

Additive Invariants of Open Petri Nets

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We classify all additive invariants of open Petri nets: these are N-valued invariants which are additive with respect to sequential and parallel composition of open Petri nets. In particular, we prove two classification theorems: one for open Petri nets and one for monically open Petri nets (i.e. open Petri nets whose interfaces are specified by monic maps). Our results can be summarized as follows. The additive invariants of open Petri nets are completely determined by their values on a particular class of singletransition Petri nets. However, for monically open Petri nets, the additive invariants are determined by their values on transitionless Petri nets and all single-transition Petri nets. Our results confirm a conjecture of John Baez (stated during the AMS' 2022 Mathematical Research Communities workshop).

1 Introduction

A Petri net is a directed bipartite graph with bipartition (S, T) consisting of a set S of *species* and a set T of *transitions*. They were invented by Carl Petri in 1939 [20] as a graphical representation of a set S of chemical compounds that can be combined by way of a set of reactions T, into new compounds. For example, consider the following Petri net representing the electrolysis $\mathcal{E}: 2H_2O \rightarrow$ $2H_2 + O_2$ of water to make oxygen and hydrogen gas. It consists of only one transition — namely \mathcal{E} — and three species: H_2O , H_2 and O_2 as drawn below.



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00000-0002-8686-2319. While preparing this article, this author also received funding by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 803421, ReduceSearch).

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In general, the applications of Petri nets need not be confined to chemistry. Indeed, they can represent many kinds of processes (concurrent, asynchronous, distributed, parallel, nondeterministic and stochastic, to name a few) in which some entities (the species) undergo transformations (transitions) in order to be converted into other kinds of entities [2, 10, 11, 14, 15, 17, 19, 24].

Formally, a **Petri net** P with *finitely many* **species** S and **transitions** T is a graph with source and target maps $s, t: T \to \mathbb{N}^S$, where \mathbb{N}^S is the free commutative monoid on S. We denote a Petri net P by the quadruple (S, T, s, t). In the example above, the source and target maps s and t are defined by:

$$\begin{split} s(\mathcal{E})(\mathrm{H}_2\mathrm{O}) &= 2, \quad s(\mathcal{E})(\mathrm{H}_2) = 0, \quad s(\mathcal{E})(\mathrm{O}_2) = 0, \\ t(\mathcal{E})(\mathrm{H}_2\mathrm{O}) &= 0, \quad t(\mathcal{E})(\mathrm{H}_2) = 2, \quad t(\mathcal{E})(\mathrm{O}_2) = 1. \end{split}$$

For a transition $\tau \in T$ and species $\sigma \in S$, the quantity $s(\tau)(\sigma)$ represents the number of **input arcs** which emanate from the species σ to the transition τ . Similarly, the quantity $t(\tau)(\sigma)$ represents the number of **output arcs** which emanate from the transition τ to the species σ . We say that $\sigma \in S$ is an **input species** (resp. **output species**) of the transition $\tau \in T$ if $s(\tau)(\sigma) > 0$ (resp. $t(\tau)(\sigma) > 0$). Accordingly, the total number of input arcs into and output arcs out of a transition τ are given by:

$$\sum_{\sigma \in S} s(\tau)(\sigma) \tag{1}$$

and

$$\sum_{\sigma \in S} t(\tau)(\sigma).$$
⁽²⁾

In the example of electrolysis, the transition \mathcal{E} has two input arcs and three output arcs. The molecule H₂O is its input species and the molecules H₂ and O₂ are its output species.

1.1 Invariants of composable Petri nets

The study of composable Petri nets was introduced by Baldan, Corradini, Ehrig, Heckel, and König [4, 5]. Baez and Pollard [3] used the formalism of decorated cospans to introduce tensoring of open Petri nets. In the same vein as our decomposition lemmas that appear in Section 3, Nielsen, Priese and Sassone [18] showed that a finite Petri net is built from single-place and single-transition Petri nets via a collection of operations on nets known as *combinators*. Gadduci and Heckel [9] presented a theorem (referred to as *the Kindred theorem*) for the decomposition of graphs into fundamental components.

Many fields have vast databases that record many kinds of chemical reactions and their associated Petri nets [13, 16, 21, 23]. These are studied empirically by computationally seeking patterns, *motifs* [22], and *numerical invariants* that arise in this data. Often due to the sheer size of the Petri nets in such databases, it is convenient to think of a large Petri net as being built out of smaller constituent nets which are "glued" together to form the entire net. This compositional structure of Petri nets is particularly useful when one wishes to study structural measurements that are *isomorphism-invariant* and which are *compositional* in the sense that they behave nicely with respect to this kind of gluing. In this article, as is customary in many areas of mathematics (cf. algebraic topology, graph theory etc.) we will adopt the term **invariant** as a more concise substitute for the term "isomorphism-invariant measurement".

The literature on Petri nets contains other uses of the term invariant (e.g. P- and T- invariants) which are topological properties which do not depend on the initial marking of a Petri net. This usage is very similar to ours; however, in this article we are not concerned with markings and instead we are interested in invariants which satisfy the following two requirements:

- 1. For a Petri net built out of smaller parts, the overall invariant value should be determinable solely from the values on the pieces.
- 2. It is *computationally cheap* to compute the invariant on the whole Petri net from its values on the pieces.



The first requirement is in a similar vein to that of the work of Baldan, Corradini, Ehrig, and Heckel [4] (which studies compositional invariants different from the ones studied in this article). On the other hand, the second requirement is born from the overarching question which, sitting in the background, motivates this paper; namely: "which structural invariants of very large, real-world Petri nets can be used to synthetically generate Petri nets with comparable structural statistics?" Although this question is beyond the scope of the present paper, it serves as a motivation to determine isomorphism-invariant measurements which are both compositional and easily computable, even on truly vast Petri nets.

There are several software systems for graphical modeling. In particular, AlgebraicPetri.jl supports the definition and composition of open Petri nets. More recently, Catcolab [6] implements a more dynamic modeling setting that includes **motif analysis** [1] for stock and flow diagrams, which is a strategy that could be generalized to identify the atoms of a Petri net. Together, these tools demonstrate the feasibility of implementing additive invariants for open Petri nets. Such an implementation would allow us to generate large synthetic Petri nets whose invariants satisfy certain constraints.

To that end, this paper focuses on one of the first, obvious requirements that one might put on an invariant so as to render its compositional computation efficient, that is, *additivity*. By this we mean that we are interested in invariants F which can be computed on any large Petri net Pas a sum $F(P) = \sum_{i=1}^{N} F(P_i)$ whenever P is built as a gluing of many small nets P_1, \ldots, P_N . In this paper we completely determine *all* of the additive invariants for composable Petri nets. Our results show that additivity forces the invariants to describe simple structural requirements. Thus one can think of a ladder of semantic complexity of isomorphism-invariant mappings on Petri nets where additive invariants occupy lower rungs compared to invariants such as mass action kinetics which is a map, defined in [3], from Petri nets into differential equations that respects gluing.

1.2 Contributions

In this paper, we will classify all N-valued invariants of open Petri nets which are additive with respect to composition and monoidal product in the category of open Petri nets, **OPetri**. In particular we show that the additive invariants of open Petri nets are completely determined by their values on a particular class of single-transition Petri nets.

We also give a similar classification for the N-valued invariants for the category of open Petri nets with monic legs, **MOPetri**, which embeds faithfully into **OPetri**. All additive invariants on **OPetri** nets are also additive invariants on **MOPetri**. However, the converse is not true. We show that the additive invariants on **MOPetri** are determined by their values on all single-transition Petri nets as well as transitionless Petri nets.

The classification of these additive invariants relies on two decomposition lemmas for open Petri nets. Given the large literature on open Petri nets in applied category theory, these lemmas are of independent significance.

