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Universal pseudomorphisms, with applications to diagrammatic coherence for braided and symmetric monoidal functors

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This work introduces a general theory of universal pseudomorphisms and develops their connection to diagrammatic coherence. The main results give hypotheses under which pseudomorphism coherence is equivalent to the coherence theory of strict algebras. Applications include diagrammatic coherence for plain, symmetric, and braided monoidal functors. The final sections include a variety of examples.

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1 Introduction

The main results of this paper are coherence theorems for pseudomorphisms between algebras over a 2-monad T. For example, T may be the 2-monad for plain, symmetric, or braided monoidal categories. Coherence theorems for pseudomorphisms are, in these cases, coherence theorems for plain, symmetric, or braided strong monoidal functors. Our main interest is what we call *diagrammatic* coherence: general conditions that guarantee commutativity of (formal) diagrams.

Example 1.1. Consider the following diagram (1.2) for a braided monoidal functor

$$f\colon (A, +, \beta) \longrightarrow (A', \bullet, \beta),$$

where A and A' are braided strict monoidal categories with braid isomorphisms β and monoidal products $+$ and $\bullet$, respectively. The two composites around the diagram apply different combinations of braidings β and monoidal constraints f_2.

$$\begin{tikzcd}
f(a) \bullet f(a) \bullet f(a) \arrow[r, "f_2 \bullet 1"] \arrow[d, "f_2"'] & f(a+a) \bullet f(a) \arrow[r, "\beta"] & f(a) \bullet f(a+a) \arrow[d, "f_2"] \\
f(a+a+a) \arrow[r, "f(1+\beta)"] & f(a+a+a) \arrow[r, "f(\beta+1)"] & f(a+a+a)
\end{tikzcd} \tag{1.2}$$

This diagram satisfies a condition called *formal* because it is determined entirely by the monoidal functor and braided monoidal category data of f, A, and A'.

Our diagrammatic coherence determines commutativity of formal diagrams like (1.2) by converting them to different—often simpler—formal diagrams that do not depend on the structure morphisms f_2. The latter are called *dissolution diagrams*, and the dissolution diagram for (1.2) is given as follows. (See Example 15.1 for further explanation.)

$$\begin{tikzcd}
\big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big) \arrow[r, "1"] \arrow[d, "1"'] & \big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big) \arrow[r, "\beta_{(\,f(a),f(a)\,)\,,\,f(a)}"] & \big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big) \arrow[d, "1"] \\
\big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big) \arrow[r, "(\,1\,,\,\beta\,)"] & \big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big) \arrow[r, "(\,\beta\,,\,1\,)"] & \big(\, f(a)\,,\, f(a)\,,\, f(a) \,\big)
\end{tikzcd}$$

The composites around the above diagram have the same underlying braid, drawn in the center, and hence the diagram commutes in the free braided monoidal category on the object $f(a)$. Our main result, Theorem 1.5, shows that the original diagram (1.2) therefore commutes in A'. ◇

In the particular example above, one can also use naturality of f_2 along with axioms for f and β to determine commutativity of (1.2) directly. Indeed, every formal diagram for f is amenable to such an approach. The purpose of the diagrammatic coherence results in this work is to provide a general theory that eliminates the need to determine, for each diagram, *which* combination of axioms is necessary.

General Overview

Suppose $\mathcal{K}$ is a 2-category and T is a 2-monad on $\mathcal{K}$. Under the respective hypotheses, our results reduce coherence for T-algebra pseudomorphisms, such as f in Example 1.1 above, to that of individual T-algebras, such as the free algebra on $f(a)$ in Example 1.1. Indeed, the conclusions of Theorems 1.5 and 1.9 are that coherence for T-algebra pseudomorphisms is *equivalent* to that of T-algebras, in the following sense.

Assuming the hypotheses of Theorems 1.5 and 1.9, each 1-cell

$$\phi\colon C \longrightarrow C' \quad \text{in} \quad \mathcal{K}$$

has an associated T-algebra $\mathsf{T}(C', \phi)$ and a *universal pseudomorphism*

$$\widetilde{\phi}\colon \mathsf{T}C \longrightarrow \mathsf{T}(C', \phi) \tag{1.3}$$

together with an equivalence of T-algebras

$$\Delta\colon \mathsf{T}(C', \phi) \xrightarrow{\simeq} \mathsf{T}C'. \tag{1.4}$$

The universality of $\widetilde{\phi}$ and the construction of Δ are explained in Section 6.

In the case $\mathcal{K} = \mathcal{C}at$, the 2-category of small categories, the universality of $\widetilde{\phi}$ gives a notion of *formal diagrams* for a T-algebra pseudomorphism f. Then, the equivalence (1.4) means that diagrams in $\mathsf{T}(C', \phi)$ commute if and only if the corresponding diagrams in $\mathsf{T}C'$ commute. Thus, the universality of $\widetilde{\phi}$ and the equivalence Δ are the essential technical channels by which the coherence theory for pseudomorphisms reduces to that of T-algebras.

Main applications

We provide three statements of main results. The first, Theorem 1.5, is the simplest. It is formulated using overly-broad hypotheses that nevertheless hold in many applications of interest. It follows as a special case of our third statement, Theorem 1.9 below. Recall that a 2-monad T is *finitary* if it preserves all filtered colimits.

Theorem 1.5 (Finitary Pseudomorphism Coherence). *Suppose T is a finitary 2-monad on a 2-category $\mathcal{K}$ that is both complete and cocomplete. Then T admits universal pseudomorphisms*

$$\widetilde{\phi}\colon \mathsf{T}C \longrightarrow \mathsf{T}(C', \phi) \quad \textit{for} \quad \phi\colon C \longrightarrow C' \quad \textit{in} \quad \mathcal{K}$$

such that, for each ϕ, the induced strict morphism of T-algebras (6.25)

$$\Delta\colon \mathsf{T}(C', \phi) \longrightarrow \mathsf{T}C' \tag{1.6}$$

is a surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ (Definition 4.6).

Our second statement of main results, Theorem 1.7, is an application with $\mathcal{K} = \mathcal{C}at$, the 2-category of small categories. We explain some notation, terminology, and motivation before stating Theorem 1.7. The hypotheses of Theorem 1.5 hold when T is one of the three 2-monads $\{\mathsf{M}^{\mathsf{g}}, \mathsf{S}^{\mathsf{g}}, \mathsf{B}^{\mathsf{g}}\}$ for plain, symmetric, or braided monoidal structure on categories (Notation 11.1). In this notation, the superscript g indicates general monoidal structure, in contrast to the strictly associative and unital structure that we will later discuss. In these cases we have the following:

- In the plain monoidal case $\mathsf{T} = \mathsf{M}^{\mathsf{g}}$, an M^{g}-algebra is a monoidal category and a pseudomorphism is a strong monoidal functor.
- In the symmetric case $\mathsf{T} = \mathsf{S}^{\mathsf{g}}$, an S^{g}-algebra is a symmetric monoidal category and a pseudomorphism is a symmetric strong monoidal functor.
- In the braided case $\mathsf{T} = \mathsf{B}^{\mathsf{g}}$, a B^{g}-algebra is a braided monoidal category and a pseudomorphism is a braided strong monoidal functor.

The statement of Theorem 1.7 uses the following terms explained further in Section 10.

- A *diagram* in a T-algebra X' is a pair $(\mathbb{D}, D)$ consisting of a small category $\mathbb{D}$ and a functor

  $$\mathbb{D} \xrightarrow{D} X'$$

  in $\mathcal{C}at$.

- A *formal diagram for a pseudomorphism* f is a diagram that lifts through a canonical strict morphism of T-algebras defined in (10.3):

  $$\Lambda \colon \mathsf{T}(\mathsf{ob}X', \phi) \longrightarrow X',$$

  where $\phi = f_{\mathsf{ob}}$ denotes the restriction of f to objects.

- Each formal diagram $(\mathbb{D}, D)$ for f has a *dissolution* diagram in the free algebra $\mathsf{T}(\mathsf{ob}X')$:

  $$\mathbb{D} \xrightarrow{|D|} \mathsf{T}(\mathsf{ob}X'),$$

  obtained by composing with Δ (1.6).

The dissolution diagram $|D|$ is generally simpler that the original diagram D. Indeed, for $\mathsf{T} \in \{\mathsf{M}^{\mathsf{g}}, \mathsf{S}^{\mathsf{g}}, \mathsf{B}^{\mathsf{g}}\}$, Explanation 14.13 (*iv*) shows that Δ sends monoidal and unit constraints of f to *identities* in $\mathsf{T}(\mathsf{ob}X')$. Example 1.1 from the beginning of this introduction shows a specific formal diagram (1.2) followed by its corresponding dissolution diagram.

In general, we have the following by Theorem 1.5.

Theorem 1.7 (Strong Monoidal Functor Coherence). *Suppose* T *is one of the three 2-monads* $\{\mathsf{M}^{\mathsf{g}}, \mathsf{S}^{\mathsf{g}}, \mathsf{B}^{\mathsf{g}}\}$ *for plain, symmetric, or braided monoidal structure on* $\mathcal{K} = \mathcal{C}at$*. Suppose given* T*-algebras* X *and* X'*, together with a* T*-algebra pseudomorphism*

$$f \colon X \longrightarrow X'$$

and a diagram

$$\mathbb{D} \xrightarrow{D} X'.$$

If $(\mathbb{D}, D)$ *is a formal diagram for* f *such that the dissolution* $|D|$ *commutes in* $\mathsf{T}(\mathsf{ob}X')$*, then the diagram* $(\mathbb{D}, D)$ *commutes in* X'*.*

The assertions of Theorem 1.7 may be summarized informally as follows.

Slogan 1.8. *In the cases* $\mathsf{T} \in \{\mathsf{M}^{\mathsf{g}}, \mathsf{S}^{\mathsf{g}}, \mathsf{B}^{\mathsf{g}}\}$*, commutativity of formal diagrams for* f *reduces to checking commutativity of the simpler dissolution diagrams, in which the monoidal and unit constraints of* f *are replaced by identities.* ◇

The definitions of $\mathsf{T}(\mathsf{ob}X', \phi)$, Λ, and Δ explain precisely how such a replacement of monoidal and unit constraints can be done. We give a variety of detailed examples and further discussion in Sections 15 and 16. The interested reader is invited to skip forward for additional motivation, and then back to the relevant definitions and constructions as needed.

Main technical result

Our third statement of main results, Theorem 1.9, is the most general and technical. It identifies more precisely how the different features of our work rely on a collection of interrelated hypotheses. In particular, Theorem 1.9 states explicitly how the existence of universal pseudomorphisms $\widetilde{\phi}$ (1.3) relates to existence of a *pseudomorphism classifier* Q for the 2-monad T. Sections 4 and 5 review those aspects of pseudomorphism classifiers that will be necessary in this work.

A pseudomorphism classifier can arise under various hypotheses, e.g., those discussed in [BKP89, Pow89, Lac02]. One aim of our treatment is to explore the relationship between existence of a pseudomorphism classifier Q, however it may arise, and existence of universal pseudomorphisms $\widetilde{\phi}$.

The proof of Theorem 1.9 is included here. It combines the essential results from the technical heart of this work, and serves as a high-level summary. Here, we use the following notation.

- $\mathsf{T}\text{-}\mathsf{Alg}$ and $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ denote the 2-categories of T-algebras with pseudomorphisms and strict morphisms, respectively, (Definition 2.15).
- $\mathbb{2}$ and $\mathbb{I}$ denote the small categories consisting of two objects and a single nonidentity morphism, respectively isomorphism (Notation 3.5).
- For each $C \in \mathcal{K}$, we write $\mathsf{P}C = \mathsf{T}(C, 1_C)$ (Definition 9.5).

Further review of 2-monads, and of the limits and colimits necessary for this work, is given in Sections 2 and 3.

Theorem 1.9 (Pseudomorphism Coherence). *Suppose* T *is a 2-monad on a 2-category* $\mathcal{K}$ *and suppose that*

(1) $\mathcal{K}$ *admits pseudolimits of 1-cells;*

(2) $\mathcal{K}$ *admits cotensors*

  (a) *of the form* $\{\mathbb{2}, C\}$ *for* $C \in \mathcal{K}$ *and*
  (b) *of the form* $\{\mathbb{I}, C\}$ *for* $C \in \mathcal{K}$*;*

(3) $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ *admits pushouts; and*

(4) $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ *admits coequalizers of* P*-free pairs (Definition 9.13).*

Then the following two conditions are equivalent.

(A) T *admits a pseudomorphism classifier* $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$*.*

(B) T *admits universal pseudomorphisms* $\widetilde{\phi}$*.*

Moreover, in this case, the following hold for each T*-algebra* Y *and each 1-cell* $\phi\colon C \longrightarrow C'$ *in* $\mathcal{K}$*.*

(C) *The components* ζ_Y *and* δ_Y *are part of an adjoint surjective equivalence.*

(D) *The induced strict morphism of* T*-algebras (6.25)* $\Delta\colon \mathsf{T}(C', \phi) \longrightarrow \mathsf{T}C'$ *is a surjective equivalence in* $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$*.*

Proof. **Theorem 4.10 [BKP89]:** Suppose $\mathcal{K}$ satisfies (1) and (2*a*). Then (A) implies (C).

Theorem 7.11: Suppose $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ satisfies (3). Then (A) and (C) together imply (B), with $\mathsf{T}(C', \phi)$ constructed as a pushout (7.5) in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$.

Theorem 8.1: Suppose $\mathcal{K}$ satisfies (2*b*). Then (A), (B), and (C) together imply (D).

Theorem 8.9: Alternate proof that (A), (B), and (C) together imply (D), under the assumption that $\mathsf{T}(C', \phi)$ is the pushout (7.5) in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$.

Theorem 9.31: Suppose $\mathcal{K}$ satisfies (2*a*) and $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ satisfies (4). Then (B) implies (A). ☐

Relation to literature

Our approach via universal pseudomorphisms in Section 6 is based on the approach to coherence for monoidal functors in [JS93, Theorem 1.7] and for pseudofunctors between bicategories in [Gur13, Theorem 2.21]. Our use of pseudomorphism classifiers is motivated by their appearance in the 2-monadic approaches to coherence in [BKP89, Pow89, Lac02].

