrow[r, "\phi_x"] & \{1\} \end{tikzcd} \tag{50b}$$

of finite sets in which the morphisms g'_x, ϕ_x, g''_x and ψ_x are the restrictions of the corresponding morphisms from (50a).

We interpret the right vertical morphism of (50b) as a corolla C_X with $X := (g'')^{-1}(x)$, imposing the trivial involution on X. Due to the definition of fibers of maps in $\mathbf{prGR}$, the diagram above represents a map of the fiber $\Phi^{-1}(x)$ in $\mathbf{prGR}$ to C_X, which makes it an object of $\mathbf{GR}$. We take it as the definition of the fiber *in* $\mathbf{GR}$. So the fiber gets its global labels from the set $(g'')^{-1}(x)$. □

Proposition 39. *The category* $\mathbf{GR}$ *is a thick cloven unital category.*

Proof. Let us start with the cloven structure. Given a graph $\Gamma = (V, g, F) \in \mathbf{GR}$ with the global labels X and an isomorphism $\phi : V \to W$ of finite sets, we define the graph $\widetilde{\Gamma}$ and an isomorphism $\widetilde{\phi}$ in the lifting square

$$\begin{tikzcd} \Gamma \arrow[r, dashed, "\widetilde{\phi}"] \arrow[d, rightsquigarrow] & \widetilde{\Gamma} \arrow[d, rightsquigarrow] \\ V \arrow[r, "\phi", "\cong"'] & W \end{tikzcd}$$

as follows. We put $\widetilde{\Gamma} = (W, \widetilde{g}, F)$, with the involution $\sigma : F \to F$ taken from Γ and $\widetilde{g} := \phi g$. The morphism $\widetilde{\phi}$ is given by the diagram

$$\begin{tikzcd} F & & F \arrow[ll, hook', "\mathbb{1}_F"'] \\ & X \arrow[ul, hook] \arrow[ur, hook'] \arrow[d] & \\ & \{1\} & \\ V \arrow[uuu, leftarrow, "g"] \arrow[ur, two heads] \arrow[rr, "\phi"] & & W. \arrow[uuu, leftarrow, "\widetilde{g}"'] \arrow[ul, two heads] \end{tikzcd}$$

The (big) set $\pi_0(\mathbf{GR})$ of connected components is $\mathbf{Fin}$. A graph $\Gamma \in \mathbf{GR}$ belongs to the component $X \in \mathbf{Fin}$ if and only if its set of global labels equals X. The chosen local terminal objects (14) are the corollas $\star_X : C_X \xrightarrow{\mathbb{1}} C_X$. In words, the chosen terminal objects are corollas such that the global labels coincide with the (labels of) the spikes. Figure 1 shows the local terminal object $\star_X$ with $X = \{a, b, c, d\}$. The labels of the legs are encircled. We leave again the details to the reader. □

The thin operadic category corresponding to $\mathbf{GR}$, realized as the restriction $\mathtt{GR} := \mathrm{R}\,\mathbf{GR}$ in the proof of Theorem 28, is a simple modification of the constructions above. The main difference

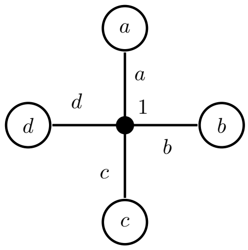


Figure 1: The local terminal object $\star_X$ of $\mathbf{GR}$ with $X = \{a,b,c,d\}$.

is that the sets of vertices of graphs in $\mathtt{GR}$ are finite ordinals $\underline{n}$, $n \geq 1$. All definitions translate verbatim, only the diagram (50b) has now the form

$$\begin{tikzcd}
(\phi g')^{-1}(i) \arrow[d, "g'_i"'] & (g'')^{-1}(i) \arrow[l, hook', "\psi_i"'] \arrow[d, "g''_i"] \\
\phi_{pb}^{-1}(i) \arrow[r, "\phi_i"] & \underline{1}
\end{tikzcd}$$

with $i \in \underline{n}$, where g'_i is the composite of the restriction $(\phi g')^{-1}(i) \to \phi^{-1}(i)$ of g' with the unique order-preserving isomorphism of the finite ordered set $\phi^{-1}(i)$ with the unique finite ordinal $\phi_{pb}^{-1}(i)$.