The paper is structured as follows. In Section 2 we introduce the category of open Petri nets, **OPetri** and the category of open Petri nets with monic legs, **MOPetri**. We then introduce many of the notions that we use in the decomposition lemmas and in classifying invariants. These include particular classes of transitionless and single transition Petri nets. In Section 3 we give the decomposition lemmas. In Section 4 we show in Theorem 4.4 that in fact all invariants of open Petri nets are additive. Finally, we classify the additive invariants of open Petri nets in Theorem 4.7 and of monically open Petri nets in Theorem 4.12.

Acknowledgements We thank John Baez for leading and guiding this project. Furthermore, we would like to thank the American Mathematical Society for its support and the organisers of the AMS' 2022 MRC (Mathematics Research Communities) on Applied Category Theory for setting up this collaboration. Additionally, we thank the referees for a number of insightful remarks that led to the expansion of the discussion on related work, improvement in exposition, and a few simplifications.

2 Preliminaries

2.1 Open Petri nets

Open Petri nets are Petri nets equipped with sets of input and output species acting as *interfaces*. Formally, an **open Petri net** is a Petri net P and a cospan of sets $X \xrightarrow{i} S \xleftarrow{o} Y$ where S is the species set of the Petri net P. The pair $\mathcal{P} = (X \xrightarrow{i} S \xleftarrow{o} Y, P)$ is an open Petri net **decorated** by the Petri net P.

Baez and Pollard show [3, Theorem 6] that open Petri nets should be thought of as *morphisms* in a *symmetric monoidal category* **OPetri** of open Petri nets. The objects of this category are finite sets, the morphisms are isomorphism classes of open Petri nets, and composition — similarly to other cospan categories [12] — is roughly obtained by pushout. One should think of this as *gluing* Petri nets together along shared interfaces. We invite the reader to consult Baez and Pollard's paper [3, Theorem 6] for a thorough account on how to define the category **OPetri** by applying Fong's theory of decorated cospans [7].

The composition and tensor product of two open Petri nets are applied in many of the proofs throughout Sections 4 and 3. For clarity, we give their explicit definition here. Let

$$\mathcal{P} = \left(X \xrightarrow{i} S \xleftarrow{o} Y, (s, t \colon T \to \mathbb{N}^S) \right), \text{ and}$$
$$\mathcal{Q} = \left(Y \xrightarrow{i'} S' \xleftarrow{o'} Z, (s', t' \colon T' \to \mathbb{N}^{S'}) \right),$$

be two open Petri nets. To compose \mathcal{P} and \mathcal{Q} , first we form the pushout:



where the maps $\overline{i'}$ and \overline{o} are the canonical morphisms from S and S' to the pushout object $S +_Y S'$. Then the composite is the open Petri net

$$\mathcal{P}; \mathcal{Q} \coloneqq \left(X \xrightarrow{\overline{i'} \circ i} S +_Y S' \xleftarrow{\overline{o} \circ o'} Z, \ \left(\bar{s}, \bar{t} \colon T + T' \to \mathbb{N}^{S +_Y S'} \right) \right)$$

where the maps $\bar{s}, \bar{t}: T + T' \to \mathbb{N}^{S+_Y S'}$ are given by

$$\bar{s}(\tau)(\bar{\sigma}) = \begin{cases} \sum_{\sigma \in S \mid \bar{i'}(\sigma) = \bar{\sigma}} s(\tau)(\sigma) & \tau \in T \\ \sum_{\sigma' \in S' \mid \bar{\rho}(\sigma') = \bar{\sigma}} s'(\tau)(\sigma') & \tau \in T' \end{cases}$$
(3)

and

$$\bar{t}(\tau)(\bar{\sigma}) = \begin{cases} \sum_{\sigma \in S \mid \bar{i'}(\sigma) = \bar{\sigma}} t(\tau)(\sigma) & \tau \in T\\ \sum_{\sigma' \in S' \mid \bar{\rho}(\sigma') = \bar{\sigma}} t'(\tau)(\sigma') & \tau \in T' \end{cases}$$
(4)

The monoidal product \oplus in **OPetri** is defined on objects as their disjoint union. For two open Petri nets $\mathcal{P} = \left(X \xrightarrow{i} S \xleftarrow{o} Y, P = \left(s, t: T \to \mathbb{N}^S\right)\right)$ and $\mathcal{Q} = \left(X' \xrightarrow{i'} S' \xleftarrow{o'} Y', Q = \left(s', t': T' \to \mathbb{N}^{S'}\right)\right)$, their monoidal product is

$$\mathcal{P} \oplus \mathcal{Q} \coloneqq \left(X + X' \xrightarrow{i+i'} S + S' \xleftarrow{o+o'} Y + Y', \left(s + s', t + t' \colon T + T' \to \mathbb{N}^{S+S'} \right) \right),$$

where S + S' and T + T' are the disjoint unions of the species and transitions sets of \mathcal{P} and \mathcal{Q} , respectively. The map $s + s' \colon T + T' \to S + S'$ sends $\tau \in T$ to $s(\tau) \in S$, and $\tau' \in T'$ to $s'(\tau') \in S'$. Similarly, the map $t + t' \colon T + T' \to S + S'$ sends $\tau \in T$ to $t(\tau) \in S$, and $\tau' \in T'$ to $t'(\tau') \in S'$. In addition to invariants of open Petri nets we are also interested in invariants for a subclass of open Petri nets defined as follows.

Definition 2.1. A monically open Petri net — or mope net for short — is an open Petri nets whose underlying cospan consists of a pair of monomorphisms.

Since the composition of two mope nets is, again, a mope net, these open Petri nets form a category **MOPetri** which is a wide subcategory of **OPetri**. In **OPetri**, composing open Petri nets can identify two species in the decoration of one of the factors. For example, an input species and an output species of a transition may be identified so that they appear as a single catalyst in the composite. This identification is not possible in **MOPetri**. Instead, composition in **MOPetri** preserves the relation between each transition and its input and output species.

2.2 Transitionless open Petri nets

In several constructions throughout the main sections of this paper, we refer to Petri nets with no transitions. For ease of notation, we introduce the following definition.

Definition 2.2. For a finite set S, let 0_S denote the unique Petri net with S species and no transitions. Then the source and target maps are both the unique morphism $!: 0 \to \mathbb{N}^S$. Explicitly, $0_S := (S, 0, !, !)$.

A transitionless open Petri net is an open Petri net whose decoration is transitionless. These are generated from several building blocks as we show in the following lemma.

Lemma 2.3. Any transitionless open Petri net is generated via the tensor product and composite from the following (transitionless) open Petri nets:

$$\begin{split} \mu &:= (2 \rightarrow 1 \leftarrow 1, 0_1), \\ \eta &:= (0 \rightarrow 1 \leftarrow 1, 0_1), \\ \delta &:= (1 \rightarrow 1 \leftarrow 2, 0_1), \\ \epsilon &:= (1 \rightarrow 1 \leftarrow 0, 0_1). \end{split}$$

Proof. An open Petri net with no transitions is equivalent to a cospan in **FinSet**. By Lemma 3.6 of [8], cospans in **FinSet** are generated by the cospans underlying the morphisms μ , η , δ , and ϵ .

Of particular interest are two types of transitionless open Petri nets.

Definition 2.4. A boundary mope net is a mope net either of the form

$$\eta_{S} \coloneqq \left(0 \to S \xleftarrow{1_{S}} S, 0_{S} \right) \text{ or } \epsilon_{S} \coloneqq \left(S \xrightarrow{1_{S}} S \leftarrow 0, 0_{S} \right),$$

for a finite set S.



Figure 1: The boundary mope net ϵ_3 .