It is important to note that this work focuses on pseudomorphism coherence rather than the more general *lax* morphism coherence. Certain special cases of the latter are treated in work of Epstein [Eps66], Lewis [Lew74], and Malkiewich-Ponto [MP22]. These coherence theorems focus on plain and symmetric monoidal structures, with Malkiewich-Ponto extending to bicategorical applications. The following example due to Lewis illustrates the potential subtlety of lax morphisms.

Non-Example 1.10 ([Lew74, Pages 5–6]). Suppose given monoidal categories $A = (A, \bullet, I)$ and $A' = (A', \bullet, I')$ with monoidal products denoted $\bullet$ and monoidal units denoted I and I', respectively. Suppose $f\colon A \longrightarrow A'$ is a lax monoidal functor. The following diagram in A' does not necessarily commute.

$$\begin{tikzcd}
f(I) \arrow[r, "\lambda^{-1}"] \arrow[d, "\rho^{-1}"'] & I' \bullet f(I) \arrow[d, "f_0 \bullet 1"] \\
f(I) \bullet I' \arrow[r, "1 \bullet f_0"] & f(I) \bullet f(I)
\end{tikzcd} \tag{1.11}$$

In the above diagram, λ and ρ are the left and right unit isomorphisms for A', respectively, and f_0 is the monoidal unit constraint for f.

For a specific case where (1.11) does not commute, let f be the forgetful functor from the category of abelian groups $A = (\mathcal{Ab}, \otimes, \mathbb{Z})$, to the category of sets $A' = (\mathcal{Set}, \times, \mathbf{1})$. This functor is lax monoidal, and the function $f_0\colon \mathbf{1} \longrightarrow \mathbb{Z}$ is given by sending the unique element of $\mathbf{1}$ to $1 \in \mathbb{Z}$. Then the two composites around the diagram are given by the functions $n \longmapsto (1, n)$ for the top/right composite and $n \longmapsto (n, 1)$ for the left/bottom composite. ◇

Thus, the theory of coherence for lax monoidal functors is *not* equivalent to that of monoidal categories, where every formal diagram commutes.

In contrast, our results show that often the coherence for T-algebra pseudomorphisms is equivalent to that of strict T-algebras. Thus, the context for our work is restricted to pseudomorphisms, but broadened to a general 2-monad T. Remark 8.14 provides further details on a key step where our restriction to pseudomorphisms is required.

Our results are related to, but somewhat different from, coherence theorems for pseudo*algebras* such as those of Power [Pow89], Hermida [Her01], and Lack [Lac02]. The latter are formulated to show that there is a left adjoint to the inclusion

$$\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s} \longrightarrow \mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathsf{Alg},$$

such that the components of the unit are equivalences in $\mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathsf{Alg}$. Here, $\mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathsf{Alg}$ is the 2-category of pseudoalgebras and pseudomorphisms for T. Such coherence results show that pseudoalgebras and pseudomorphisms for T can be replaced with equivalent strict algebras and strict morphisms. They do not directly address the diagrammatic coherence questions that are resolved by Theorem 1.7 for pseudomorphisms.

Outline

This work is organized into three parts. Part I consists of Sections 2 through 5 and reviews relevant parts of 2-monad theory. Sections 2 and 3 recall basic definitions, limits, and colimits. Sections 4 and 5 recall essential parts of the theory of pseudomorphism classifiers.

Part II consists of Sections 6 through 9 and contains the core technical work. The definition of universal pseudomorphisms $\widetilde{\phi}$ and their basic properties are given in Section 6. Section 7 gives a construction of $\mathsf{T}(C', \phi)$ as a pushout of a pseudomorphism classifier Q, in the case that $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ admits pushouts. Section 8 proves that Δ is an equivalence in each of two separate results with slightly different hypotheses. Section 9 identifies hypotheses under which the existence of universal pseudomorphisms $\widetilde{\phi}$ implies the existence of a pseudomorphism classifier Q.

Part III contains applications to diagrammatic coherence for 2-monads over $\mathcal{C}at$. Section 10 gives a general definition of formal diagrams for such 2-monads T, and the remaining sections focus on three special cases for plain, symmetric, and braided monoidal structures. Section 11 recalls the relevant definitions and the standard coherence theorems in those cases. Section 12 contains a novel simplification in the symmetric monoidal case. Sections 13 and 14 give detailed explanations of the abstract constructions from Part II for plain, symmetric, and braided monoidal structures.

Section 15 contains a number of examples that apply the results above to check commutativity of various diagrams for symmetric and braided strong monoidal functors. Section 16 treats two specific monoidal functors and a diagram (16.12) that is *not* simplified by the dissolution approach developed in this work. Both Sections 15 and 16 have been written to minimize explicit dependence on the preceding theory, and to be read as independently as possible. Some readers may find it interesting to read those sections immediately after this introduction, and then follow the references from there back to the main body as necessary.

Part I: Background

2 2-monads

For basic theory of categories and 2-categories, we refer the reader to [ML98, Lac10, Gur13, JY21].

Convention 2.1. Throughout this work, we let $\mathcal{K}$ denote a 2-category. We denote 1-cells as

$$\phi\colon C \longrightarrow C' \quad \text{or} \quad \psi\colon D \longrightarrow D'.$$

We use a *relative dimension convention* and denote 2-cells as

$$\Gamma\colon \phi \longrightarrow \phi' \quad \text{or} \quad \begin{tikzcd} C \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Gamma\Downarrow"] & C'. \end{tikzcd}$$

◇

Definition 2.2. Suppose $\mathcal{K}$ is a 2-category. A *2-monad* on $\mathcal{K}$ is a triple (T, μ, η) consisting of

- a 2-functor $\mathsf{T}\colon \mathcal{K} \longrightarrow \mathcal{K}$,
- a 2-natural transformation $\mu\colon \mathsf{T}^2 \longrightarrow \mathsf{T}$, and
- a 2-natural transformation $\eta\colon 1_{\mathcal{K}} \longrightarrow \mathsf{T}$.

These data are required to make the following unity and associativity diagrams commute.

$$\begin{tikzcd} \mathsf{T}^3 \arrow[r, "1_{\mathsf{T}} * \mu"] \arrow[d, "\mu * 1_{\mathsf{T}}"'] & \mathsf{T}^2 \arrow[d, "\mu"] \\ \mathsf{T}^2 \arrow[r, "\mu"] & \mathsf{T} \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{K}}\mathsf{T} \arrow[r, "\eta * 1_{\mathsf{T}}"] \arrow[d, equal] & \mathsf{T}^2 \arrow[d, "\mu"] & \mathsf{T}1_{\mathcal{K}} \arrow[l, "1_{\mathsf{T}} * \eta"'] \arrow[d, equal] \\ \mathsf{T} \arrow[r, equal] & \mathsf{T} \arrow[r, equal, leftarrow] & \mathsf{T} \end{tikzcd}$$

We often write a 2-monad as T, leaving μ, η implicit. ◇

Definition 2.3. Suppose T is a 2-monad on $\mathcal{K}$. A T-*algebra* is a pair (X, x) consisting of

- an object $X \in \mathcal{K}$ and
- a structure 1-cell $x \colon \mathsf{T}X \longrightarrow X$ in $\mathcal{K}$

such that the following two diagrams commute.

$$\begin{tikzcd}
X \arrow[d, "\eta_X"'] \arrow[dr, "1_X"] & \\
\mathsf{T}X \arrow[r, "x"'] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{T}^2X \arrow[r, "\mu_X"] \arrow[d, "\mathsf{T}x"'] & \mathsf{T}X \arrow[d, "x"] \\
\mathsf{T}X \arrow[r, "x"'] & X
\end{tikzcd}
\tag{2.4}$$

◇

Definition 2.5. Suppose (X, x) and (Y, y) are T-algebras for a 2-monad T on $\mathcal{K}$. A T-*algebra pseudomorphism*, or T-*map*, is a pair

$$(f, f_\bullet) \colon (X, x) \longrightarrow (Y, y)$$

consisting of

- a 1-cell $f \colon X \longrightarrow Y$ in $\mathcal{K}$ called the *underlying 1-cell* and
- an invertible 2-cell $f_\bullet$ in $\mathcal{K}$ as shown below, called the *algebra constraint* of f.

$$\begin{tikzcd}
\mathsf{T}X \arrow[r, "\mathsf{T}f"] \arrow[d, "x"'] \arrow[dr, phantom, "f_\bullet \swArrow \cong"] & \mathsf{T}Y \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{2.6}$$

These data are required to satisfy unit and multiplication axioms indicated by the two equalities of pasting diagrams below. In these diagrams, the unlabeled regions commute because X and Y are assumed to be T-algebras.

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[dl, "\eta_X"'] & Y \arrow[dl, "\eta_Y"'] \arrow[ddl, bend left=40, "1_Y"] \\
\mathsf{T}X \arrow[r, "\mathsf{T}f"] \arrow[d, "x"'] \arrow[dr, phantom, "f_\bullet \swArrow"] & \mathsf{T}Y \arrow[d, "y"] & \\
X \arrow[r, dash, "f"'] & Y &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[r, "f"] \arrow[dl, "\eta_X"'] \arrow[ddl, bend left=40, "1_X"] & Y \arrow[ddl, bend left=40, "1_Y"] \\
\mathsf{T}X \arrow[d, "x"'] & & \\
X \arrow[r, dash, "f"'] & Y &
\end{tikzcd}
\tag{2.7}$$

$$\begin{tikzcd}
& \mathsf{T}X \arrow[rr, "x"] & & X \arrow[dr, "f"] & \\
\mathsf{T}^2X \arrow[ur, "\mu_X"] \arrow[rr, "\mathsf{T}x"] \arrow[dr, "\mathsf{T}^2f"'] & & \mathsf{T}X \arrow[ur, "x"] \arrow[dr, "\mathsf{T}f"] \arrow[rr, phantom, "\Uparrow f_\bullet"] & & Y \\
& \mathsf{T}^2Y \arrow[rr, "\mathsf{T}y"'] \arrow[ur, phantom, "\nearrow \mathsf{T}f_\bullet"] & & \mathsf{T}Y \arrow[ur, "y"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{T}X \arrow[rr, "x"] \arrow[dr, "\mathsf{T}f"'] & & X \arrow[dr, "f"] \arrow[dl, phantom, "f_\bullet \nearrow"] & \\
\mathsf{T}^2X \arrow[ur, "\mu_X"] \arrow[dr, "\mathsf{T}^2f"'] & & \mathsf{T}Y \arrow[rr, "y"] & & Y \\
& \mathsf{T}^2Y \arrow[rr, "\mathsf{T}y"'] \arrow[ur, "\mu_Y"] & & \mathsf{T}Y \arrow[ur, "y"'] &
\end{tikzcd}
\tag{2.8}$$

We often abbreviate the pair $(f, f_\bullet)$ as f. We say that f is a *strict* T-*map* if $f_\bullet$ is an identity 2-cell, so that (2.6) commutes. We will sometimes say "map" or "strict map" when T is clear from context. ◇

Remark 2.9. In the context of Definition 2.5, let $\mathcal{K}_0$ denote the underlying 1-category of $\mathcal{K}$ and let T_0 denote the monad on $\mathcal{K}_0$ underlying T. Suppose that (X, x) and (Y, y) are T-algebras. Then a 1-cell $f\colon X \longrightarrow Y$ in $\mathcal{K}$ is a strict T-map if and only if f is a morphism of T_0-algebras. ◇

Remark 2.10. Our terms "T-map", respectively "strict T-map," are convenient abbreviations for what are called *pseudo* or *strong* T*-morphism*, respectively *strict* T*-morphism*, in the literature. The more general notion of *lax* T*-morphism*, where $f_\bullet$ is not assumed to be invertible, will not be used in this present work. ◇

Definition 2.11. Suppose $(f, f_\bullet)$ and $(g, g_\bullet)$ are two T-maps $(X, x) \longrightarrow (Y, y)$ for T-algebras X and Y in $\mathcal{K}$. A T*-algebra 2-cell*

$$\alpha\colon f \longrightarrow g$$

is a 2-cell $\alpha\colon f \longrightarrow g$ in $\mathcal{K}$ such that the following equality holds.

$$\begin{tikzcd}
\mathsf{T}X \arrow[r, bend left, "\mathsf{T}f"] \arrow[d, "x"'] \arrow[dr, phantom, "f_\bullet \Swarrow"] & \mathsf{T}Y \arrow[d, "y"] \\
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{T}X \arrow[r, bend left, "\mathsf{T}f"] \arrow[r, bend right, "\mathsf{T}g"'] \arrow[r, phantom, "\Downarrow\mathsf{T}\alpha"] \arrow[d, "x"'] \arrow[dr, phantom, "g_\bullet \Swarrow"] & \mathsf{T}Y \arrow[d, "y"] \\
X \arrow[r, bend right, "g"'] & Y
\end{tikzcd}$$

We will also say that α is an *algebra 2-cell* when T is clear from context. ◇

Definition 2.12. The composite of T-maps

$$X \xrightarrow{f} X' \xrightarrow{f'} X''$$

is defined as follows.

- The underlying 1-cell of $f' \circ f$ is the composite of underlying 1-cells.
- The algebra constraint $(f' \circ f)_\bullet$ is given by the pasting in $\mathcal{K}$ indicated below.

$$\begin{tikzcd}
\mathsf{T}X \arrow[r, "\mathsf{T}f"] \arrow[d, "x"'] \arrow[dr, phantom, "f_\bullet \Swarrow \cong"] & \mathsf{T}Y \arrow[r, "\mathsf{T}f'"] \arrow[d, "y"] \arrow[dr, phantom, "f'_\bullet \Swarrow \cong"] & \mathsf{T}Z \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "f'"'] & Z
\end{tikzcd} \tag{2.13}$$

That is,

$$(f' \circ f)_\bullet = (f' * f_\bullet) \circ (f'_\bullet * \mathsf{T}f). \tag{2.14}$$

Horizontal and vertical composition of algebra 2-cells is given by that of the underlying 2-category, $\mathcal{K}$. ◇

Definition 2.15. Suppose $\mathcal{K}$ is a 2-category and T is a 2-monad on $\mathcal{K}$. We use the notations

$$\mathsf{T}\text{-}\mathsf{Alg} \quad \text{and} \quad \mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$$

to denote the 2-categories consisting of

- T-algebras as 0-cells,

- T-maps, respectively strict T-maps, as 1-cells, and
- T-algebra 2-cells as 2-cells.