As stated in [4, Proposition 5.11], algebras for the constant operad over the category $\mathtt{Gr}$ are modular operads with the arities in finite ordinals. Let us see what happens in our setup. Let thus $\boldsymbol{\zeta}$ be the constant unital $\mathbf{GR}$-operad given by $\boldsymbol{\zeta}(\Gamma) := \mathbb{1}_{\mathtt{V}}$ for each $\Gamma \in \mathbf{GR}$ and the composition law

$$\mu_\Phi : \boldsymbol{\zeta}(\Gamma'') \otimes \boldsymbol{\zeta}(\Phi) = \mathbb{1}_{\mathtt{V}} \otimes \mathbb{1}_{\mathtt{V}} \longrightarrow \mathbb{1}_{\mathtt{V}} = \boldsymbol{\zeta}(\Gamma')$$

for $\Phi : \Gamma' \to \Gamma''$ a morphism in $\mathbf{GR}$.

It will not be difficult to investigate $\boldsymbol{\zeta}$-operads directly, but we can further simplify our life by looking at its restriction ζ to $\mathtt{GR}$. Both operads $\boldsymbol{\zeta}$ and ζ have equivalent, "almost" isomorphic, categories of algebras by Proposition 34. An algebra for a $\mathtt{GR}$-operad is, by definition, a collection indexed by $\pi_0(\mathtt{GR}) = \mathbf{Fin}$. A ζ-algebra thus sits on a collection $\mathcal{M} = \{M(X) \mid X \in \mathbf{Fin}\}$.

Let us investigate the structure operations, starting with the one-vertex graph, the corolla $\star_\omega \in \mathtt{GR}$ given, in the representation used in (49), by the diagram

$$\begin{tikzcd}
X' \arrow[d, "!"'] & X'' \arrow[l, "\omega"', "\cong"] \arrow[d, "!"] \\
\underline{1} \arrow[r, equal] & \underline{1},
\end{tikzcd}$$

in which ω is an isomorphism of finite sets and the involutions on X' and X'' are trivial. Its list of sources contains only X', its target is X''. The associated operations

$$\alpha_{\star_\omega} : \mathcal{M}(X') \cong \zeta(X') \otimes \mathcal{M}(X') \longrightarrow \mathcal{M}(X'') \tag{51}$$

make $\mathcal{M}$ a module over the groupoid of finite sets and their isomorphism. Let X, Y be disjoint finite sets and $x \notin X$, $y \notin Y$, $x \neq y$. Consider the two-vertex graph given by

$$\begin{tikzcd}
X \cup \{x,y\} \cup Y \arrow[d, "g"'] & X \cup Y \arrow[l, hook', "\psi"'] \arrow[d, "!"] \\
\underline{2} \arrow[r, two heads, "!"] & \underline{1},
\end{tikzcd} \tag{52}$$

with the involution σ acting trivially on $X \cup Y$, $\sigma(x) = y$, ψ the inclusion and the vertex map g which equals 1 on $X \cup \{x\}$ and 2 on $Y \cup \{y\}$. It has the sources $X \cup \{x\}, Y \cup \{y\}$, and the target $X \cup Y$. The associated operation is a map

$${}_x\circ_y : \mathcal{M}(X \cup \{x\}) \otimes \mathcal{M}(Y \cup \{y\}) \longrightarrow \mathcal{M}(X \cup Y). \tag{53}$$

Consider finally the one vertex graph, the tadpole

$$\begin{tikzcd} X \cup \{a,b\} \arrow[d, "!"'] & X \arrow[l, hook', "\psi"'] \arrow[d, "!"] \\ \underline{1} \arrow[r, equal] & \underline{1}, \end{tikzcd} \tag{54}$$

with the involution σ acting trivially on X, $\sigma(a) = b$, and ψ the inclusion. It has only one source $X \cup \{a, b\}$ and the target X. The corresponding operation has the form

$$\circ_{ab} : \mathcal{M}(X \cup \{a, b\}) \longrightarrow \mathcal{M}(X). \tag{55}$$

Proposition 40. *The operations* (51)*,* (53) *and* (55) *induce on* $\mathcal{M}$ *the structure of a modular operad in the presentation of [4, Definition A.1].*

Proof. The proof is practically the same as that of [4, Proposition 5.11]. We are not going to repeat it here. The graph in (52) is an analog of the graph in (69b) of [4], and the graph in (54) an analog of the graph in (69a) loc. cit. $\square$