Remark 2.5. Note that the boundary mope nets η_1 and ϵ_1 are in fact the generators η and ϵ defined in Lemma 2.3. Additionally, there are several useful relations between these boundary mope nets.

$$\delta; \mu = \mathrm{id}_1,$$
$$\eta_2; \mu = \eta_1$$
$$\delta; \epsilon_2 = \epsilon_1$$

Furthermore for a finite set S, we have

$$\underbrace{\eta_1 \oplus \cdots \oplus \eta_1}_{|S| \text{ times}} = \eta_S$$

and

$$\underbrace{\epsilon_1 \oplus \cdots \oplus \epsilon_1}_{|S| \text{ times}} = \epsilon_S$$

2.3 Atomic Petri nets

Next we introduce classes of open Petri nets which will form the building blocks for our decomposition lemmas in Section 3.

Definition 2.6. An **atomic** Petri net is a Petri net with a single transition such that each species is connected to the transition as an input and/or as an output.

Definition 2.7. For integers m, n define $P_{m,n}$ to be the atomic Petri net whose transition has m distinct input species and n distinct output species.

Explicitly, $P_{m,n}$ has a single transition τ and m + n distinct species.

$$S = \{1, ..., m + n\}$$

with source and target maps

$$s(\tau)(i) = \begin{cases} 1 & i = 1, ..., m \\ 0 & i = m + 1, ..., m + n \end{cases}, \quad t(\tau)(i) = \begin{cases} 0 & i = 1, ..., m \\ 1 & i = m + 1, ..., m + n \end{cases}$$

Example 2.8. Figure 2 gives four examples of Petri nets with a single transition. The Petri nets in (a), (b), and (c) are all atomic. The Petri nets in Figure 2 (a) and (b) are not of the form $P_{m,n}$ for any m, n. In (a), this is because there is a species that is both an input species and an output species for the transition. We call this type of species a **catalyst** of the transition. On the other hand, (b) is not of the form $P_{m,n}$ because there is a species connected to the transition by two output arcs. Finally, (c) depicts the Petri net $P_{2,2}$. It has two distinct input species, each of which is connected to the transition by a single input arc. Likewise, it has two distinct output species, each of which is connected to the transition by a single output arc.

Finally the Petri net in (d) it not atomic because it contains a species that does not participate in the transition.

Each atomic Petri net has a type signature called a **body type** which we define below.

Definition 2.9. Let $P = (s, t : T \to \mathbb{N}^S)$ by any Petri net. For a transition $\tau \in T$, let $S_\tau \subseteq S$ be the subset of species which participate in the transition τ . So $S_\tau = \{\sigma \in S | s(\tau)(\sigma) \neq 0 \text{ or } t(\tau)(\sigma) \neq 0\}$. Then the transition $\tau \in T$ is represented by the multiset

$$\{[(s(\tau)(\sigma), t(\tau)(\sigma))] | \sigma \in S_{\tau}\}$$

of pairs of natural numbers $\mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$. We refer to this multiset as the **body type** of a transition τ .



Figure 2: (a) An atomic Petri net containing a catalyst. (b) An atomic Petri net with a single transition that has two output arcs connected to the same species. (c) The atomic Petri net $P_{2,2}$. (d) A non-atomic Petri net with a single transition that has a species that does not participate in the transition.

There is a 1-1 correspondence between body types and isomorphism classes of atomic Petri nets. Indeed, consider the body type $\beta = [(a_1, b_1), ..., (a_n, b_n)]$. Then, let P_β be an atomic Petri net with a transition τ and n distinct species $S = \{1, ..., n\}$ with source and target maps $s(\tau)(i) = a_i$, $t(\tau)(i) = b_i$. The Petri net P_β is a canonical representative of the isomorphism class of atomic Petri nets that correspond to body type β .

Remark 2.10. Note that $P_{m,n} = P_{\beta}$ for body type

$$\beta = [\underbrace{(1,0),\cdots,(1,0)}_{m\text{-times}},\underbrace{(0,1)\cdots,(0,1)}_{n\text{-times}}].$$

Definition 2.11. For natural numbers m, n, let $\mathcal{P}_{m,n}$ be the open Petri net decorated by $P_{m,n}$ and whose underlying cospan is the identity.

Likewise for body type β , let \mathcal{P}_{β} be the open Petri net decorated by P_{β} and whose underlying cospan is the identity. We call an open Petri net of this form a **body net**.

Figure 3 gives an example of the body net $\mathcal{P}_{1,1}$.

Remark 2.12. If \mathcal{P} is any open Petri net whose decoration has a single transition and whose underlying cospan is the identity, then \mathcal{P} is isomorphic to the monoidal product of a body net \mathcal{P}_{β} and an identity open Petri net.

3 Decomposition Lemmas

Petri nets in the wild are often quite complicated, with hundreds of transitions. In this section we first prove a decomposition theorem for open Petri nets which decomposes an open Petri net into transitionless and single-transition open Petri nets. One of the main advantages of this decomposition theorem is that it applies to mope nets as well as generic open Petri nets. Then we will prove Lemma 3.7 which gives a canonical decomposition of open Petri nets into transitionless open Petri nets decorated with Petri nets of the form $P_{m,n}$. This factorization involves both composition and the monoidal product.

We start first by showing that any open Petri net is canonically the composite of transitionless open Petri nets and an open Petri net whose underlying cospan is the identity.





Figure 3: The open atomic Petri net $\mathcal{P}_{1,1}$.

Lemma 3.1. Let \mathcal{P} be an open Petri net. Then \mathcal{P} can be decomposed as

$$\mathcal{P} = \mathcal{Q}; \mathcal{R}; \mathcal{Q}'$$

where Q and Q' are decorated by transitionless Petri nets and the legs of \mathcal{R} are identities. If \mathcal{P} is monic, then so are Q and Q'.

Proof. Let \mathcal{P} be the Petri net defined by

$$\mathcal{P} = \left(X \xrightarrow{i} S \xleftarrow{o} Y, P = \left(S, T, s, t \colon T \to \mathbb{N}^S \right) \right).$$

Consider the Petri nets

$$\mathcal{Q} = \left(X \xrightarrow{i} S \xleftarrow{1_S} S, 0_S \right),$$
$$\mathcal{R} = \left(S \xrightarrow{1_S} S \xleftarrow{1_S} S, P \right),$$
$$\mathcal{Q} = \left(S \xrightarrow{1_S} S \xleftarrow{1_S} S, P \right),$$

and

$$\mathcal{Q}' = \left(S \xrightarrow{1_S} S \xleftarrow{o} Y, 0_S\right).$$

Note that if \mathcal{P} is monic, then so are \mathcal{Q} and \mathcal{Q}' .

First, we show that $\mathcal{Q}; \mathcal{R} = (X \xrightarrow{i} S \xleftarrow{1_S} S, P)$. Consider the composition of cospans:



That the decoration is P follows straightforwardly from the explicit definition of the composition of open Petri nets given in Section 2.

Similarly $(\mathcal{Q};\mathcal{R});\mathcal{Q}'$ consists of the cospan $X \xrightarrow{i} S \xleftarrow{o} Y$ and the decoration P.

We are now ready to prove our first decomposition Lemma of open Petri nets into atomic factors.

Lemma 3.2. If \mathcal{P} is an open Petri net with N transitions, then

$$\mathcal{P} = \mathcal{Q}; \mathcal{G}_1; \mathcal{G}_2; \cdots; \mathcal{G}_N; \mathcal{Q}',$$

where Q and Q' are transitionless open Petri nets and each G_i satisfies:

• Its underlying cospan is the identity.

• Its decoration has a single transition.

Proof. By Lemma 3.1 is suffices to prove the theorem for open Petri nets whose underlying cospan is the identity. We do this by induction on N. Suppose \mathcal{P} is of the form

$$\mathcal{P} = (S \xrightarrow{1_S} S \xleftarrow{1_S} S, P = (S, T, s, t: T \to \mathbb{N}^S)),$$

where T has elements $\tau_1, \tau_2, \ldots, \tau_N$.