Because every strict T-map is a T-map with identity algebra constraints, there is an identity-on-objects, locally full and faithful inclusion denoted

$$\mathtt{i}\colon \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \hookrightarrow \mathsf{T}\text{-}\mathsf{Alg}\,. \tag{2.16}$$

Moreover, each T-algebra, T-map, or T-algebra 2-cell has an underlying object, 1-cell, or 2-cell in $\mathcal{K}$, respectively. We let $\mathtt{u}$ denote the forgetful 2-functors as indicated in the following diagram, with $\mathtt{u} = \mathtt{u} \circ \mathtt{i}$.

$$\begin{tikzcd}
\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[rr, hook, "\mathtt{i}"] \arrow[dr, "\mathtt{u}"'] & & \mathsf{T}\text{-}\mathsf{Alg} \arrow[dl, "\mathtt{u}"] \\
& \mathcal{K} &
\end{tikzcd} \tag{2.17}$$

◇

Convention 2.18. The 2-functor $\mathtt{i}\colon \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \hookrightarrow \mathsf{T}\text{-}\mathsf{Alg}$ (2.16) is the identity on objects, 1-cells, and 2-cells. Therefore, we will sometimes leave $\mathtt{i}$ implicit and omit it from the notation. For example, any time that a strict T-map is composed with a general T-map, there may be an implicit usage of $\mathtt{i}$. ◇

Definition 2.19. In the context of Definition 2.15, we use the notations

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "\mathsf{T}"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[l, bend left, "\mathtt{u}"]
\end{tikzcd} \tag{2.20}$$

for the free-forgetful 2-adjunction with left 2-adjoint T and right 2-adjoint $\mathtt{u}$. We let η and ε denote, respectively, the unit and counit of $\mathsf{T} \dashv \mathtt{u}$. For each T-algebra (X, x),

- the unit component η_X is the unit of the T-algebra structure on X and
- the counit component ε_X is the algebra structure cell $x\colon \mathsf{T}X \longrightarrow X$. ◇

Convention 2.21. Beginning here, and throughout the rest of this document, we will write

$$f\colon X \rightsquigarrow Y,$$

using a zigzag arrow, to denote that f is the 1-cell part of a T-map $(f, f_\bullet)$. If $f_\bullet$ is known to be an identity, so that f is a strict T-map, we use a straight arrow and write

$$f\colon X \longrightarrow Y.$$

◇

Remark 2.22 (Uniqueness of mates). The following elementary detail about 2-adjunctions will be useful below. Suppose given a 2-adjunction

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend left, "R"]
\end{tikzcd}$$

between 2-categories $\mathcal{K}$ and $\mathcal{A}$, with unit η and counit ε. For objects $C \in \mathcal{K}$ and $Y \in \mathcal{A}$, the isomorphism of categories

$$\mathcal{A}(LC, Y) \xrightarrow{\cong} \mathcal{K}(C, RY) \tag{2.23}$$

is given by the right adjoint R and composition or whiskering with η:

$$\begin{aligned} f &\longmapsto Rf \circ \eta \\ \alpha &\longmapsto R\alpha * \eta \end{aligned} \tag{2.24}$$

where $\alpha\colon f \longrightarrow f'$ in $\mathcal{A}(LC, Y)$. In particular, if f and g are two 1-cells in $\mathcal{A}(LC, Y)$ such that $Rf \circ \eta = Rg \circ \eta$ as 1-cells in $\mathcal{K}$, then f and g are equal as 1-cells in $\mathcal{K}$. ◇

3 Cotensors and coequalizers

Completeness and cocompleteness for 2-categories generally refers to the $\mathcal{C}at$-enriched sense, meaning not just conical limits and colimits but also including all small $\mathcal{C}at$-weighted limits and colimits. The only non-conical such we will employ, in Sections 6 and 8, is that of a cotensor (also called a power). Below, we recall their defining property and a key application. For the more general theory of 2-dimensional limits and colimits, we refer the reader to [Kel89, Bor94].

Later in this section we discuss various coequalizers and their relation to T-algebra structures. These will be used in Section 9.

Definition 3.1. Suppose $\mathcal{K}$ is a 2-category, X is an object of $\mathcal{K}$, and C is a small category. The *cotensor* of C and X is an object of $\mathcal{K}$, denoted $\{C, X\}$, equipped with a 2-natural isomorphism

$$\mathcal{C}at\big(C, \mathcal{K}(-, X)\big) \cong \mathcal{K}\big(-, \{C, X\}\big) \tag{3.2}$$

of 2-functors $\mathcal{K}^{op} \longrightarrow \mathcal{C}at$. If the cotensor $\{C, X\}$ exists in $\mathcal{K}$ for every object X and every small category C, we say that $\mathcal{K}$ *has all cotensors.* ◇

Remark 3.3. If $\mathcal{K} = \mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ or $\mathcal{K} = \mathsf{T}\text{-}\mathsf{Alg}$ for a 2-monad T on $\mathcal{C}at$, then $\{C, X\}$ will be the ordinary functor category $\mathcal{C}at\big(C, \mathsf{u}X\big)$ equipped with the pointwise T-algebra structure. ◇

Notation 3.4. If $\mathcal{K}$ is a 2-category, we let $\mathcal{K}_0$ denote the underlying category of $\mathcal{K}$. If $F\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a 2-functor, we let $F_0\colon \mathcal{K}_0 \longrightarrow \mathcal{L}_0$ denote the functor obtained by restricting F to the underlying categories. ◇

Notation 3.5. We let $\mathbb{2} = \{0 \longrightarrow 1\}$ denote the free arrow category, consisting of two objects and one non-identity morphism. Similarly, let $\mathbb{I} = \{0 \cong 1\}$ denote the free isomorphism category, consisting of two objects and an isomorphism between them. ◇

Remark 3.6. Most of our work below will depend only on cotensors of the form $\{\mathbb{2}, X\}$ and $\{\mathbb{I}, X\}$. Unpacking Definition 3.1 and Notation 3.5 in these cases gives the following direct descriptions of (3.2) on 1-cells.

i. 1-cells $f\colon W \longrightarrow \{\mathbb{2}, X\}$ in $\mathcal{K}$ are in bijection with triples (f_1, f_2, α) where $f_1, f_2\colon W \longrightarrow X$ are 1-cells and $\alpha\colon f_1 \longrightarrow f_2$ is a 2-cell in $\mathcal{K}$.

ii. 1-cells $f\colon W \longrightarrow \{\mathbb{I}, X\}$ in $\mathcal{K}$ are described similarly, with α being an isomorphism. ◇

Recall from (2.17) the forgetful functors u from $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ and $\mathsf{T}\text{-}\mathsf{Alg}$ to $\mathcal{K}$ and the inclusion $\mathtt{i}$ from $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ to $\mathsf{T}\text{-}\mathsf{Alg}$. We need the following two facts about cotensor products; proofs of both can be found in [BKP89].

Proposition 3.7.

i. *[BKP89, Proposition 2.5] Suppose $\mathcal{K}$ is a 2-category, and T is a 2-monad on $\mathcal{K}$. If C is a small category and $\mathcal{K}$ admits all cotensors of the form $\{C, X\}$, then so do $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ and $\mathsf{T}\text{-}\mathsf{Alg}$. Moreover, the inclusion $\mathtt{i}$ and both forgetful functors u preserve those cotensors.*

ii. *[BKP89, Proposition 3.1] Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories such that $\mathcal{A}$ admits cotensors of the form $\{\mathbb{2}, X\}$. Suppose $V\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a 2-functor that preserves those cotensors. Then the underlying functor $V_0\colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ has a left adjoint if and only if V has a left 2-adjoint.*

Now we turn to a discussion of various coequalizers and their relation to T-algebra structures.

Definition 3.8 (Split coequalizers and u-split pairs). Suppose C is a category, and $\mathsf{u}\colon C \longrightarrow C'$ is a functor.

i. A *split coequalizer* in C is a diagram of the form below,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right=50, "t"'] \arrow[r, "h"'] & Z \arrow[l, bend right=40, "s"'] \end{tikzcd} \tag{3.9}$$

such that the following equations hold.

$$\begin{aligned} hf &= hg & \qquad sh &= gt \\ hs &= 1_Z & \qquad ft &= 1_Y \end{aligned} \tag{3.10}$$

In this case, h is said to be *a split coequalizer* of f and g.

ii. Suppose $f, g\colon X \longrightarrow Y$ are parallel arrows in C. This pair is called a u-*split pair* if there exists an object Z' together with morphisms h', s', and t' in C' such that

$$\begin{tikzcd} \mathsf{u}X \arrow[r, shift left, "\mathsf{u}f"] \arrow[r, shift right, "\mathsf{u}g"'] & \mathsf{u}Y \arrow[l, bend right=50, "t'"'] \arrow[r, "h'"'] & Z' \arrow[l, bend right=40, "s'"'] \end{tikzcd} \tag{3.11}$$

is a split coequalizer in C'. $\diamond$

Remark 3.12 (Split coequalizers are coequalizers). Suppose given a split coequalizer as in (3.9) and a morphism $p\colon Y \longrightarrow W$ such that $pf = pg$. Then the unique morphism $\widetilde{p}\colon Z \longrightarrow W$ such that $p = \widetilde{p}h$ is given by the formula

$$\widetilde{p} = ps.$$

Therefore, h is the coequalizer of f and g. $\diamond$

Remark 3.13 (Split coequalizers are absolute). Suppose given a split coequalizer in C as in (3.9), and a functor $F\colon C \longrightarrow D$. Then applying F to the entire diagram gives a split coequalizer in D. $\diamond$

Example 3.14 (The canonical u-split pair for a T-algebra). Suppose T is a monad on a category C, and $x\colon \mathsf{T}X \longrightarrow X$ is a T-algebra structure on an object X. Then the pair $\mu, \mathsf{T}x\colon \mathsf{T}^2X \longrightarrow \mathsf{T}X$ has $x\colon \mathsf{T}X \longrightarrow X$ as its coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$, and is u-split for u the forgetful functor from $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ back to C. An explicit splitting in C, with the forgetful functor u suppressed, is given below.

$$\begin{tikzcd} \mathsf{T}^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}x"'] & \mathsf{T}X \arrow[l, bend right=50, "\eta_{\mathsf{T}X}"'] \arrow[r, "x"'] & X \arrow[l, bend right=40, "\eta_X"'] \end{tikzcd} \tag{3.15}$$

This observation is a key component of Beck's Monadicity Theorem [Bec67] and related variants. See [ML98, Section VI.7] and [Rie16, Section 5.5]. $\diamond$

We require an analogue of the previous example in the 2-category $\mathsf{T}\text{-}\mathsf{Alg}$ for a 2-monad T on a 2-category $\mathcal{K}$.

Lemma 3.16. *Suppose $\mathcal{K}$ is a 2-category, and that*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right=50, "t"'] \arrow[r, "h"'] & Z \arrow[l, bend right=40, "s"'] \end{tikzcd} \tag{3.17}$$

is a split coequalizer in $\mathcal{K}_0$, the underlying category of $\mathcal{K}$. Then Z is also the $\mathcal{C}at$-enriched colimit of the same diagram, meaning it also satisfies the following 2-dimensional universal property.

2-dimensional universality of split coequalizers: *Suppose given 1-cells*

$$p, q\colon Y \longrightarrow W$$

such that $pf = pg$ and $qf = qg$. Let $\widetilde{p}, \widetilde{q}\colon Z \longrightarrow W$ be the unique 1-cells induced by the universal property of h as the coequalizer of f, g in $\mathcal{K}_0$. Then the functions given by whiskering with h and s

$$(- * h)\colon \mathcal{K}(Z, W)(\widetilde{p}, \widetilde{q}) \rightleftarrows \mathcal{K}(Y, W)(p, q) : (- * s)$$

induce inverse bijections between the set of 2-cells $\widetilde{\alpha}\colon \widetilde{p} \longrightarrow \widetilde{q}$ and the subset

$$\big\{\alpha\colon p \longrightarrow q \,\big|\, \alpha * f = \alpha * g\big\} \subseteq \mathcal{K}(Y, W)(p, q).$$

Proof. Suppose $\alpha\colon p \longrightarrow q$ such that $\alpha * f = \alpha * g$. Recall (Remark 3.12) that $\widetilde{p} = ps$ and $\widetilde{q} = qs$, and define $\widetilde{\alpha}\colon \widetilde{p} \longrightarrow \widetilde{q}$ to be $\alpha * s$. Then

$$\widetilde{\alpha} * h = \alpha * sh = \alpha * gt = \alpha * ft = \alpha * 1_Y = \alpha \tag{3.18}$$

by the definition of $\widetilde{\alpha}$, the assumption $\alpha * f = \alpha * g$, and the equations in (3.10). It remains to prove that $\alpha * s$ is the only 2-cell $\beta\colon \widetilde{p} \longrightarrow \widetilde{q}$ such that $\beta * h = \alpha$. Indeed, if $\beta * h = \alpha$, then

$$\beta = \beta * 1_Z = \beta * hs = \alpha * s = \widetilde{\alpha}.$$

□

Remark 3.19. Note, in the context of Lemma 3.16 above, that the 2-cell

$$\alpha = \widetilde{\alpha} * h$$

is invertible if and only if $\widetilde{\alpha}$ is invertible. This follows because the inverse bijection to $(- * h)$ is $(- * s)$ and whiskering preserves invertibility of 2-cells. ◇

We adopt the following temporary notation to distinguish between the two different versions of u for a 2-monad T.