Suppose for a moment that the base monoidal category $\mathtt{V}$ is the category of graded vector spaces. For a vector space A of dimension k, we denote by $\det(A) := \bigwedge^k(A)$ the top-dimensional piece of its Grassmann algebra. If S is a nonempty finite set, we let $\det(S)$ to be the determinant of the vector space spanned by S. Let $\rotatebox[origin=c]{180}{ζ}$ (ζ turned upside down) be the **GR**-operad defined by $\rotatebox[origin=c]{180}{ζ}(\Gamma) := \det(\mathrm{Edg}(\Gamma))$, the determinant of the internal edges of Γ, with the composition law given by the natural isomorphism

$$\mu_\Phi : \rotatebox[origin=c]{180}{ζ}(\Gamma'') \otimes \rotatebox[origin=c]{180}{ζ}(\Phi) = \det(\mathrm{Edg}(\Gamma'') \otimes \bigotimes_{v \in \mathrm{Ver}(\Gamma'')} \mathrm{Edg}(\Gamma_v)) \xrightarrow{\cong} \det(\mathrm{Edg}(\Gamma') = \rotatebox[origin=c]{180}{ζ}(\Gamma'), \tag{56}$$

where Γ_v, $v \in \mathrm{Ver}(\Gamma'')$ are the fibers of a map $\Phi : \Gamma' \to \Gamma''$. The isomorphism uses the fact that the set of edges of Γ' is the union of the set of edges of Γ'' with the sets of edges of the fibers of Φ. In (56) we thus see an isomorphism of one-dimensional vector spaces given by multiplication with the sign resulting from the reordering the factors. We formulate

Proposition 41. *The algebras for the constant* **GR***-operad* ζ *are modular operads in the presentation [4, Definition A.1]. The algebras for the odd* **GR***-operad* $\rotatebox[origin=c]{180}{ζ}$ *are odd modular operads in the presentation [4, Definition A.2].*

Proof. The first part is a reformulation of Proposition 40. The second part can be established in a similar way, taking into account the nontrivial signs coming from the isomorphism in (56). $\square$

In the same way we can get the thick versions of the operadic categories adapted for classical operads, wheeled PROPs, dioperads and $\frac{1}{2}$PROPs respectively, listed in diagram (37) of [3].

Appendix

A Semi-ordered operadic categories

It turns out that *every* – no cleavage needed – standard (thin) operadic category has a thick counterpart with cardinalities in *ordered* finite sets. But this fact is not very interesting since it does not remove orders from the picture. We include the related material just to finish our story. Let us denote by $\underline{\mathbf{Fin}}$ the category of ordered sets and their *all*, not necessarily order-preserving, maps.

Definition 42. A thick operadic category $\mathbf{O}$ is *semi-ordered* if a factorization of the cardinality functor $|\text{-}| : \mathbf{O} \to \mathbf{Fin}$ through the category $\underline{\mathbf{Fin}}$ is specified. A semi-ordered operadic category is *cloven* if functorial lifts $\widetilde{\sigma}$ in (7) of order-preserving isomorphisms $\sigma : X \to Y$ are specified.

Since every finite set can be ordered, under a mild categorial assumptions, every functor to finite sets factorizes through $\underline{\mathbf{Fin}}$, every operadic category admits a semi-ordered structure. However, the chosen factorization determines the class of isomorphisms required to have functorial lifts, so not every semi-ordered operadic category is cloven. The notion of a cloven **O**-module is easily translated to the cloven semi-ordered case; we require the action of $\widetilde{\sigma}$ in (20) only if σ is order-preserving. The definitions of operads and their algebras remain the same.

Notice that thin operadic categories are always "cloven semi-ordered," since the image of the cardinality functor already consists of ordered sets, and the only order-preserving isomorphism $\sigma : \underline{n} \to \underline{n}$ that must admit functorial lift is the identity automorphism. A semi-ordered operadic category (thin or thick) is *ordered*, if the image of the chosen factorization lies within the category of ordered sets and their order-preserving maps.

Theorem 43. *There is a natural correspondence* $\mathbf{E} : \mathtt{O} \leftrightarrow \mathbf{O} : \mathrm{R}$ *(extension and restriction) which extends to an equivalence of the categories of (standard, thin) operadic categories with the category of semi-ordered cloven thick operadic categories. The corresponding operadic categories have equivalent categories of operads and their algebras. This equivalence restricts to the subcategories of ordered operadic categories.*

Proof. A very simplified version of the proof of Theorem 28. The simplification comes from the fact that the construction of the extension $\mathbf{E}$ does not use equivalence classes, since the objects of $\mathbf{E}\,\mathtt{O}$ are diagrams (35) with σ the unique order-preserving isomorphism. □

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