If N = 0 or N = 1 then the result is trivial. Now suppose that $N \ge 2$ and that the result holds for all open Petri nets with fewer than N transitions.

Define $T = \{\tau_1, ..., \tau_{N-1}\}$ and the Petri nets

$$\tilde{P} = (S, \tilde{T}, s|_{\tilde{T}}, t|_{\tilde{T}}), \quad G_N = (S, \{\tau_N\}, s|_{\{\tau_N\}}, t|_{\{\tau_N\}}).$$

Let \mathcal{G}_N be the open Petri net $(S \xrightarrow{1_S} S \xleftarrow{1_S} S, G_N)$. By construction \mathcal{G}_N satisfies the requisite criteria. Then $\mathcal{P} = (S \xrightarrow{1_S} S \xleftarrow{1_S} S, \tilde{P}); \mathcal{G}_N$. This result is a straightforward application of the definition of composing open Petri nets.

By the induction hypothesis there exist open Petri nets $\mathcal{G}_1, \dots, \mathcal{G}_{N-1}$ satisfying the criteria such that

$$(S \xrightarrow{1_S} S \xleftarrow{1_S} S, \tilde{P}) = \mathcal{G}_1; \cdots; \mathcal{G}_{N-1}.$$

Thus $\mathcal{P} = \mathcal{G}_1; \cdots; \mathcal{G}_{N-1}; \mathcal{G}_N.$

Remark 3.3. The order of the Petri nets \mathcal{G}_i depended on the choice of ordering of the transitions. Therefore, this decomposition is unique up to isomorphism and permutations of \mathcal{G}_i . Since composition involves pushouts, everything is unique up to isomorphism.

Remark 3.4. By Lemma 3.1, if \mathcal{P} is a mope net then its factors according to the decomposition in Lemma 3.2 are mope nets as well.

Remark 3.5. By Remark 2.12, each \mathcal{G}_i is the monoidal product of a body net and an identity open Petri net.



Figure 4: A mope net with three transitions.

Example 3.6. Consider the mope net $\mathcal{P} = \left(X \xrightarrow{i} S \xleftarrow{o} Y, P = (S, T, s, t: T \to \mathbb{N}^S)\right)$, where $S = \{A, B, C, D, E\}$ and $T = \{\alpha, \beta, \gamma\}$ shown in Figure 4. Let

$$\mathcal{Q} = \left(X \xrightarrow{i} S \xleftarrow{1_S} S, 0_S \right), \text{ and}$$
$$\mathcal{Q}' = \left(S \xrightarrow{1_S} S \xleftarrow{o} Y, 0_S \right).$$

For i = 1, 2, 3, let

$$\mathcal{G}_i = \left(S \xrightarrow{1_S} S \xleftarrow{1_S} S, P_i = \left(S, T_i, s | _{T_i}, t | _{T_i} \colon T_i \to \mathbb{N}^S \right) \right),$$

where $T_1 = \{\alpha\}, T_2 = \{\beta\}$, and $T_3 = \{\gamma\}$. Using Lemma 3.2, \mathcal{P} can be decomposed as follows

$$\mathcal{P} = \mathcal{Q}; \mathcal{G}_1; \mathcal{G}_2; \mathcal{G}_3; \mathcal{Q}'.$$

See Figure 5 for a depiction of this decomposition.

As an example of Remark 3.3, this decomposition does not depend on the order of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$. Composing them in a permuted order yields the same composite mope net.



Figure 5: The decomposition of the mope net depicted in Figure 4 into transitionless mope nets Q and Q' (farthest left and farthest right) and single-transition mope nets G_1 , G_2 , and G_3 (middle mope nets).

Next we state and prove our second decomposition Lemma of open Petri nets into $P_{n,m}$ factors.

Lemma 3.7. Any open Petri net \mathcal{P} can be factored as

$$\mathcal{Q}; (\mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_N); \mathcal{Q}',$$

where:

- Q and Q' are transitionless open Petri nets,
- \mathcal{G}_0 is an identity morphism in **OPetri**, and
- For $i = 1, \dots, N$, \mathcal{G}_i is a body net decorated with an atomic Petri net isomorphic to P_{m_i,n_i} .

Proof. First, we define the Petri nets that will be involved in our composite.

For each transition $\tau \in T$ we will define an open Petri net G_{τ} which has a unique input species for each input arc incoming to τ and a unique output species for each output arc outgoing from τ . In particular, the sets

$$I_{\tau} \coloneqq \sum_{\sigma \in S} s(\tau)(\sigma), \quad O_{\tau} \coloneqq \sum_{\sigma \in S} t(\tau)(\sigma),$$

represent the input and output arcs to the transition τ . Define $S_{\tau} := I_{\tau} + O_{\tau}$. Define the Petri net G_{τ} to have a single transition and species S_{τ} . Its source and target maps are defined so that there is a single input arc from each species in I_{τ} to the transition and there is a single output arc from the transition to each species in O_{τ} . Note that G_{τ} is isomorphic to $P_{|I_{\tau}|,|O_{\tau}|}$. Let \mathcal{G}_{τ} be the open Petri net

$$\left(S_{\tau} \xrightarrow{1_{S_{\tau}}} S_{\tau} \xleftarrow{1_{S_{\tau}}} S_{\tau}, G_{\tau} = (S_{\tau}, 1, s_{\tau}, t_{\tau})\right).$$

Next, let S_0 be the species in S which are neither the input species nor the output species of any transition. Let \mathcal{G}_0 be the identity morphism on S_0 .

Consider the monoidal product $\mathcal{G} = \mathcal{G}_0 \oplus (\bigoplus_{\tau \in T} \mathcal{G}_{\tau})$. Explicitly,

$$\mathcal{G} = \left(\tilde{S} \xrightarrow{1_{\tilde{S}}} \tilde{S} \xleftarrow{1_{\tilde{S}}} \tilde{S}, \left(G = \tilde{S}, T, \tilde{s}, \tilde{t} \right) \right),$$

where $\tilde{S} = S_0 + \sum_{\tau \in T} S_{\tau}$. Next we want to create transitionless open Petri nets Q and Q' which glue together the species in G which arose from the same species in the original Petri net P. We begin by defining a map $\tilde{h}: \tilde{S} \to S$ which maps each species of G to the species in P to which it corresponds.

For each transition τ , let $f_{\tau} \colon I_{\tau} \to S$ be the unique map such that for each $\sigma \in S$, f_{τ} is the constant σ map on $s(\tau)(\sigma)$. Likewise, let $g_{\tau}: O_{\tau} \to S$ be the unique map such that for each $\sigma \in S$, g_{τ} is the constant σ map on $t(\tau)(\sigma)$. Together these induce a map $h_{\tau} \coloneqq [f_{\tau}, g_{\tau}] \colon S_{\tau} \to S$. Let h_0 be the inclusion of S_0 into S. Then let $\tilde{h}: \tilde{S} \to S$ be the universal morphism induced by h_0 and h_{τ} for $\tau \in T$.