Notation 3.20. Suppose T is a 2-monad on a 2-category $\mathcal{K}$. We write $\mathsf{u}_s\colon \mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s} \longrightarrow \mathcal{K}$ for the forgetful functor when considering only the strict T-maps, and $\mathsf{u}\colon \mathsf{T}\text{-}\mathsf{Alg} \longrightarrow \mathcal{K}$ when considering all T-maps. In this notation, the commutative diagram (2.17) is an equality $\mathsf{u} \circ \mathsf{i} = \mathsf{u}_s$ as 2-functors $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s} \longrightarrow \mathcal{K}$. ◇

Proposition 3.21. *Suppose $f, g\colon (X, x) \longrightarrow (Y, y)$ is a u_s-split pair of strict T-maps. Let $h\colon Y \longrightarrow Z$ be the split coequalizer of $\mathsf{u}_s f, \mathsf{u}_s g$ in $\mathcal{K}$. Then h is the underlying 1-cell of a strict T-map, also denoted h, and is the coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ of the pair f, g.*

Proof. This follows from the analogous standard result for 1-monads, e.g., [Rie16, Proposition 5.4.9], and Remark 2.9. □

Lemma 3.22. *Suppose given f, g, and h as in Proposition 3.21 and suppose given a 1-cell $\widetilde{k}$ and a 2-cell $\widetilde{k}_\bullet$ in $\mathcal{K}$*

$$\widetilde{k}\colon Z \longrightarrow W \quad \text{and} \quad \widetilde{k}_\bullet\colon w \circ \mathsf{T}\widetilde{k} \longrightarrow \widetilde{k} \circ z$$

for some T-algebra (W, w). Then $(\widetilde{k}, \widetilde{k}_\bullet)$ is a T-map $(Z, z) \rightsquigarrow (W, w)$ if and only if the composite

$$(k, k_\bullet) = (\widetilde{k}, \widetilde{k}_\bullet) \circ h = (\widetilde{k} \circ h, \widetilde{k}_\bullet * \mathsf{T}h) \tag{3.23}$$

is a T-map $(Y, y) \rightsquigarrow (W, w)$.

Proof. If $(\widetilde{k}, \widetilde{k}_\bullet)$ is a T-map, then the composite $(\widetilde{k}, \widetilde{k}_\bullet) \circ h$ is a T-map. In this case, the composition formula (2.14) simplifies to the right hand side of (3.23) because h is a strict T-map.

For the reverse implication, let $(k, k_\bullet)$ be defined via the formula on the right hand side of (3.23). Since h is a split coequalizer, recall from Remark 3.13 that $\mathsf{T}h$ is too. Therefore, applying Remark 3.19 to $\mathsf{T}h$, invertibility of $k_\bullet$ implies that of $\widetilde{k}_\bullet$. Now it remains to show that the T-map axioms (2.7) and (2.8) for $(k, k_\bullet)$ imply those for $(\widetilde{k}, \widetilde{k}_\bullet)$. This verification uses the hypothesis that h is a split coequalizer in $\mathcal{K}$ and, separately, the implication that T^2h is also a split coequalizer in $\mathcal{K}$ by Remark 3.13. The applications of both of these facts use the 2-dimensional universality from Lemma 3.16.

For the unit axiom (2.7), we must verify that $\widetilde{k}_\bullet * \eta_Z = 1_{\widetilde{k}}$. Note that the source and target of $\widetilde{k}_\bullet * \eta_Z$ are both equal to $\widetilde{k}$ by naturality of η and the unit axioms for (Z, z) and (W, w), respectively:

$$\begin{aligned} z \circ \eta_Z &= 1_Z, \\ w \circ \eta_W &= 1_W. \end{aligned}$$

The two-dimensional part of the universal property of the split coequalizer $h\colon Y \longrightarrow Z$ (Lemma 3.16) implies that the 2-cell $\widetilde{k}_\bullet * \eta_Z$ is an identity if and only if it is the identity 1_k after applying $- * h$. The following computation uses naturality of η, the defining equality $k_\bullet = \widetilde{k}_\bullet * \mathsf{T}h$ (3.23), and the unit axiom for $(k, k_\bullet)$, respectively:

$$\begin{aligned} \widetilde{k}_\bullet * \eta_Z * h &= \widetilde{k}_\bullet * \mathsf{T}h * \eta_Y \\ &= k_\bullet * \eta_Y \\ &= 1_k. \end{aligned}$$

This verifies the unit axiom (2.7) for $(\widetilde{k}, \widetilde{k}_\bullet)$.

For the multiplication axiom (2.8), we must check the equality of pastings below.

$$\begin{tikzcd}
 & \mathsf{T}Z \arrow[rr, "z"] & & Z \arrow[dr, "\widetilde{k}"] & \\
\mathsf{T}^2Z \arrow[ur, "\mu_Z"] \arrow[rr, "\mathsf{T}z"] \arrow[dr, swap, "\mathsf{T}^2\widetilde{k}"] \arrow[drrr, phantom, "\Nearrow \mathsf{T}\widetilde{k}_\bullet"] & & \mathsf{T}Z \arrow[ur, "z"] \arrow[dr, "\mathsf{T}\widetilde{k}"] \arrow[rr, phantom, "\Uparrow \widetilde{k}_\bullet"] & & W \\
 & \mathsf{T}^2W \arrow[rr, swap, "\mathsf{T}w"] & & \mathsf{T}W \arrow[ur, swap, "w"] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathsf{T}Z \arrow[rr, "z"] \arrow[dr, swap, "\mathsf{T}\widetilde{k}"] & & Z \arrow[dr, "\widetilde{k}"] & \\
\mathsf{T}^2Z \arrow[ur, "\mu_Z"] \arrow[dr, swap, "\mathsf{T}^2\widetilde{k}"] & & \mathsf{T}W \arrow[rr, "w"] \arrow[ur, phantom, "\widetilde{k}_\bullet\Nearrow"] & & W \\
 & \mathsf{T}^2W \arrow[ur, "\mu_W"] \arrow[rr, swap, "\mathsf{T}w"] & & \mathsf{T}W \arrow[ur, swap, "w"] &
\end{tikzcd}
\tag{3.24}$$

Once again using that h is a split coequalizer, and therefore T^2h is also (Remark 3.13), the desired equality holds if and only if it holds after applying $- * \mathsf{T}^2h$.

Whiskering the left pasting diagram in (3.24) with T^2h gives the left diagram below, where the additional regions commute because h is a strict T-map by Proposition 3.21, $k = \widetilde{k}h$ by definition (3.23), μ is 2-natural, and T is 2-functorial. The equality of pastings is immediate as the only difference between the diagrams is how commutative regions are displayed.

$$\begin{tikzcd}
 & & & Y \arrow[drr, "h"] & & & \\
 & \mathsf{T}Y \arrow[urr, "y"] \arrow[drr, "\mathsf{T}h"] & & & & Z \arrow[ddr, "\widetilde{k}"] & \\
 & & & \mathsf{T}Z \arrow[urr, "z"] & & & \\
\mathsf{T}^2Y \arrow[uur, "\mu_Y"] \arrow[rr, "\mathsf{T}^2h"] \arrow[drrr, swap, "\mathsf{T}^2k"] & & \mathsf{T}^2Z \arrow[ur, "\mu_Z"] \arrow[rr, "\mathsf{T}z"] \arrow[dr, "\mathsf{T}^2\widetilde{k}"] \arrow[drrr, phantom, "\Nearrow \mathsf{T}\widetilde{k}_\bullet"] & & \mathsf{T}Z \arrow[uur, "z"] \arrow[dr, "\mathsf{T}\widetilde{k}"] \arrow[rr, phantom, "\Uparrow \widetilde{k}_\bullet"] & & W \\
 & & & \mathsf{T}^2W \arrow[rr, swap, "\mathsf{T}w"] & & \mathsf{T}W \arrow[ur, swap, "w"] &
\end{tikzcd}
=
\begin{tikzcd}
 & & & Y \arrow[drr, "h"] & & & \\
 & \mathsf{T}Y \arrow[urr, "y"] & & & & Z \arrow[ddr, "\widetilde{k}"] & \\
 & & & \mathsf{T}Y \arrow[uu, "y"] \arrow[dr, "\mathsf{T}h"] & & & \\
\mathsf{T}^2Y \arrow[uur, "\mu_Y"] \arrow[urrr, "\mathsf{T}y"] \arrow[rr, "\mathsf{T}^2h"] \arrow[drrr, swap, "\mathsf{T}^2k"] & & \mathsf{T}^2Z \arrow[rr, "\mathsf{T}z"] \arrow[dr, "\mathsf{T}^2\widetilde{k}"] \arrow[drrr, phantom, "\Nearrow \mathsf{T}\widetilde{k}_\bullet"] & & \mathsf{T}Z \arrow[uur, "z"] \arrow[dr, "\mathsf{T}\widetilde{k}"] \arrow[rr, phantom, "\Uparrow \widetilde{k}_\bullet"] & & W \\
 & & & \mathsf{T}^2W \arrow[rr, swap, "\mathsf{T}w"] & & \mathsf{T}W \arrow[ur, swap, "w"] &
\end{tikzcd}$$

The pasting in the diagram at right above is equal to that of the diagram at left below by applying T to the defining equality $k_\bullet = \widetilde{k}_\bullet * \mathsf{T}h$ (3.23). Another application of the same equality shows that the two pastings below are equal.

$$\begin{tikzcd}
& & Y & & & \\
& \mathsf{T}Y & & & Z & \\
& & \mathsf{T}Y & & & \\
\mathsf{T}^2Y & & \mathsf{T}k_{\bullet}\Nearrow & \mathsf{T}Z & \Uparrow\widetilde{k}_{\bullet} & W \\
& & \mathsf{T}^2W & & \mathsf{T}W &
\arrow[from=4-1, to=2-2, "\mu_Y"]
\arrow[from=2-2, to=1-3, "y"]
\arrow[from=4-1, to=3-3, "\mathsf{T}y"]
\arrow[from=3-3, to=1-3, "y"']
\arrow[from=1-3, to=2-5, "h"]
\arrow[from=3-3, to=4-4, "\mathsf{T}h"]
\arrow[from=3-3, to=5-5, "\mathsf{T}k"']
\arrow[from=4-4, to=2-5, "z"]
\arrow[from=4-4, to=5-5, "\mathsf{T}\widetilde{k}"]
\arrow[from=2-5, to=4-6, "\widetilde{k}"]
\arrow[from=5-5, to=4-6, "w"']
\arrow[from=4-1, to=5-3, "\mathsf{T}^2k"']
\arrow[from=5-3, to=5-5, "\mathsf{T}w"']
\end{tikzcd}
=
\begin{tikzcd}
& & Y & & & \\
& \mathsf{T}Y & & & Z & \\
& & \mathsf{T}Y & & & \\
\mathsf{T}^2Y & & \mathsf{T}k_{\bullet}\Nearrow & \Uparrow k_{\bullet} & & W \\
& & \mathsf{T}^2W & & \mathsf{T}W &
\arrow[from=4-1, to=2-2, "\mu_Y"]
\arrow[from=2-2, to=1-3, "y"]
\arrow[from=4-1, to=3-3, "\mathsf{T}y"]
\arrow[from=3-3, to=1-3, "y"']
\arrow[from=1-3, to=2-5, "h"]
\arrow[from=1-3, to=4-6, "k"']
\arrow[from=3-3, to=5-5, "\mathsf{T}k"']
\arrow[from=2-5, to=4-6, "\widetilde{k}"]
\arrow[from=5-5, to=4-6, "w"']
\arrow[from=4-1, to=5-3, "\mathsf{T}^2k"']
\arrow[from=5-3, to=5-5, "\mathsf{T}w"']
\end{tikzcd}$$

Lastly, the pasting in the diagram at right above is equal to that of the diagram at left below by the multiplication axiom (2.8) for $(k, k_\bullet)$. Equality of the two pastings below holds by another application of (3.23).

$$\begin{tikzcd}
& & & Y & & & \\
& \mathsf{T}Y & & & & Z & \\
& & \mathsf{T}Z & \Uparrow k_{\bullet} & & & \\
\mathsf{T}^2Y & \mathsf{T}^2Z & & & \mathsf{T}W & & W \\
& & & \mathsf{T}^2W & & \mathsf{T}W &
\arrow[from=4-1, to=2-2, "\mu_Y"]
\arrow[from=2-2, to=1-4, "y"]
\arrow[from=1-4, to=2-6, "h"]
\arrow[from=1-4, to=4-7, "k"']
\arrow[from=2-6, to=4-7, "\widetilde{k}"]
\arrow[from=2-2, to=3-3, "\mathsf{T}h"']
\arrow[from=2-2, to=4-5, "\mathsf{T}k"]
\arrow[from=3-3, to=4-5, "\mathsf{T}\widetilde{k}"']
\arrow[from=4-1, to=4-2, "\mathsf{T}^2h"]
\arrow[from=4-2, to=3-3, "\mu_Z"']
\arrow[from=4-2, to=5-4, "\mathsf{T}^2\widetilde{k}"]
\arrow[from=4-1, to=5-4, "\mathsf{T}^2k"']
\arrow[from=5-4, to=4-5, "\mu_W"']
\arrow[from=4-5, to=4-7, "w"']
\arrow[from=5-4, to=5-6, "\mathsf{T}w"']
\arrow[from=5-6, to=4-7, "w"']
\end{tikzcd}
=
\begin{tikzcd}
& & & Y & & & \\
& \mathsf{T}Y & & & & Z & \\
& & \mathsf{T}Z & & \Nearrow\widetilde{k}_{\bullet} & & \\
\mathsf{T}^2Y & \mathsf{T}^2Z & & & \mathsf{T}W & & W \\
& & & \mathsf{T}^2W & & \mathsf{T}W &
\arrow[from=4-1, to=2-2, "\mu_Y"]
\arrow[from=2-2, to=1-4, "y"]
\arrow[from=1-4, to=2-6, "h"]
\arrow[from=3-3, to=2-6, "z"]
\arrow[from=2-6, to=4-7, "\widetilde{k}"]
\arrow[from=2-2, to=3-3, "\mathsf{T}h"']
\arrow[from=3-3, to=4-5, "\mathsf{T}\widetilde{k}"']
\arrow[from=4-1, to=4-2, "\mathsf{T}^2h"]
\arrow[from=4-2, to=3-3, "\mu_Z"']
\arrow[from=4-2, to=5-4, "\mathsf{T}^2\widetilde{k}"]
\arrow[from=4-1, to=5-4, "\mathsf{T}^2k"']
\arrow[from=5-4, to=4-5, "\mu_W"']
\arrow[from=4-5, to=4-7, "w"']
\arrow[from=5-4, to=5-6, "\mathsf{T}w"']
\arrow[from=5-6, to=4-7, "w"']
\end{tikzcd}$$

The final pasting at right above is the whiskering of the right hand diagram in (3.24) with T^2h.