Finally, we define

$$\mathcal{Q} \coloneqq \left(X \xrightarrow{i} S \xleftarrow{h} \tilde{S}, 0_S \right), \quad \mathcal{Q}' \coloneqq \left(\tilde{S} \xrightarrow{h} S \xleftarrow{o} Y, 0_S \right).$$

We will show that $\mathcal{P} = \mathcal{Q}; \mathcal{G}; \mathcal{Q}'$. First consider $\mathcal{Q}; \mathcal{G}$. The composition of the underlying cospans is as follows:



The decoration of the composite is (S, T, s', t') where $s': T \to \mathbb{N}^S$ is defined by

$$s'(\tau)(\sigma) = \sum_{\tilde{\sigma}\in\tilde{S}|h(\tilde{\sigma})=\sigma} \tilde{s}(\tau)(\tilde{\sigma})$$

$$= \sum_{\tilde{\sigma}\in S_{0}|h_{0}(\tilde{\sigma})=\sigma} + \sum_{\upsilon\in T} \sum_{\tilde{\sigma}\in S_{\upsilon}|h_{\upsilon}(\tilde{\sigma})=\sigma} \tilde{s}(\tau)(\tilde{\sigma})$$

$$= \sum_{\tilde{\sigma}\in S_{\tau}|h_{\tau}(\tilde{\sigma})=\sigma} \tilde{s}(\tau)(\tilde{\sigma})$$

$$= \sum_{\tilde{\sigma}\in I_{\tau}|f_{\tau}(\tilde{\sigma})=\sigma} s_{\tau}(\tilde{\sigma}) + \sum_{\tilde{\sigma}\in O_{\tau}|g_{\tau}(\tilde{\sigma})=\sigma} s_{\tau}(\tilde{\sigma})$$

$$= \sum_{s(\tau)(\sigma)} 1 + \sum_{t(\tau)(\sigma)} 0$$

$$= s(\tau)(\sigma).$$
(5)

Note that Equation 5 follows from the fact that for $\tilde{s}(\tau)(\tilde{\sigma})$ is 0 unless $\tilde{\sigma} \in S_{\tau}$. Equation 6 follows from the fact that f_{τ} is the constant $\tilde{\sigma}$ map on $s(\tau)(\tilde{\sigma})$, and thus $f_{\tau}(\tilde{\sigma}) = \sigma$ if and only if $\tilde{\sigma} \in s(\tau)(\sigma).$

The above calculations prove that s' = s. An identical argument shows that t' = t as well. Therefore the composite $\mathcal{Q}; \mathcal{G}$ is decorated by the original Petri net P.

Finally, we must show that $(\mathcal{Q};\mathcal{G});\mathcal{Q}'$ is \mathcal{P} . First we examine the composite of their underlying cospans:



Note that $S +_{\tilde{S}} S = S$ because $h: \tilde{S} \to S$ is surjective. That the decoration of the composite is P again is a straightforward application of the definition of composing open Petri nets.

Example 3.8. Consider the open Petri net depicted in Figure 6.

Applying the Lemma 3.7 this Petri net can be decomposed as

$$\mathcal{Q}$$
;($\mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$); \mathcal{Q}

where \mathcal{G}_0 is decorated with a transitionless Petri net and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are decorated with the atomic Petri nets. Here, α has two input arcs and one output arc, so the atomic Petri net $P_{2,1}$ decorates \mathcal{G}_1 ; likewise, β has one input arc and one output arc, so $P_{1,1}$ decorates \mathcal{G}_2 ; and γ has one input arc and two output arcs, so $P_{1,2}$ decorates \mathcal{G}_3 . The decomposition is depicted in Figure 7.



Figure 6: An open Petri net with three transitions.



Figure 7: The decomposition of the open Petri net depicted in Figure 6 into atomic Petri nets as defined by Lemma 3.7. The decomposition is Q; $(\mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3)$; Q'. Graphically each \mathcal{G}_i is enclosed by a grey box and they are shown in numerical order from top to bottom.

4 Additive Invariants of Open Petri Nets

This section is devoted to the classification theorems of additive invariants for open Petri nets and monically open Petri nets. These are discussed in Sections 4.2 and 4.3, respectively. We begin by establishing that all functors from **OPetri** to **B** \mathbb{N} are monoidal. First a definition.

Definition 4.1. An additive invariant of open Petri nets is a monoidal functor **OPetri** \rightarrow **B** \mathbb{N} where **B** \mathbb{N} is the one-object category induced by the monoid (\mathbb{N} , +). The monoidal product of **B** \mathbb{N} is also given by +.

4.1 Every invariant of OPetri is additive

Before proving our main classification theorems, we first prove that all functors **OPetri** \rightarrow **B**N are in fact monoidal. Therefore our classification of monoidal functors **OPetri** \rightarrow **B**N is in fact a classification of *all* functors **OPetri** \rightarrow **B**N.

We remind the reader of the Eckmann-Hilton argument, which states that in any monoidal category, composition and the monoidal product coincide when restricted to endomorphisms of the unit object. As a consequence, for open Petri nets with empty interfaces, composition coincides with their direct sum. Specifically, if an open Petri net Q has an empty codomain, then: $Q \circ id_{\emptyset} = Q \oplus id_{\emptyset}$. Likewise if Q' has an empty domain. Then:

$$\mathcal{Q} \circ \mathcal{Q}' = (\mathcal{Q} \oplus \mathrm{id}_{\emptyset}) \circ (\mathrm{id}_{\emptyset} \oplus \mathcal{Q}') = (\mathcal{Q} \circ \mathrm{id}_{\emptyset}) \oplus (\mathrm{id}_{\emptyset} \circ \mathcal{Q}') = \mathcal{Q} \oplus \mathcal{Q}'.$$

This follows naturally from the interchange law of monoidal categories. With this in mind, we begin with the following Lemma.

Lemma 4.2. Let $F: \mathbf{OPetri} \to \mathbf{B}\mathbb{N}$ be a functor, and let \mathcal{P} be an open Petri net. If \mathcal{P} is transitionless, then $F(\mathcal{P}) = 0$.

Proof. By Lemma 2.3, any transitionless open Petri nets can be generated by the open Petri nets μ , η , δ , and ϵ . Therefore it is sufficient to show that F is trivial on these open Petri nets. Since F is functorial, $F(id_S) = 0$ for all species set S. Thus, $\delta; \mu = id_1$ implies that

$$0 = F(\delta; \mu) = F(\delta) + F(\mu).$$

Since $F(\delta)$ and $F(\mu)$ are natural numbers, this implies that $F(\delta) = 0$ and $F(\mu) = 0$.

It remains to show that F vanishes on $\eta = \eta_1$ and $\epsilon = \epsilon_1$ as well. From the standard relations $\eta_1 = \eta_2; \mu$ and $\epsilon_1 = \delta; \epsilon_2$ as in 2.5, we obtain

$$F(\eta_1;\epsilon_1) = F(\eta_2) + F(\mu) + F(\delta) + F(\epsilon_2)$$

= $F(\eta_2) + F(\epsilon_2)$
= $F(\eta_2;\epsilon_2).$

As composition and direct sum coincide when both open Petri nets have empty domain and codomain, and since $\eta_m; \epsilon_m$ decomposes as the direct sum $\eta_m; \epsilon_m = \bigoplus_{i=1}^m (\eta_1; \epsilon_1)$, it follows that

$$F(\eta_2;\epsilon_2) = 2F(\eta_1;\epsilon_1).$$

Since $(\mathbb{N}, +)$ is a cancellative monoid, this implies $0 = F(\eta_1; \epsilon_1) = F(\eta_1) + F(\epsilon_1)$. Therefore, $F(\eta) = 0$ and $F(\epsilon) = 0$, as required.

Remark 4.3. Notice that the result above holds for any cancellative monoid where the identity is the only element with an inverse. Further, the classification of invariants as a linear combination of generators relies on the monoid being abelian and cancellative.

Theorem 4.4. Every functor **OPetri** \rightarrow **B** \mathbb{N} is monoidal.

Proof. Let \mathcal{P} and \mathcal{Q} be the open Petri nets

$$\mathcal{P} = \left(X \xrightarrow{i} S \xleftarrow{o} Y, \ \left(s, t \colon T \to \mathbb{N}^S \right) \right)$$
$$\mathcal{Q} = \left(U \xrightarrow{i'} S' \xleftarrow{o'} V, \ \left(s', t' \colon T' \to \mathbb{N}^{S'} \right) \right).$$

Then

$$\mathcal{P} \oplus \mathcal{Q} = (\mathcal{P}; \mathrm{id}_Y) \oplus (\mathrm{id}_U; Q) = (\mathcal{P} \oplus \mathrm{id}_U); (\mathrm{id}_Y \oplus Q).$$

Therefore, it suffices to show for each open Petri net \mathcal{P} and each finite set M that $F(\mathcal{P} \oplus id_M) = F(\mathcal{P})$.

We begin by considering the open Petri net

$$\mathcal{R} \coloneqq (\eta_X \oplus \eta_M); (\mathcal{P} \oplus \mathrm{id}_M); (\epsilon_Y \oplus \epsilon_M),$$

whose domain and codomain are both \emptyset . Since composition and direct sum coincide in such cases and using the interchange law, we have

$$\mathcal{R} = (\eta_X; \mathcal{P}; \epsilon_Y) \oplus (\eta_M; \mathrm{id}_M; \epsilon_M), = (\eta_X; \mathcal{P}; \epsilon_Y); (\eta_M; \mathrm{id}_M; \epsilon_M).$$

It follows that,

$$F(\eta_{X+M}) + F(\mathcal{P} \oplus \mathrm{id}_M) + F(\epsilon_{Y+M}) = F(\mathcal{R}) = F(\eta_X) + F(\mathcal{P}) + F(\epsilon_Y) + F(\eta_M) + F(\epsilon_M).$$

This derivation uses that $\eta_X \oplus \eta_M = \eta_{X+M}$, and likewise that $\epsilon_Y \oplus \epsilon_M = \epsilon_{Y+M}$.

Finally, by Lemma 4.2, all of the boundary terms are 0, revealing that $F(\mathcal{P} \oplus \mathrm{id}_M) = F(\mathcal{P})$.

4.2 Classifying additive invariants of open Petri nets

We begin by proving the functoriality of the maps $F_{m,n}$: **OPetri** \rightarrow **B**N that take an open Petri net to the number of transitions having *m* input arcs and *n* output arcs. These will form the building blocks of our classification of additive invariants.

Lemma 4.5. There is a monoidal functor $F_{m,n}$: **OPetri** \to **B** \mathbb{N} such that for an open Petri net \mathcal{P} with decoration P, $F_{m,n}(\mathcal{P})$ is the number of transitions in P with m input arcs and n output arcs.

Proof. For an object S in **OPetri**, the identity open Petri net is $\operatorname{id}_{S} = \left(S \xrightarrow{1_{S}} S \xleftarrow{1_{S}} S, 0_{S}\right)$. Since 0_{S} has no transitions, we obtain $F_{m,n}\left(S \xrightarrow{1_{S}} S \xleftarrow{1_{S}} S, 0_{S}\right) = 0$.

Consider two open Petri nets $\mathcal{P} = (X \to S \leftarrow Y, P)$ and $\mathcal{P}' = (Y \to S' \leftarrow Z, P')$, such that Pand P' have T and T' as transition sets, respectively. Recall that the Petri net decorating $\mathcal{P}; \mathcal{P}'$ has transition set T + T'. For $\tau \in T$, suppose that τ has k input arcs in P. Equations 1 and 3 imply that τ has k input arcs in Petri net decorating $\mathcal{P}'; \mathcal{P}$. Likewise for output arcs and for $\tau' \in T'$. Therefore, $F_{m,n}(\mathcal{P}; \mathcal{P}') = F_{m,n}(\mathcal{P}') + F_{m,n}(\mathcal{P})$. Similarly, for open Petri nets \mathcal{P} and \mathcal{Q} , we have $F_{m,n}(\mathcal{P} \oplus \mathcal{Q}) = F_{m,n}(\mathcal{P}) + F_{m,n}(\mathcal{Q})$.

Example 4.6. Consider the open Petri net in Figure 6. The transition labeled α has two input arcs and one output arc, so $F_{2,1}$ applied to this open Petri net is 1. The transition β has one input arc and one output arc. Therefore, $F_{1,1}$ applied to this open Petri net is 1. Finally, the transition γ has one input arc and two output arcs; $F_{1,2}$ applied to this open Petri net is 1. For all other m, n, $F_{m,n}$ applied to these open Petri nets is 0.

We are now ready to present the proof of one of our main results, Theorem 4.7.

Theorem 4.7. For any two $m, n \in \mathbb{N}$, there is a functor

$$F_{m,n}: \mathbf{OPetri} \to \mathbf{BN}$$

which maps each open Petri net to the total number of transitions with exactly n input arcs and m output arcs. Furthermore, any additive invariant $G: \mathbf{OPetri} \to \mathbf{BN}$ is completely determined — as a linear combination of the functors $F_{m,n}$ defined above — by its values on the family $\mathcal{P}_{m,n}$. In particular, we have

$$G(-) = \sum_{m,n \in \mathbb{N}} G(\mathcal{P}_{m,n}) F_{m,n}(-)$$

Proof. In Lemma 4.5 we verified that the maps $F_{m,n}$: **OPetri** \to **B**N as defined in the first sentence of the Theorem statement are indeed monoidal functors. Next we show that any monoidal functor F: **OPetri** \to **B**N satisfies

$$F = \sum_{m,n \in \mathbb{N}} F(\mathcal{P}_{m,n}) F_{m,n}.$$

First, for the open Petri net $\mathcal{P}_{\bar{m},\bar{n}}$ observe that $\sum_{m,n\in\mathbb{N}} F(\mathcal{P}_{m,n})F_{m,n}(\mathcal{P}_{\bar{m},\bar{n}}) = F(\mathcal{P}_{\bar{m},\bar{n}})$ because $F_{m,n}(\mathcal{P}_{\bar{m},\bar{n}})$ is 1 if $m = \bar{m}$ and $n = \bar{n}$ and 0 otherwise.

Now let $\mathcal{P} = (X \to S \leftarrow Y, P)$ be any open Petri net. Let \mathcal{Q} ; $\left(\bigoplus_{i=0}^{N} \mathcal{G}_{i}\right)$; \mathcal{Q}' be the decomposition of \mathcal{P} as given in Lemma 3.7. Thus, using this decomposition of \mathcal{P} , we can invoke the additivity of F together with Lemma 4.2 to obtain

$$F(\mathcal{P}) = F(\mathcal{Q}) + F(\mathcal{G}_0) + \sum_{i=1}^{N} F(\mathcal{G}_i) + F(\mathcal{Q}') = \sum_{i=1}^{N} F(\mathcal{G}_i).$$

Likewise $\sum_{m,n} F(\mathcal{P}_{m,n}) F_{m,n}$ is an additive invariant and so

$$\sum_{m,n} F(\mathcal{P}_{m,n}) F_{m,n}(\mathcal{P}) = \sum_{i=1}^{N} \sum_{m,n} F(\mathcal{P}_{m,n}) F_{m,n}(\mathcal{G}_i)$$

For i = 1, ..., N, \mathcal{G}_i is decorated with an atomic Petri net P_{m_i,n_i} . Therefore, \mathcal{G}_i is isomorphic to \mathcal{P}_{m_i,n_i} and hence $F(\mathcal{G}_i) = \sum_{m,n} F(\mathcal{P}_{m,n}) F_{m,n}(\mathcal{G}_i)$. Thus, as desired, we have that

$$\sum_{m,n} F(\mathcal{P}_{m,n})F_{m,n}(\mathcal{P}) = \sum_{i=1}^{N} \sum_{m,n} F(\mathcal{P}_{m,n})F_{m,n}(\mathcal{G}_i) = \sum_{i=1}^{N} F(\mathcal{G}_i) = F(\mathcal{P}).$$

Classifying additive invariants of monically open Petri nets

In this section, we show that the invariants of mope nets are linear combinations of some countable generating sets. These play the role of $F_{m,n}$ in the classification of additive invariants for all open Petri nets. In this case, the generators correspond to the two subclasses of mope nets which we have already encountered, namely *body nets* and *boundary nets*, which were defined in Definitions 2.9 and 2.4, respectively.