This shows that the two sides of (3.24) are equal after applying $- * \mathsf{T}^2h$, and hence completes the proof that the two pastings in (3.24) are equal. This completes the proof that $(\widetilde{k}, \widetilde{k}_\bullet)$ satisfies the axioms of a T-map. □

Proposition 3.25. *Suppose* T *is a 2-monad on a 2-category* $\mathcal{K}$*. The 2-functor* $\mathtt{i}\colon \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}$ *sends coequalizers of* $\mathtt{u}_s$*-split pairs to coequalizers of* $\mathtt{u}$*-split pairs.*

Proof. Suppose $h\colon (Y, y) \longrightarrow (Z, z)$ is the coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ of a $\mathtt{u}_s$-split pair $f, g\colon (X, x) \longrightarrow (Y, y)$. Let $h'\colon Y \longrightarrow Z'$ be the split coequalizer in $\mathcal{K}$ of $\mathtt{u}_s f$ and $\mathtt{u}_s g$. By Proposition 3.21, h' is the underlying 1-cell of a strict T-map, so by uniqueness of coequalizers we assume $Z' = Z$ and $h' = \mathtt{u}_s h$.

Thus, there are 1-cells s and t in $\mathcal{K}$ such that the following is a split coequalizer in $\mathcal{K}$.

$$\begin{tikzcd}
\mathtt{u}_s(X, x) \arrow[r, shift left, "\mathtt{u}_s f"] \arrow[r, shift right, "\mathtt{u}_s g"'] & \mathtt{u}_s(Y, y) \arrow[l, bend right=50, "t"'] \arrow[r, "h' = \mathtt{u}_s h"'] & \mathtt{u}_s(Z, z) \arrow[l, bend right=40, "s"']
\end{tikzcd} \tag{3.26}$$

We will show that $\mathtt{i}h$ is the coequalizer of $\mathtt{i}f$ and $\mathtt{i}g$ in $\mathsf{T}\text{-}\mathsf{Alg}$. Since $\mathtt{u}_s = \mathtt{u} \circ \mathtt{i}$, the same s and t will then make $\mathtt{i}f, \mathtt{i}g$ a $\mathtt{u}$-split pair.

To prove that $\mathtt{i}h$ is the coequalizer of $\mathtt{i}f$ and $\mathtt{i}g$ in $\mathsf{T}\text{-}\mathsf{Alg}$, suppose given a T-map

$$(k, k_\bullet)\colon (Y, y) \rightsquigarrow (W, w)$$

such that

$$(k, k_\bullet) \circ \mathtt{i}f = (k, k_\bullet) \circ \mathtt{i}g. \tag{3.27}$$

We will show that there exists a unique T-map

$$(\widetilde{k}, \widetilde{k}_\bullet)\colon (Z, z) \rightsquigarrow (W, w) \tag{3.28}$$

such that

$$(k, k_\bullet) = (\widetilde{k}, \widetilde{k}_\bullet) \circ \mathtt{i}h. \tag{3.29}$$

Applying $\mathtt{u}$ to (3.27), we have $kf = kg$. Since h is the coequalizer of f, g in $\mathcal{K}$, we define $\widetilde{k}$ as the unique 1-cell in $\mathcal{K}$ induced by the universal property of the coequalizer. Thus, we have an equality in $\mathcal{K}$:

$$k = \widetilde{k} \circ h. \tag{3.30}$$

Next we note that, because (3.27) is an equality of T-maps, the two sides have the same algebra constraints. Recalling the formula (2.14) for algebra constraints of a composite, we have

$$k_\bullet * \mathsf{T}f = k_\bullet * \mathsf{T}g \tag{3.31}$$

because both f and g are strict T-maps. The algebra constraint $k_\bullet$ is shown in the rectangle below, where each of the triangles commutes by the equality (3.30).

$$\begin{tikzcd}
& \mathsf{T}Z \arrow[dr, "\mathsf{T}\widetilde{k}"] & \\
\mathsf{T}Y \arrow[ur, "\mathsf{T}h"] \arrow[rr, "\mathsf{T}k"] \arrow[d, "y"'] \arrow[drr, phantom, "k_\bullet \swarrow"] & & \mathsf{T}W \arrow[d, "w"] \\
Y \arrow[rr, "k"'] \arrow[dr, "h"'] & & W \\
& Z \arrow[ur, "\widetilde{k}"'] &
\end{tikzcd} \tag{3.32}$$

Since h is a strict T-map, we have $z \circ \mathsf{T}h = h \circ y$ and, therefore, $k_\bullet$ has target

$$\widetilde{k} \circ h \circ y = \widetilde{k} \circ z \circ \mathsf{T}h. \tag{3.33}$$

Since h is a split coequalizer in $\mathcal{K}$, so is $\mathsf{T}h$ by Remark 3.13. Therefore, by Lemma 3.16, $\mathsf{T}h$ satisfies an additional two-dimensional aspect to its universal property: the whiskering function $- * \mathsf{T}h$ induces an isomorphism between the set of 2-cells $\mathcal{K}(\mathsf{T}Z, W)(w \circ \mathsf{T}\widetilde{k}, \widetilde{k} \circ z)$ and the subset

$$\begin{aligned} S &= \left\{\alpha \colon w \circ \mathsf{T}\widetilde{k} \circ \mathsf{T}h \longrightarrow \widetilde{k} \circ z \circ \mathsf{T}h \,\middle|\, \alpha * \mathsf{T}f = \alpha * \mathsf{T}g\right\} \\ &\subseteq \mathcal{K}\big(\mathsf{T}Y, W\big)\big(w \circ \mathsf{T}\widetilde{k} \circ \mathsf{T}h, \widetilde{k} \circ z \circ \mathsf{T}h\big). \end{aligned}$$

Combining (3.31) through (3.33) shows that the algebra constraint $k_\bullet$ is a member of the subset S. Therefore, by the two-dimensional aspect of the universal property for $\mathsf{T}h$, there is a unique 2-cell in $\mathcal{K}$

$$\widetilde{k}_\bullet \colon w \circ \mathsf{T}\widetilde{k} \longrightarrow \widetilde{k} \circ z$$

such that

$$k_\bullet = \widetilde{k}_\bullet * \mathsf{T}h. \tag{3.34}$$

Since $(k, k_\bullet)$ is a T-map, the equalities (3.30) and (3.34) imply, by Lemma 3.22, that $(\widetilde{k}, \widetilde{k}_\bullet)$ is a T-map.

The calculation above verifies that there is a unique T-map $(\widetilde{k}, \widetilde{k}_\bullet)$ such that

$$(k, k_\bullet) = (\widetilde{k}, \widetilde{k}_\bullet) \circ \mathtt{i}h.$$

This completes the proof that $\mathtt{i}h$ is the coequalizer of $\mathtt{i}f$ and $\mathtt{i}g$, as desired. □

4 Pseudomorphism classifiers

For many 2-monads T of interest, the inclusion (2.16)

$$\mathtt{i}\colon \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \hookrightarrow \mathsf{T}\text{-}\mathsf{Alg}$$

has a left 2-adjoint. In such cases, the left 2-adjoint can be used to develop strictification and coherence results, as we will do in Section 7.

This section and the next recall the basic terminology and related properties. Much of this content comes from [BKP89], and we refer the reader there for further development. Examples, in the special case of monads that encode strict monoidal structures, are explained in Section 13.

Definition 4.1 (Pseudomorphism Classifier). Suppose given a 2-monad T on a 2-category $\mathcal{K}$. A *pseudomorphism classifier* for T is a left 2-adjoint $\mathsf{Q} \dashv \mathtt{i}$ as shown below.

$$\begin{tikzcd} \mathsf{T}\text{-}\mathsf{Alg} \arrow[r, bend left, "\mathsf{Q}"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[l, bend left, "\mathtt{i}"] \end{tikzcd} \tag{4.2}$$

The unit $\zeta\colon 1 \longrightarrow \mathtt{i}\mathsf{Q}$ has components that are T-maps

$$\zeta_X\colon X \rightsquigarrow \mathtt{i}\mathsf{Q}X \quad \text{for} \quad X \in \mathsf{T}\text{-}\mathsf{Alg}\,.$$

The counit $\delta\colon \mathsf{Q}\mathtt{i} \longrightarrow 1$ has components that are *strict* T-maps

$$\delta_Y\colon \mathsf{Q}\mathtt{i}Y \longrightarrow Y \quad \text{for} \quad Y \in \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}\,.$$

$\diamond$

The unit and counit of a pseudomorphism classifier Q satisfy triangle identities that lead to a 2-natural isomorphism of categories

$$\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}(\mathsf{Q}X, Y) \cong \mathsf{T}\text{-}\mathsf{Alg}(X, \mathtt{i}Y)$$

for every pair of T-algebras X and Y. This is the standard translation between the hom-set and unit/counit expressions for an adjunction. In this context, we use the following notation.

Definition 4.3. For each T-map $f\colon X \rightsquigarrow \mathtt{i}Y$, let $f^{\perp}\colon \mathsf{Q}X \longrightarrow Y$ be the strict T-map that is the mate of f. Thus, f factors uniquely as follows.

$$\begin{tikzcd} \mathtt{i}\mathsf{Q}X \arrow[r, "f^{\perp}"] & Y \\ X \arrow[u, squiggly, "\zeta_X"] \arrow[ur, squiggly, "f"'] & \end{tikzcd} \tag{4.4}$$

$\diamond$

Remark 4.5. The triangle identities for $\mathsf{Q} \dashv \mathtt{i}$ consist of the following equalities for each $Y \in \mathsf{T}\text{-}\mathsf{Alg}$ and $X \in \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$:

$$\mathtt{i}\delta_Y \circ \zeta_{\mathtt{i}Y} = 1_{\mathtt{i}Y} \quad \text{and} \quad \delta_{\mathsf{Q}X} \circ \mathsf{Q}\zeta_X = 1_{\mathsf{Q}X}\,.$$

Thus, omitting the inclusion $\mathtt{i}$, as discussed in Convention 2.18, we have $\delta_Y \zeta_Y = 1_Y$ for each T-algebra Y.

The composite $\zeta_Y \delta_Y$ is generally not equal to 1_Y, but it often has other useful structure. This additional structure is described in Definition 4.6 and Theorem 4.10 below. $\diamond$

We will use the following terminology in the 2-categories $\mathcal{A} = \mathsf{T}\text{-}\mathsf{Alg}$ and $\mathcal{A} = \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$.

Definition 4.6. Suppose given a pair of 1-cells

$$\zeta\colon Y \longrightarrow Z \quad \text{and} \quad \delta\colon Z \longrightarrow Y$$

in a 2-category $\mathcal{A}$.

Surjective equivalence: We say that (ζ, δ) is a *surjective equivalence in* $\mathcal{A}$ if δ is a retraction, so that $\delta\zeta = 1_Y$, and there is 2-cell isomorphism

$$\Theta\colon \zeta\delta \xrightarrow{\cong} 1_Z \quad \text{in} \quad \mathcal{A}.$$

Thus, (ζ, δ) is a surjective equivalence in $\mathcal{A}$ if and only if there is a 2-cell isomorphism Θ such that $(\zeta, \delta, 1_{1_Y}, \Theta)$ is an internal equivalence in $\mathcal{A}$. We say that δ is a surjective equivalence if it has a section ζ such that (ζ, δ) is a surjective equivalence.

Adjoint surjective equivalence: We say that (ζ, δ, Θ) is an *adjoint surjective equivalence* if (ζ, δ) is a surjective equivalence with $\Theta\colon \zeta\delta \cong 1_Z$ such that $\Theta * \zeta = 1_\zeta$. Thus, (ζ, δ, Θ) is an adjoint surjective equivalence if and only if $(\zeta, \delta, 1_{1_Y}, \Theta)$ is an internal adjoint equivalence in $\mathcal{A}$. ◇

Definition 4.7. Suppose T has a pseudomorphism classifier $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$. We say that $(\mathsf{Q}, \mathtt{i})$ is *effective* if, for each T-algebra Y, there is a T-algebra 2-cell isomorphism

$$\Theta\colon \zeta_Y \delta_Y \xrightarrow{\cong} 1_{\mathsf{Q}Y}$$

such that $(\zeta_Y, \delta_Y, \Theta)$ is an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}$. In this case, Θ is sometimes called the *efficacy* of $(\mathsf{Q}, \mathtt{i})$. ◇

Theorem 4.8 ([BKP89, Theorem 3.13]). *Suppose that $\mathcal{K}$ is a complete and cocomplete 2-category and suppose that T is a finitary monad on $\mathcal{K}$. Then T has a pseudomorphism classifier.*

Remark 4.9. The hypotheses of Theorem 4.8 are convenient, but not necessary. See [BKP89, Remark 3.14] for a discussion of the completeness hypothesis. The results of Power [Pow89] and Lack [Lac02] give an alternate approach under varying hypotheses, studying a more general coherence for pseudo-algebras. Remark 13.19 below discusses aspects of their work in relation to the applications in Section 13. ◇

Theorem 4.10 ([BKP89, Theorem 4.2]). *Suppose T is a 2-monad on a 2-category $\mathcal{K}$ and suppose that $\mathcal{K}$ admits pseudolimits of 1-cells. If T has a pseudomorphism classifier $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ then it is an effective pseudomorphism classifier in the sense of Definition 4.7.*

Remark 4.11. We note that even though δ_Y in Definition 4.7 and Theorem 4.10 is a strict T-map, it is not guaranteed to have a strict T-map for a pseudoinverse, a condition that would make δ_Y an equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. When there is a strict T-map that is pseudoinverse to δ, then Y is said to be a *flexible* T-algebra. While the full theory of flexible algebras will not be necessary in this work, we will use several results related to flexibility of free algebras from [BKP89, Section 4]. These results are described in Section 5. ◇

5 Effective pseudomorphism classifiers

Throughout this section we suppose that T has an effective pseudomorphism classifier $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ in the sense of Definition 4.7. So, for each T-algebra Y there is a T-algebra 2-cell isomorphism

$$\Theta\colon \zeta_Y \delta_Y \xrightarrow{\cong} 1_{\mathsf{Q}Y}$$

such that the following equalities hold, making $(\zeta_Y, \delta_Y, \Theta)$ an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}$:

$$\delta_Y \zeta_Y = 1_Y \quad \text{and} \quad \Theta * \zeta_Y = 1_{\zeta_Y}. \tag{5.1}$$

In this section, we prove a number of elementary properties that will be used in Sections 7 and 8.