Our first set of generating functors consist of functors F_{β} for body types β . These are defined in Lemma 4.8 and capture invariants of body nets. The second set of generating functors consist of functors $F_{a,z}$ where $\{a_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$ are non-decreasing sequences satisfying the law in Equation 7. These are defined in Lemma 4.9 and capture invariants having to do with the underlying cospan of the open Petri net.

Lemma 4.8. Let β be a body type. There is a monoidal functor F_{β} : **MOPetri** \rightarrow **B** \mathbb{N} such that for a more net \mathcal{M} with decoration P, $F_{\beta}(\mathcal{M})$ is the number of transitions in P with type β .

Proof. That F_{β} is 0 on the identity and respects the monoidal product follows the same reasoning as in the proof of Lemma 4.5.

Consider two mope nets $\mathcal{M} = (X \to S \leftarrow Y, P)$ and $\mathcal{M}' = (Y \to S' \leftarrow Z, P')$ such that P and P' have T and T' as transition sets, respectively. Recall that the Petri net decorating $\mathcal{M}; \mathcal{M}'$ has transition set T + T'. For $\tau \in T$, suppose that τ has body type β . Since the legs of \mathcal{M} and \mathcal{M}' are monic, no two input and/or output species to τ are identified. Thus, τ also has body type β in the

4.3

Petri net decorating $\mathcal{M}; \mathcal{M}'$. Likewise if τ does not have body type β in the Petri net decorating $\mathcal{M},$ then τ does not have body type β in the Petri net decorating $\mathcal{M}; \mathcal{M}'$. The same is true for transitions in the Petri net decorating \mathcal{M}' . Therefore $F_{\beta}(\mathcal{M}; \mathcal{M}') = F_{\beta}(\mathcal{M}) + F_{\beta}(\mathcal{M}')$.

Lemma 4.9. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$ be non-decreasing sequences of natural numbers that satisfy

$$k(a_1 + z_1) = a_k + z_k \tag{7}$$

for all $k \in \mathbb{N}$. Then there is a functor $F_{a,z}$: **MOPetri** \to **B** \mathbb{N} defined such that for a more net $\mathcal{M} = (X \to S \leftarrow Y, P)$

$$F_{a,z}(\mathcal{M}) = (a_{|S|} - a_{|X|}) + (z_{|S|} - z_{|Y|}).$$

Proof. First, $F_{a,z}$ is not only integer-valued but in fact natural-valued, since a, z are non-decreasing. And by direct computation $F_{a,z}(id_S) = (a_{|S|} - a_{|S|}) + (z_{|S|} - z_{|S|}) = 0.$

We now verify that $F_{a,z}$ respects composition. Consider two mope nets $\mathcal{M} = (X \to S \leftarrow Y, P)$ and $\mathcal{M}' = (Y \to S' \leftarrow Z, P')$. By the monotonicity of the legs of \mathcal{M} and \mathcal{M}' , there are |S|+|S'|-|Y|species in the decoration of \mathcal{M} ; \mathcal{M}' . Therefore,

$$\begin{aligned} F(\mathcal{M}) + F(\mathcal{M}') &= (a_{|S|} - a_{|X|} + z_{|S|} - z_{|Y|}) + (a_{|S'|} - a_{|Y|} + z_{|S'|} - z_{|Z|}) \\ &= (|S| + |S'| - |Y|)(a_1 + z_1) - a_{|X|} - z_{|Z|} \\ &= (a_{|S| + |S'| - |Y|} + z_{|S| + |S'| - |Y|}) - a_{|X|} - z_{|Z|} \\ &= (a_{|S| + |S'| - |Y|} - a_{|X|}) + (z_{|S| + |S'| - |Y|} - z_{|Z|}) \\ &= F(\mathcal{M}; \mathcal{M}'). \end{aligned}$$

Hence, $F_{a,z}$ is functorial.

We next turn to show that any functor must satisfy a certain law and the value of this functor on transitionless mope net is determined by its value on boundary mope nets.

Lemma 4.10. Let $F: \mathbf{MOPetri} \to \mathbf{BN}$ be a functor and let $\mathcal{M} = (X \to S \leftarrow Y, 0_S)$ be a transitionless mope net. Then the following holds

(a) $k(F(\eta_1) + F(\epsilon_1)) = F(\eta_k) + F(\epsilon_k)$ for all natural k. (b) $F(\mathcal{M}) = (F(\eta_S) - F(\eta_X)) + (F(\epsilon_S) - F(\epsilon_Y)).$

Proof. The composite mope net

$$\eta_{|X|}; \mathcal{M}; \epsilon_{|Y|} = (0 \to S \leftarrow 0, 0_S)$$

is isomorphic to $(\eta_1; \epsilon_1)$ composed with itself |S| times. Therefore,

$$|S|(F(\eta_1) + F(\epsilon_1)) = F(\eta_X) + F(\mathcal{M}) + F(\epsilon_Y).$$
(8)

Applying Equation 8 to the case where \mathcal{M} is the identity mope net on S and |S| = k proves the first assertion that $k(F(\eta_1) + F(\epsilon_1)) = F(\eta_k) + F(\epsilon_k)$.

Finally to prove (b), applying the equality $|S|(F(\eta_1) + F(\epsilon_1)) = F(\eta_S) + F(\epsilon_S)$ to Equation 8 and rearranging implies that $F(\mathcal{M}) = (F(\eta_S) - F(\eta_X)) + (F(\epsilon_S) - F(\epsilon_Y))$.

We will next prove a crucial result that is similar in nature to Theorem 4.4. Note that the proof technique in Theorem 4.4 (specifically Lemma 4.2) cannot be extended to mope nets since it requires the use of the non-monic open Petri nets μ and δ .

Lemma 4.11. Let $F: \mathbf{MOPetri} \to \mathbf{BN}$ be a functor. Let \mathcal{M} be any endomorphism in $\mathbf{MOPetri}$ and let id_S be the identity more net with S species. Then $F(\mathcal{M} \oplus id_S) = F(\mathcal{M})$.



Proof. Let \mathcal{M} be a mope net with domain and codomain X. Define

$$\mathcal{P} \coloneqq (\eta_X \oplus \eta_S); (\mathcal{M} \oplus \mathrm{id}_S); (\epsilon_X \oplus \epsilon_S)$$

Note that $\eta_X \oplus \eta_S = \eta_{X+S}$ and likewise $\epsilon_X \oplus \epsilon_S = \epsilon_{X+S}$. Applying F shows that

$$F(\mathcal{P}) = F(\eta_{|X+S|}) + F(\mathcal{M} \oplus \mathrm{id}_S) + F(\epsilon_{|X+S|}) = |X+S|(F(\eta_1) + F(\epsilon_1)) + F(\mathcal{M} \oplus \mathrm{id}_S).$$
(9)

The second equality follows from the equality proven in Lemma 4.10.

On the other hand, rearranging the factors of \mathcal{P} yields

$$\mathcal{P} = (\eta_X; \mathcal{M}; \epsilon_X) \oplus (\eta_S; \mathrm{id}_S; \epsilon_S).$$

The monoidal product of two open Petri nets whose feet are both the empty set is isomorphic to their composite. Since the two factors of \mathcal{P} satisfy this criteria, we have

$$\mathcal{P} = (\eta_X; \mathcal{M}; \epsilon_X); (\eta_S; \mathrm{id}_S; \epsilon_S)$$

Applying F to this equality yields

$$F(\mathcal{P}) = F(\eta_X) + F(\mathcal{M}) + F(\epsilon_X) + F(\eta_X) + F(\mathrm{id}_S) + F(\epsilon_S) = |X + S|(F(\eta_1) + F(\epsilon_1)) + F(\mathcal{M}).$$
(10)

Again, the second equality follows from applying the equality proven in Lemma 4.10 twice. Comparing Equation 9 and Equation 10 and cancelling in \mathbb{N} , we establish the claim.