Lemma 5.2. *Suppose C is an object of $\mathcal{K}$. There is a strict T-map*

$$\zeta^\flat_{\mathsf{T}C} \colon \mathsf{T}C \longrightarrow \mathsf{QT}C$$

together with an isomorphism

$$\Theta^\flat \colon \zeta^\flat_{\mathsf{T}C}\delta_{\mathsf{T}C} \xrightarrow{\cong} 1_{\mathsf{QT}C}$$

such that $(\zeta^\flat_{\mathsf{T}C}, \delta_{\mathsf{T}C}, \Theta^\flat)$ is an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg_s}$.

Proof. Consider the composite

$$C \xrightarrow{\eta_C} \mathsf{uT}C \xrightarrow{\mathsf{u}\zeta_{\mathsf{T}C}} \mathsf{uiQT}C \tag{5.3}$$

and define $\zeta^\flat_{\mathsf{T}C}$ as the indicated composite in the following diagram.

$$\begin{tikzcd}
\mathsf{T}C \arrow[rrr, "\zeta^\flat_{\mathsf{T}C}"] \arrow[dr, "\mathsf{T}\eta_C"'] & & & \mathsf{iQT}C \\
& \mathsf{TuT}C \arrow[r, "\mathsf{Tu}\zeta_{\mathsf{T}C}"] & \mathsf{TuiQT}C \arrow[ur, "\varepsilon_{\mathsf{iQT}C}"'] &
\end{tikzcd} \tag{5.4}$$

That is, $\zeta^\flat_{\mathsf{T}C}$ is the mate of (5.3) under the adjunction $\mathsf{T} \dashv \mathsf{u}$ (2.20). For the remainder of this proof, we omit the subscripts $\mathsf{T}C$ on $\delta_{\mathsf{T}C}$, $\zeta_{\mathsf{T}C}$, and $\zeta^\flat_{\mathsf{T}C}$.

Next we consider the composite $\delta\zeta^\flat$. Using the definition of $\zeta^\flat$ in (5.4), naturality of ε with respect to the strict T-map δ gives the first equality below. The second follows from 2-functoriality of uT, the left hand side of (5.1) with $Y = \mathsf{T}C$, and a triangle identity for η and ε.

$$\delta\zeta^\flat = \varepsilon_{\mathsf{T}C} \circ (\mathsf{Tu}\delta) \circ (\mathsf{Tu}\zeta) \circ (\mathsf{T}\eta_C) = 1_{\mathsf{T}C}. \tag{5.5}$$

Next, define

$$\Gamma_\zeta = \Theta * \zeta^\flat \colon \zeta \xrightarrow{\cong} \zeta^\flat \tag{5.6}$$

as shown in the following diagram. Here and below, we omit the notation $\mathtt{i}$, as discussed in Convention 2.18.

$$\begin{tikzcd}
\mathsf{T}C \arrow[rr, "\zeta^\flat"] \arrow[dr, "1"'] & & \mathsf{QT}C \arrow[dl, "\delta"'] \arrow[dr, "1"] \arrow[d, phantom, "\Theta \Nearrow"] & \\
& \mathsf{T}C \arrow[rr, rightsquigarrow, "\zeta"'] & {} & \mathsf{QT}C
\end{tikzcd}$$

The T-algebra 2-cell isomorphism Γ_ζ has the following two properties.

i. The following diagram commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & \mathsf{uT}C \arrow[d, "\mathsf{u}\zeta^\flat"] \\
\mathsf{uT}C \arrow[r, rightsquigarrow, "\mathsf{u}\zeta"] & \mathsf{uQT}C
\end{tikzcd} \tag{5.7}$$

This holds by definition of $\zeta^\flat$ as the mate of $\mathsf{u}\zeta \circ \eta_C$ (5.3). Let χ denote the two equal composites in (5.7):

$$\chi = \mathsf{u}\zeta \circ \eta_C = \mathsf{u}\zeta^\flat \circ \eta_C. \tag{5.8}$$

ii. The whiskering $\mathsf{u}\Gamma_\zeta * \eta_C$ is equal to the identity 2-cell in $\mathcal{K}$ of the 1-cell χ (5.8). This follows from the definition of Γ_ζ (5.6), the commutativity of (5.7), and the right hand side of (5.1):

$$\begin{aligned} \mathsf{u}\Gamma_\zeta * \eta_C &= \mathsf{u}\Theta * (\zeta^\flat \circ \eta_C) \\ &= \mathsf{u}\Theta * (\zeta \circ \eta_C) \\ &= 1_\zeta * \eta_C = 1_\chi. \end{aligned} \tag{5.9}$$

Now we define

$$\Theta^\flat = \Theta \circ \left(\Gamma_\zeta^{-1} * \delta\right) = \Theta \circ \left(\Theta_\zeta^{-1} * (\zeta^\flat \delta)\right) : \zeta^\flat \delta \xrightarrow{\cong} 1_{\mathsf{QT}C}$$

as shown in the following diagram.

$$\begin{tikzcd} & \mathsf{T}C \arrow[rr, "\zeta^\flat"] \arrow[dr, "1"'] & & \mathsf{QT}C \arrow[dl, "\delta"'] \arrow[dddr, "1"] & \\ & & \mathsf{T}C \arrow[ddrr, squiggly, "\zeta"] & \overset{\Theta^{-1}}{\Leftarrow} & \\ & & \Downarrow\Theta & & \\ \mathsf{QT}C \arrow[uuur, "\delta"] \arrow[uurr, "\delta"] \arrow[rrrr, "1"'] & & & & \mathsf{QT}C \end{tikzcd} \tag{5.10}$$

Using the definition of $\Theta^\flat$ and (5.5), we have

$$\begin{aligned} \Theta^\flat * \zeta^\flat &= \left(\Theta * \zeta^\flat\right) \circ \left(\Theta^{-1} * (\zeta^\flat \delta \zeta^\flat)\right) \\ &= \left(\Theta * \zeta^\flat\right) \circ \left(\Theta^{-1} * \zeta^\flat\right) \\ &= 1_{\zeta^\flat}. \end{aligned}$$

This completes the proof that $(\zeta^\flat, \delta, \Theta^\flat)$ is an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. ☐

Remark 5.11. Recalling Remark 4.11, the conclusion of Lemma 5.2 implies that each free T-algebra $\mathsf{T}C$ is flexible. Beyond this, it identifies the adjoint surjective equivalence $(\zeta^\flat_{\mathsf{T}C}, \delta_{\mathsf{T}C}, \Theta^\flat)$ that will be necessary in Sections 7 and 8 below. Moreover, Lemma 5.12 makes use of Γ_ζ and the two properties noted in (5.7) and (5.9). ◇

Lemma 5.12. *Suppose given an object $C \in \mathcal{K}$ and a T-algebra Y together with a T-map*

$$\psi \colon \mathsf{T}C \rightsquigarrow Y.$$

Then there is a unique pair $(\psi^\flat, \Gamma_\psi)$ consisting of a strict T-map $\psi^\flat$ together with an invertible T-algebra 2-cell Γ_ψ

$$\psi^\flat \colon \mathsf{T}C \longrightarrow Y \quad \text{and} \quad \Gamma_\psi \colon \psi \xrightarrow{\cong} \psi^\flat$$

such that the following statements hold.

i. The following diagram commutes in $\mathcal{K}$;

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & \mathsf{uT}C \arrow[d, "\mathsf{u}\psi^\flat"] \\ \mathsf{uT}C \arrow[r, squiggly, "\mathsf{u}\psi"] & \mathsf{u}Y \end{tikzcd} \tag{5.13}$$

let χ_ψ denote either of the two equal composites in (5.13):

$$\chi_\psi = \mathsf{u}\psi \circ \eta_C = \mathsf{u}\psi^\flat \circ \eta_C. \tag{5.14}$$

ii. The whiskering $\mathsf{u}\Gamma_\psi * \eta_C$ is equal to the identity 2-cell in $\mathcal{K}$ of the 1-cell χ_ψ (5.14).

Proof. In the case $Y = \mathsf{QT}C$ and $\psi = \zeta_{\mathsf{T}C} \colon \mathsf{T}C \rightsquigarrow \mathsf{QT}C$, the proof of Lemma 5.2 defines $\zeta^\flat_{\mathsf{T}C}$ and $\Gamma_{\zeta_{\mathsf{T}C}}$ in (5.4) and (5.6). The desired conditions are (5.7) and (5.9).

For general $\psi \colon \mathsf{T}C \rightsquigarrow Y$, let $\psi^\perp \colon \mathsf{QT}C \longrightarrow Y$ be the strict T-map factoring ψ, as in (4.4). This provides the commutative triangle at right in the diagram below.

$$\begin{tikzcd}
\mathsf{iQT}C \arrow[rr, "\psi^{\perp}"] & & Y \\
\mathsf{T}C \arrow[u, bend left=50, "\zeta^{\flat}_{\mathsf{T}C}"] \arrow[u, bend right=50, squiggly, "\zeta_{\mathsf{T}C}"'] \arrow[u, phantom, "\overset{\Gamma_{\zeta_{\mathsf{T}C}}}{\Leftarrow}"] \arrow[urr, squiggly, "\psi"'] & &
\end{tikzcd}$$

We now define

$$\psi^\flat = \psi^\perp \circ \zeta^\flat_{\mathsf{T}C} \quad \text{and} \quad \Gamma_\psi = \psi^\perp * \Gamma_{\zeta_{\mathsf{T}C}}.$$

Thus, Γ_ψ provides a T-algebra 2-cell isomorphism

$$\psi = \psi^\perp \zeta_{\mathsf{T}C} \xrightarrow[\cong]{\Gamma_\psi} \psi^\perp \zeta^\flat_{\mathsf{T}C} = \psi^\flat$$

as desired. The required conditions for $\psi^\flat$ and Γ_ψ now follow from the corresponding ones for $\zeta^\flat$ and Γ_ζ in (5.7) and (5.9):

$$\begin{aligned}
\left(\mathsf{u}\psi^\flat\right)\eta_C &= \left(\mathsf{u}\psi^\perp\right)\left(\mathsf{u}\zeta^\flat\right)\eta_C \\
&= \left(\mathsf{u}\psi^\perp\right)\left(\mathsf{u}\zeta\right)\eta_C \\
&= \left(\mathsf{u}\psi\right)\eta_C
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\left(\mathsf{u}\Gamma_\psi\right) * \eta_C &= \left(\mathsf{u}\psi^\perp\right) * \left(\mathsf{u}\Gamma_{\zeta_{\mathsf{T}C}}\right) * \eta_C \\
&= \left(\mathsf{u}\psi^\perp\right) * 1_\chi \\
&= 1_{\chi_\psi},
\end{aligned}$$

where χ and χ_ψ are the composites in (5.8) and (5.14), respectively. This completes the proof. □

Part II: Universal pseudomorphisms

6 Universal pseudomorphisms

In this section we provide the definition and basic properties of universal pseudomorphisms

$$\widetilde{\phi} \colon \mathsf{T}C \longrightarrow \mathsf{T}(C', \phi)$$

for 1-cells $\phi \colon C \longrightarrow C'$ in $\mathcal{K}$. Recall from Notation 3.5 that $\mathbb{2} = \{0 \longrightarrow 1\}$ denotes the free arrow category.

Definition 6.1. Suppose $\mathsf{T} = (\mathsf{T}, \mu, \eta)$ is a 2-monad on a 2-category $\mathcal{K}$. Recall the free/forgetful adjunction $\mathsf{T} \dashv \mathsf{u}$ from (2.20).

Arrow category: The *arrow category* of $\mathcal{K}$ is denoted $\mathcal{K}^{\mathbb{2}}$. Its objects are 1-cells $\phi \colon C \longrightarrow C'$ in $\mathcal{K}$ and its morphisms $(R, S) \colon \phi \longrightarrow \psi$ are pairs of 1-cells such that $\psi R = S\phi$ in $\mathcal{K}$, as in the diagram at left in (6.2) below.

Strict arrow category of T-maps: The *strict arrow category of* T*-maps* is denoted $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$. Its objects are T-maps $f \colon X \rightsquigarrow X'$ in $\mathsf{T}\text{-}\mathsf{Alg}$ and its morphisms $(j, k) \colon f \longrightarrow g$ are pairs of strict T-maps such that $jf = gk$ in $\mathsf{T}\text{-}\mathsf{Alg}$, as in the diagram at right in (6.2) below.