We are now ready for the main result of this section.

Theorem 4.12. Every functor $F: \mathbf{MOPetri} \to \mathbf{B}\mathbb{N}$ decomposes as

$$F_{a,z} + \sum_{\beta} d_{\beta} F_{\beta}$$

as β ranges over body types for coefficients $d_{\beta} \in \mathbb{N}$ and non-decreasing sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{z_k\}_{k \in \mathbb{N}}$ satisfying the condition in Equation 7.

Proof. The functors have been introduced in Lemma 4.8 and Lemma 4.9.

Now let $F: \mathbf{MOPetri} \to \mathbf{BN}$ be any functor. Define sequences $a_k = F(\eta_k)$ and $z_k = F(\epsilon_k)$ along with coefficients $d_\beta = F(\mathcal{P}_\beta)$.

First we show that the sequences a and z are non-decreasing. Note that

$$\eta_{k+1} = \eta_k; (k \to k+1 \leftarrow k+1, 0_{k+1})$$

Applying F to both sides and appealing to functoriality implies that $a_{k+1} = F(\eta_{k+1})$ is greater than or equal to $a_k = F(\eta_k)$. Therefore, the sequence a is non-decreasing and likewise for z. Furthermore, they also satisfy the condition in Equation 7 by Lemma 4.10.

Let $\mathcal{M} = (X \to S \leftarrow Y, P)$ be any mope net. By Corollary 3.1, $\mathcal{M} = \mathcal{Q}; \mathcal{G}_1; \dots; \mathcal{G}_n; \mathcal{Q}'$ where \mathcal{Q} and \mathcal{Q}' are transitionless mope nets and the \mathcal{G}_i are body nets. Since \mathcal{Q} and \mathcal{Q}' are transitionless, $F_{\beta}(\mathcal{Q}) = 0$ and $F_{\beta}(\mathcal{Q}') = 0$ for all body types β . Therefore, functoriality of F, $F_{a,z}$, and the F_{β} implies that it suffices to show that $F(\mathcal{Q}) + F(\mathcal{Q}') = F_{a,z}(\mathcal{Q}) + F_{a,z}(\mathcal{Q}')$ and that F and $F_{a,z} + \sum_{\beta} d_{\beta} F_{\beta}$ agree on \mathcal{G}_i for all i.

The proof of Lemma 3.1 in fact shows that $\mathcal{Q} = (X \to S \xleftarrow{1_S} S, 0_S)$ and $\mathcal{Q}' = (S \xrightarrow{1_S} S \leftarrow Y, 0_S)$. Note that $F(\mathcal{Q}; \mathcal{Q}') = F(X \to S \leftarrow Y, 0_S)$. By Lemma 4.10, this equals

$$(F(\eta_S) - F(\eta_X)) + (F(\epsilon_S) - F(\epsilon_Y))$$

which by definition of a and z equals $(a_{|S|} - a_{|X|}) + (z_{|S|} - z_{|Y|})$. Therefore,

$$F(\mathcal{Q}) + F(\mathcal{Q}') = F(\mathcal{Q}; \mathcal{Q}')$$

= $(a_{|S|} - a_{|X|}) + (z_{|S|} - z_{|Y|})$
= $F_{a,z}(\mathcal{Q}) + F_{a,z}(\mathcal{Q}').$

For each i, let τ_i be the single transition of \mathcal{G}_i and let β_i be the body type of τ_i . Let S_i be the species in the decoration of \mathcal{G}_i which are neither source nor target species of τ_i . Then $\mathcal{G}_i = \mathcal{P}_{\beta_i} \oplus \mathbb{1}_{S_i}$. In other words, since each \mathcal{G}_i is body mope net, it is isomorphic to the monoidal product of an irreducible body net and an identity mope net. Recall that the canonical irreducible body nets \mathcal{P}_{β} are defined in Definition 2.9. By Lemma 4.11 any functor **MOPetri** $\to \mathbb{B}\mathbb{N}$ agree on \mathcal{G}_i and \mathcal{P}_{β_i} .

Since F and F_{β_i} are 0 on the identity and $F_{\beta_i}(\mathcal{P}_{\beta_i}) = 1$ we have,

$$F(\mathcal{G}_i) = F(\mathcal{P}_{\beta_i}) = F(\mathcal{P}_{\beta_i})F_{\beta_i}(\mathcal{P}_{\beta_i}) = F(\mathcal{P}_{\beta_i})F_{\beta_i}(\mathcal{G}_i).$$

For body types $\beta \neq \beta_i$,

$$F_{\beta}(\mathcal{G}_i) = F_{\beta}(\mathcal{P}_{\beta_i}) = 0.$$

Finally, the underlying cospan of \mathcal{G}_i is the identity, so $F_{a,z}(\mathcal{G}_i) = 0$. All together, this implies that F and $F_{a,z} + \sum_{\beta} d_{\beta} F_{\beta}$ agree on \mathcal{G}_i .

5 Final Remarks

This paper presents an in-depth study of additive invariants for open Petri nets, conceptualized as a monoidal functor from a symmetric monoidal cospan category, **OPetri**, to **B** \mathbb{N} . Our research makes significant strides in both the development and classification of all \mathbb{N} -valued additive invariants.

We have shown that all functors from **OPetri** to \mathbb{BN} are monoidal, and thus additive invariants. Our study characterizes these invariants for both general open nets in **OPetri** and the subcategory **MOPetri**, which comprises those open Petri nets whose underlying cospan has monomorphic legs. Further, we found that the additive invariants are completely characterized by their values on specific classes of Petri nets: for general open Petri nets, the generators are single-transition nets, whereas for **MOPetri**, they include both all single-transition and transitionless Petri nets. In exploring the broader applicability for our main classification theorems, we identify a special subclass of cancellative monoids, characterized by the unique property that the identity is the sole element with an inverse.

The classification of these additive invariants relies strongly on key decomposition lemmas for open Petri nets, which assert that any open Petri net can be canonically factorized, using tensor or composite operations, into single-transition and transitionless open Petri nets. These Lemmas are of independent interest given the significant literature on open Petri nets in applied category theory.

In this study, we focused on the invariants of open Petri nets with the codomain $\mathbf{B}\mathbb{N}$. The classification problem of functors out of **OPetri**, however seems very challenging in general. Despite the progress made in this paper, several avenues remain open for future research, particularly in exploring invariants within different monoids. Investigating invariants beyond numerical values, such as $\mathbb{Z}/2$ -valued of open Petri nets, raises further inquiry into whether such invariants could provide new insights not captured by \mathbb{N} -valued invariants, similar to how $\mathbb{Z}/2$ -valued co/homology invariants in topology reveal aspects not visible in \mathbb{Z} -valued invariants.

Furthermore, our exploration of additive invariants confirms their role as compositional semantics for Petri nets. Mass action kinetics, while a highly descriptive invariant on open Petri nets, are computationally impractical for verification and specification. Conversely the relative simplicity of the additive invariants presented in this paper, make them particularly practical for developing domain-specific Petri nets. An additive invariant can represent constraints either imposed by domain experts or discovered empirically through real-world applications. The compositional nature of these invariants is critical for constraint verification and for modularly constructing networks to fulfill specific constraint requirements. These features are well-suited for integration into the software packages AlgebraicPetri.jl and Catcolab [6]. Although not currently implemented in full, we have developed a working prototype. A full implementation would involve (1) a decomposition for open Petri nets into atomic components and (2) a mechanism to define additive invariants by assigning values to each atomic net. To sum up, this paper promote for a more algebraic and compositional understanding of Petri nets. It lays the groundwork of valuable theoretical insights with potential applications in computational biology, computer science, network theory, and systems design with a deeper understanding of complex systems.

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