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "R"'] & C' \arrow[d, "S"] & X \arrow[r, rightsquigarrow, "f"] \arrow[d, "j"'] & X' \arrow[d, "k"] \\
D \arrow[r, "\psi"] & D' & Y \arrow[r, rightsquigarrow, "g"] & Y'
\end{tikzcd} \tag{6.2}$$

The forgetful $\mathsf{u}\colon \mathsf{T}\text{-}\mathsf{Alg} \longrightarrow \mathcal{K}$ induces a functor on arrow categories that we also denote

$$\mathsf{u}\colon \mathsf{T}\text{-}\mathsf{Alg}^{2,s} \longrightarrow \mathcal{K}^{2}. \tag{6.3}$$

◇

Remark 6.4. Both $\mathcal{K}^{2}$ and $\mathsf{T}\text{-}\mathsf{Alg}^{2,s}$ are the underlying 1-categories of 2-categories, with 2-cells given by pairs of 2-cells in $\mathcal{K}$ and $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$,

$$(\Gamma, \Omega)\colon (R, S) \longrightarrow (R', S') \quad \text{and} \quad (\alpha, \gamma)\colon (j, k) \longrightarrow (j', k'),$$

respectively, that satisfy equalities as in (6.2):

$$\Omega * \phi = \psi * \Gamma \quad \text{and} \quad \gamma * f = g * \alpha.$$

Most of our discussion below will restrict to the underlying 1-categories as written in Definition 6.1, but we will refer to the ambient 2-categories using the same notation in Lemma 6.19 below. ◇

Definition 6.5. In the context of Definition 6.1, we say that T admits *universal pseudomorphisms* if, for each 1-cell

$$\phi\colon C \longrightarrow C' \quad \text{in} \quad \mathcal{K}$$

there is a T-algebra $\mathsf{T}(C', \phi)$ and T-map

$$\widetilde{\phi}\colon \mathsf{T}C \rightsquigarrow \mathsf{T}(C', \phi) \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg} \tag{6.6}$$

together with a *unit morphism* in $\mathcal{K}^{2}$

$$(\eta_C, \kappa_\phi)\colon \phi \longrightarrow \mathsf{u}\widetilde{\phi} \quad \text{in} \quad \mathcal{K}^{2}, \tag{6.7}$$

where η is the unit structure transformation of $\mathsf{T} = (\mathsf{T}, \mu, \eta)$, such that the following holds.

Universal property: For each T-map $f\colon X \rightsquigarrow Y$ there is a bijection of sets

$$\mathsf{T}\text{-}\mathsf{Alg}^{2,s}(\widetilde{\phi}, f) \xrightarrow{\cong} \mathcal{K}^{2}(\phi, \mathsf{u}f) \tag{6.8}$$

induced by u and composition with (η_C, κ_ϕ).

In this case, we say that $\widetilde{\phi}\colon \mathsf{T}C \rightsquigarrow \mathsf{T}(C', \phi)$ is the *universal pseudomorphism* for ϕ. ◇

Remark 6.9. In the context of Definition 6.5, the universal property (6.8) is equivalent to the following. For each $f\colon X \rightsquigarrow X'$ in $\mathsf{T}\text{-}\mathsf{Alg}$ and each pair of 1-cells R and S such that $(R, S)\colon \phi \longrightarrow \mathsf{u}f$ in $\mathcal{K}^{2}$, there are unique strict T-maps $\overline{R}$ and $\overline{S}$ so that $(\overline{R}, \overline{S})\colon \widetilde{\phi} \longrightarrow f$ in $\mathsf{T}\text{-}\mathsf{Alg}^{2,s}$ and the diagram below commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "R"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "S"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, rightsquigarrow, "\mathsf{u}\widetilde{\phi}"] \arrow[dl, dashed, "\mathsf{u}\overline{R}"', "\exists!"] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dr, dashed, "\mathsf{u}\overline{S}", "\exists!"'] & \\
\mathsf{u}X \arrow[rrr, rightsquigarrow, "\mathsf{u}f"] & & & \mathsf{u}X'
\end{tikzcd} \tag{6.10}$$

Observe that uniqueness and commutativity of the triangle at left above implies that $\overline{R}$ depends only on R. In contrast, $\overline{S}$ depends on both S and ϕ. ◇

Remark 6.11. For a universal pseudomorphism $\widetilde{\phi}$, the bijection of sets (6.8) implies a certain 1-categorical adjunction that we explain in Lemma 6.14 below. Then, Lemma 6.19 shows, under mild additional hypotheses, that the adjunction extends to a 2-adjunction. However, as we explain further in Remark 6.20, such an extension is (a) not needed for this work and (b) more difficult to verify in practice. These are the reasons that the universal property (6.8) is defined as a mere bijection of sets.◇

Notation 6.12. In the context of Definition 6.5 and Remark 6.9, the mate of κ_ϕ under the adjunction $\mathsf{T} \dashv \mathsf{u}$ is denoted κ and is uniquely determined such that the following commutes.

$$\begin{tikzcd}
C' \arrow[rr, "\kappa_\phi"] \arrow[dr, "\eta_{C'}"'] & & \mathsf{u}\mathsf{T}(C',\phi) \\
& \mathsf{u}\mathsf{T}C' \arrow[ur, "\mathsf{u}\kappa"'] &
\end{tikzcd} \tag{6.13}$$

◇

Recall from Definition 2.19 that η and ε denote, respectively, the unit and counit of the adjunction $\mathsf{T} \dashv \mathsf{u}$.

Lemma 6.14. *Suppose*

- *C, C' are objects of $\mathcal{K}$,*
- *$\phi\colon C \longrightarrow C'$ is an object of $\mathcal{K}^{\mathbb{2}}$,*
- *X, X' are objects of $\mathsf{T}\text{-}\mathsf{Alg}$, and*
- *$f\colon X \rightsquigarrow X'$ is an object of $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$.*

In the context of Definition 6.5, the assignment

$$\phi \longmapsto \widetilde{\phi}$$

is functorial with respect to morphisms in $\mathcal{K}^{\mathbb{2}}$ and is left adjoint to the forgetful $\mathsf{u}\colon \mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s} \longrightarrow \mathcal{K}^{\mathbb{2}}$ from (6.3). The unit and counit of the adjunction $\widetilde{(-)} \dashv \mathsf{u}$ are given, respectively, by

$$\widetilde{\eta}_\phi = (\eta_C, \kappa_\phi) \quad \text{and} \quad \widetilde{\varepsilon}_f = (\varepsilon_X, \overline{1_{\mathsf{u}X'}}). \tag{6.15}$$

Proof. First we define $\widetilde{(-)}$ on morphisms of $\mathcal{K}^{\mathbb{2}}$. Suppose that

$$(R, S)\colon \phi \longrightarrow \psi$$

is a morphism of $\mathcal{K}^{\mathbb{2}}$, where

$$\begin{gathered}
\phi\colon C \longrightarrow C', \quad \psi\colon D \longrightarrow D', \\
R\colon C \longrightarrow D, \quad \text{and} \quad S\colon C' \longrightarrow D'
\end{gathered}$$

are 1-cells of $\mathcal{K}$. Recall from (6.7) the unit morphisms for ϕ and ψ are

$$(\eta_C, \kappa_\phi)\colon \phi \longrightarrow \widetilde{\phi} \quad \text{and} \quad (\eta_D, \kappa_\psi)\colon \psi \longrightarrow \widetilde{\psi}.$$

We now use the universal property (6.8) of $\widetilde{(-)}$, in the form described in Remark 6.9. Composition in $\mathcal{K}^{\mathbb{2}}$ yields the outer vertical morphisms in the diagram below, and the universal property gives the two dashed extensions such that the diagram commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "R"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "S"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[ddl, dashed, "\mathsf{u}(\overline{\eta_D R})"', "\exists!"] & \mathsf{u}\mathsf{T}(C',\phi) \arrow[ddr, dashed, "\mathsf{u}(\overline{\kappa_\psi S})", "\exists!"'] & \\
D \arrow[d, "\eta_D"'] & & & D' \arrow[d, "\kappa_\psi"] \\
\mathsf{u}\mathsf{T}D \arrow[rrr, squiggly, "\mathsf{u}\widetilde{\psi}"] & & & \mathsf{u}\mathsf{T}(D',\psi)
\end{tikzcd} \tag{6.16}$$

By uniqueness, we have $\overline{\eta R} = \mathsf{T}R$. Thus, $\widetilde{(-)}$ is defined on morphisms by

$$\widetilde{R} = \overline{\eta_D R} = \mathsf{T}R \quad \text{and} \quad \widetilde{S} = \overline{\kappa_\psi S}.$$

Uniqueness shows that this assignment is functorial, and commutativity of the triangles at left and right of (6.16) shows that the components (η_C, κ_ϕ) define a natural transformation

$$1_{\mathcal{K}^{\mathbb{2}}} \longrightarrow \mathsf{u}\widetilde{(-)}.$$

This justifies the name *unit* for (η_C, κ_ϕ) in Definition 6.5 and we define

$$\widetilde{\eta}_\phi = (\eta_C, \kappa_\phi).$$

If $f\colon X \rightsquigarrow X'$ is a T-map, we define the counit component

$$\widetilde{\varepsilon}_f = (\varepsilon_X, \overline{1_{\mathsf{u}X'}}).$$

Naturality of $\widetilde{\varepsilon}$ with respect to morphisms $(j,k)\colon f \longrightarrow g$ in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$ follows from uniqueness in the universal property (6.8).

The triangle identities for $\widetilde{\eta}$ and $\widetilde{\varepsilon}$ follow from the definitions and the triangle identities for η and ε. This completes the proof that there is an adjunction $\widetilde{(-)} \dashv \mathsf{u}$ with unit and counit given by (6.15).□

Definition 6.17. Define the *source functors*

$$\mathsf{s}\colon \mathcal{K}^{\mathbb{2}} \longrightarrow \mathcal{K} \quad \text{and} \quad \mathsf{s}\colon \mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$$

by the assignments

$$\begin{aligned} \mathsf{s}(\phi) &= C & \qquad \mathsf{s}(R,S) &= R \\ \mathsf{s}(f) &= X & \qquad \mathsf{s}(j,k) &= j \end{aligned}$$

where

$$\phi\colon C \longrightarrow C', \qquad (R,S)\colon \phi \longrightarrow \psi$$

are 0-, respectively 1-cells in $\mathcal{K}^{\mathbb{2}}$, and

$$f\colon X \rightsquigarrow X', \qquad (j,k)\colon f \longrightarrow g$$

are 0-, respectively 1-cells in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$. ◇

The next result follows from Lemma 6.14 along with Definitions 6.1, 6.5, and 6.17.

Proposition 6.18. *In the context of Definition 6.5, the following diagram of adjunctions serially commutes.*

$$\begin{tikzcd}
\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s} \arrow[r, "\mathsf{s}"] \arrow[d, bend left, "\mathsf{u}"] & \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[d, bend left, "\mathsf{u}"] \\
\mathcal{K}^{\mathbb{2}} \arrow[r, "\mathsf{s}"] \arrow[u, bend left, "\widetilde{(-)}"] \arrow[u, phantom, "\dashv"] & \mathcal{K} \arrow[u, bend left, "\mathsf{T}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

That is, the following equalities hold:

$$\mathsf{s}\mathsf{u} = \mathsf{u}\mathsf{s} \qquad \qquad \mathsf{s}\widetilde{(-)} = \mathsf{T}\mathsf{s}$$

$$\mathsf{s} * \widetilde{\eta} = \eta * \mathsf{s} \qquad \qquad \mathsf{s} * \widetilde{\varepsilon} = \varepsilon * \mathsf{s}.$$

For the next result, we let $\mathcal{K}^{\mathbb{2}}$ and $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$ denote the ambient 2-categories, as described in Remark 6.4. The following 2-dimensional extension is included for completeness and context, but will not be necessary in our further work; Remark 6.20 gives further explanation.

Lemma 6.19. *In the context of Definition 6.5, suppose furthermore that $\mathcal{K}$ admits cotensors of the form $\{\mathbb{2}, -\}$. Then the adjunction $\widetilde{(-)} \dashv \mathsf{u}$ of Lemma 6.14 extends to a 2-adjunction.*

Proof. The hypothesis that $\mathcal{K}$ admits cotensors $\{\mathbb{2}, -\}$ implies, by Proposition 3.7 (*i*) that $\mathsf{T}\text{-}\mathsf{Alg}$ and $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ both admit those cotensors and that the functors $\mathtt{i}$ and u preserve them. The cotensors $\{\mathbb{2}, -\}$ in $\mathcal{K}$ induce the cotensors $\{\mathbb{2}, -\}$ in $\mathcal{K}^{\mathbb{2}}$ pointwise, and the cotensors $\{\mathbb{2}, -\}$ in $\mathsf{T}\text{-}\mathsf{Alg}$ and $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ induce the cotensors $\{\mathbb{2}, -\}$ in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$ pointwise. Moreover, $\mathsf{u}\colon \mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s} \longrightarrow \mathcal{K}^{\mathbb{2}}$ preserves those cotensors. Therefore, by Proposition 3.7 (*ii*), the universal pseudomorphism functor $\widetilde{(-)}$ extends uniquely to a left 2-adjoint of u. ☐

Remark 6.20. As written in Definition 6.5, the universal property of a universal pseudomorphism is a 1-categorical property. The 2-categorical extension that appears in Lemma 6.19 is not needed for any of our work below. It does not appear to simplify any of the proofs of the results used in Theorem 1.9. Furthermore, the 1-categorical version is simpler to verify in cases where one proves that a 2-monad has universal pseudomorphisms. This occurs, for example, in the proof of Theorem 7.11. ◇

Recall from Notation 3.5 that $\mathbb{I}$ denotes the category consisting of two objects and an isomorphism between them. The following is a generalization of [Gur13, Lemma 2.22].

Lemma 6.21. *Suppose that T is a 2-monad on a 2-category $\mathcal{K}$. Suppose that T admits universal pseudomorphisms (Definition 6.5) and suppose given $C, C' \in \mathcal{K}$ and $X, X' \in \mathsf{T}\text{-}\mathsf{Alg}$ together with*

$$\begin{aligned} \phi\colon C &\longrightarrow C' && \text{in} && \mathcal{K}, \\ R\colon C &\longrightarrow \mathsf{u}X && \text{in} && \mathcal{K}, \\ \beta\colon S_1 &\longrightarrow S_2 && \text{in} && \mathcal{K}(C', \mathsf{u}X'), \quad \text{and} \\ \alpha\colon f_1 &\longrightarrow f_2 && \text{in} && \mathsf{T}\text{-}\mathsf{Alg}(X, X') \end{aligned}$$

as shown at left in (6.22) below, such that

$$\beta * \phi = (\mathsf{u}\alpha) * R.$$

$$\begin{tikzcd}
C \arrow[rr, "\phi"] \arrow[d, "R"'] & & C' \arrow[d, bend right=50, "S_1"', ""{name=A}] \arrow[d, bend left=50, "S_2", ""'{name=B}] \arrow[from=A, to=B, phantom, "\overset{\beta}{\Rightarrow}"] \\
\mathsf{u}X \arrow[rr, bend left=20, squiggly, "\mathsf{u}f_1", ""'{name=C}] \arrow[rr, bend right=20, squiggly, "\mathsf{u}f_2"', ""{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow \mathsf{u}\alpha"] & & \mathsf{u}X'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}C \arrow[rr, squiggly, "\widetilde{\phi}"] \arrow[d, "\overline{R}"'] & & \mathsf{T}(C', \phi) \arrow[d, bend right=50, "\overline{S}_1"', ""{name=A}] \arrow[d, bend left=50, "\overline{S}_2", ""'{name=B}] \arrow[from=A, to=B, phantom, "\overset{\overline{\beta}}{\Rightarrow}"] \\
X \arrow[rr, bend left=20, squiggly, "f_1", ""'{name=C}] \arrow[rr, bend right=20, squiggly, "f_2"', ""{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow \alpha"] & & X'
\end{tikzcd}
\tag{6.22}$$

Then the following statements hold.

i. If $\mathcal{K}$ admits cotensors of the form $\{\mathbb{2}, -\}$, then there is a unique T-algebra 2-cell $\overline{\beta}\colon \overline{S}_1 \longrightarrow \overline{S}_2$ at right in (6.22) such that

$$\overline{\beta} * \widetilde{\phi} = \alpha * \overline{R}.$$

Here, for $i = 1, 2$,

$$(\overline{R}, \overline{S}_i)\colon \widetilde{\phi} \longrightarrow f_i$$

is the pair of unique strict T-maps determined by the universal property (6.8) of $\widetilde{\phi}$.

ii. If $\mathcal{K}$ admits cotensors of the form $\{\mathbb{I}, -\}$, and if α and β are invertible, then there is a unique T*-algebra 2-cell $\overline{\beta}$ as above, and $\overline{\beta}$ is invertible.*

Proof. We begin with the first assertion. Recalling Proposition 3.7 (*i*), the assumption that $\mathcal{K}$ admits cotensors $\{\mathbb{2}, -\}$ implies the same for both $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ and $\mathsf{T}\text{-}\mathsf{Alg}$. Furthermore, the inclusion $\mathtt{i}$ and the forgetful functors u preserve those cotensors.

Using the fact that u preserves cotensor products and unpacking the definition of cotensors for $\{\mathbb{2}, X'\}$, as in Remark 3.6, the diagrams in (6.22) correspond to the diagrams in (6.23) below, with F and $\mathsf{u}F$ corresponding, respectively, to the triples (f_1, f_2, α) and $(\mathsf{u}f_1, \mathsf{u}f_2, \mathsf{u}\alpha)$. Likewise, S and $\overline{S}$ correspond, respectively, to the triples (S_1, S_2, β) and $(\overline{S}_1, \overline{S}_2, \overline{\beta})$.

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "R"'] & C' \arrow[d, "S"] & & \mathsf{T}C \arrow[r, squiggly, "\widetilde{\phi}"] \arrow[d, "\overline{R}"'] & \mathsf{T}(C', \phi) \arrow[d, "\overline{S}"] \\
\mathsf{u}X \arrow[r, squiggly, "\mathsf{u}F"] & \{\mathbb{2}, \mathsf{u}X'\} & & X \arrow[r, squiggly, "F"] & \{\mathbb{2}, X'\}
\end{tikzcd} \tag{6.23}$$

With this reformulation, the assertion (*i*) follows by applying the universal property (6.16) to (R, S) at left in (6.23): there is a unique morphism $(\overline{R}, \overline{S})\colon \widetilde{\phi} \rightsquigarrow F$ in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$, at right in (6.23), such that

$$(R, S) = \mathsf{u}(\overline{R}, \overline{S}) \circ (\eta_C, \kappa_\phi).$$

Unpacking the above equation, via Remark 3.6 and a similar description of 2-cells in (3.2), yields the data $(\overline{S_1}, \overline{S_2}, \overline{\beta})$ at right in (6.22) satisfying the desired equations. Using cotensors with the category $\mathbb{I} = \{0 \cong 1\}$ instead of $\mathbb{2}$ yields the second assertion, in which all of the 2-cells are invertible. ☐

Definition 6.24. Suppose T is a 2-monad on $\mathcal{K}$ that admits universal pseudomorphisms. For each 1-cell $\phi\colon C \longrightarrow C'$ in $\mathcal{K}$, define a strict T-map

$$\Delta = \overline{\eta_{C'}}\colon \mathsf{T}(C', \phi) \longrightarrow \mathsf{T}C' \tag{6.25}$$

as follows. The unit η defines a morphism

$$(\eta_C, \eta_{C'})\colon \phi \longrightarrow \mathsf{u}\mathsf{T}\phi \quad \text{in} \quad \mathcal{K}^{\mathbb{2}}.$$

Therefore, by the universal property (6.8) there is a unique morphism $(\overline{\eta_C}, \overline{\eta_{C'}})$ in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$ as shown in the diagram below. By uniqueness, $\overline{\eta_C}$ is the identity $1_{\mathsf{T}C}$.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "\eta_C"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "\eta_{C'}"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[dl, dashed, "\exists!"', "\mathsf{u}\overline{\eta_C} = 1"] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dr, dashed, "\exists!", "\mathsf{u}\Delta = \mathsf{u}\overline{\eta_{C'}}"'] & \\
\mathsf{u}\mathsf{T}C \arrow[rrr, "\mathsf{u}\mathsf{T}\phi"] & & & \mathsf{u}\mathsf{T}C'
\end{tikzcd} \tag{6.26}$$

Define $\Delta = \overline{\eta_{C'}}$. ◇

7 Universal pseudomorphisms via pushouts

Throughout Sections 7 and 8, T is assumed to have an effective pseudomorphism classifier (Definition 4.7). The goal of this section is to show that universal pseudomorphisms for T (Definition 6.5) can be constructed as pushouts in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. In Section 14 we explain applications in the case that T is one of the 2-monads for strict monoidal structures (Notation 11.1).

Definition 7.1. Suppose T is a 2-monad on a 2-category $\mathcal{K}$ such that T has an effective pseudomorphism classifier and $\mathsf{T}\text{-}\mathsf{Alg_s}$ admits pushouts. For each 1-cell $\phi\colon C \longrightarrow C'$ in $\mathcal{K}$, define a T-algebra $\mathsf{T}(C',\phi)$ together with

- a T-map $\widetilde{\phi}\colon \mathsf{T}C \rightsquigarrow \mathsf{T}(C',\phi)$ and
- a 1-cell $\kappa_\phi\colon C' \longrightarrow \mathtt{u}\mathsf{T}(C',\phi)$ in $\mathcal{K}$.

as follows.

The unit of (4.2) is a T-map

$$\zeta_{\mathsf{T}C}\colon \mathsf{T}C \rightsquigarrow \mathtt{i}\mathsf{QT}C. \tag{7.2}$$

By Lemma 5.12, $\zeta_{\mathsf{T}C}$ is isomorphic to a unique strict T-map

$$\zeta^\flat\colon \mathsf{T}C \longrightarrow \mathtt{i}\mathsf{QT}C \tag{7.3}$$

such that the diagram below commutes.

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "\eta_C"'] & \mathtt{u}\mathsf{T}C \arrow[d, "\mathtt{u}\zeta^\flat"] \\
\mathtt{u}\mathsf{T}C \arrow[r, rightsquigarrow, "\mathtt{u}\zeta_{\mathsf{T}C}"] & \mathtt{u}\mathtt{i}\mathsf{QT}C
\end{tikzcd} \tag{7.4}$$

Define $\mathsf{T}(C',\phi)$ as the pushout in $\mathsf{T}\text{-}\mathsf{Alg_s}$ of $\zeta^\flat$ and $\mathsf{T}\phi$, with structure morphisms $\widehat{\phi}$ and κ as shown in the square below.

$$\begin{tikzcd}
\mathsf{T}C \arrow[r, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & \mathsf{T}C' \arrow[d, "\kappa"] \\
\mathtt{i}\mathsf{QT}C \arrow[r, "\widehat{\phi}"] & \mathsf{T}(C',\phi)
\end{tikzcd} \tag{7.5}$$

Moreover, define $\widetilde{\phi}$ and κ_ϕ by the following composites in $\mathsf{T}\text{-}\mathsf{Alg}$ and $\mathcal{K}$, respectively.

$$\begin{tikzcd}
\mathsf{T}C \arrow[rr, rightsquigarrow, "\widetilde{\phi}"] \arrow[dr, rightsquigarrow, "\zeta_{\mathsf{T}C}"'] & & \mathsf{T}(C',\phi) & C' \arrow[rr, "\kappa_\phi"] \arrow[dr, "\eta_{C'}"'] & & \mathtt{u}\mathsf{T}(C',\phi) \\
& \mathtt{i}\mathsf{QT}C \arrow[ur, "\mathtt{i}\widehat{\phi}"'] & & & \mathtt{u}\mathsf{T}C' \arrow[ur, "\mathtt{u}\kappa"'] &
\end{tikzcd} \tag{7.6}$$

This completes the definition of

$$\widetilde{\phi}\colon \mathsf{T}C \longrightarrow \mathsf{T}(C',\phi) \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg}$$

and the unit

$$(\eta_C, \kappa_\phi)\colon \phi \longrightarrow \mathtt{u}\widetilde{\phi} \quad \text{in} \quad \mathcal{K}^2.$$

We show that these satisfy the universal property (6.8) in Theorem 7.11 below. ◇

In the following, we use Convention 2.18 and implicitly apply the inclusion $\mathtt{i}$ to compose a general T-map with a strict one.

Lemma 7.7. *In the context of Definition 7.1, suppose given*

- *a* T*-map* $f\colon X \rightsquigarrow X'$ *in* $\mathsf{T}\text{-}\mathsf{Alg}$*,*
- *1-cells* $R\colon C \longrightarrow \mathtt{u}X$ *and* $S\colon C' \longrightarrow \mathtt{u}X'$ *in* $\mathcal{K}$*, and*

- *strict* T*-maps*

$$\overline{R} \colon \mathsf{T}C \longrightarrow X$$
$$\overline{S}_1,\, \overline{S}_2 \colon \mathsf{T}(C', \phi) \longrightarrow X'$$

such that, for each $i = 1, 2$,

$$\overline{S}_i\, \widetilde{\phi} = f\, \overline{R} \colon \mathsf{T}C \longrightarrow X' \quad \textit{in} \quad \mathsf{T}\text{-}\mathsf{Alg} \tag{7.8}$$

and the following diagram commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "R"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "S"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[dl, "\mathsf{u}\overline{R}"'] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dr, "\mathsf{u}\overline{S}_i"] & \\
\mathsf{u}X \arrow[rrr, squiggly, "\mathsf{u}f"] & & & \mathsf{u}X'
\end{tikzcd} \tag{7.9}$$

Then $\overline{S}_1 = \overline{S}_2$ *in* $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$.

Proof. To use the universal property of the pushout (7.5) defining $\mathsf{T}(C', \phi)$, we will show

$$\overline{S}_1 \kappa = \overline{S}_2 \kappa \quad \text{and} \quad \overline{S}_1 \widehat{\phi} = \overline{S}_2 \widehat{\phi}. \tag{7.10}$$

For the first of these, we obtain

$$\mathsf{u}\big(\overline{S}_1 \kappa\big) \circ \eta_{C'} = S = \mathsf{u}\big(\overline{S}_2 \kappa\big) \circ \eta_{C'}$$

using 2-functoriality of u, the definition $\kappa_\phi = \mathsf{u}\kappa \circ \eta_{C'}$ from (7.6), and commutativity of the triangle at right in (7.9). Then the uniqueness of mates noted in Remark 2.22 implies that $\overline{S}_1 \kappa = \overline{S}_2 \kappa$.

For the second desired equality in (7.10), we obtain

$$\big(\overline{S}_1 \widehat{\phi}\big) \circ \zeta_{\mathsf{T}C} = f\, \overline{R} = \big(\overline{S}_2 \widehat{\phi}\big) \circ \zeta_{\mathsf{T}C}$$

using the associativity of 1-cell composition, the definition $\widetilde{\phi} = \widehat{\phi} \zeta_{\mathsf{T}C}$ in (7.6), and the hypothesis (7.8). Then uniqueness of mates, for the adjunction $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$, implies that $\overline{S}_1 \widehat{\phi} = \overline{S}_2 \widehat{\phi}$. The result $\overline{S}_1 = \overline{S}_2$ then follows from the universal property of the pushout (7.5). ☐

Theorem 7.11. *In the context of Definition 7.1, the pushouts* $\mathsf{T}(C', \phi)$ *in (7.5) determine universal pseudomorphisms for* T.

Proof. We show that $\widetilde{\phi}$ and κ_ϕ, as defined in (7.6), satisfy the universal property (6.8) for each 1-cell

$$\phi \colon C \longrightarrow C' \quad \text{in} \quad \mathcal{K}$$

and each T-map

$$f \colon X \rightsquigarrow X' \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg}\,.$$

For this purpose, suppose given 1-cells R and S in $\mathcal{K}$, as in the outer diagram (7.12) below. Following Remark 6.9, we will show that there are unique strict T-maps $\overline{R}$ and $\overline{S}$ such that

$$\overline{S}\, \widetilde{\phi} = f\, \overline{R} \colon \mathsf{T}C \rightsquigarrow X' \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg}$$

and the following diagram commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "R"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "S"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[dl, dashed, "\mathsf{u}\overline{R}"', "\exists!"] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dr, dashed, "\mathsf{u}\overline{S}", "\exists!"'] & \\
\mathsf{u}X \arrow[rrr, squiggly, "\mathsf{u}f"] & & & \mathsf{u}X'
\end{tikzcd} \tag{7.12}$$

Recall from Definition 2.3 that $x \colon \mathsf{T}X \longrightarrow X$ denotes the T-algebra structure 1-cell for X. We define

$$\overline{R} = x \circ \mathsf{T}R \colon \mathsf{T}C \longrightarrow X \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg_s}$$

and note that each of the following diagrams commutes by naturality of η and the unit condition (2.4) for X.

$$\begin{tikzcd}
C \arrow[r, "R"] \arrow[d, "\eta_C"'] & \mathsf{u}X \arrow[d, "\eta_X"'] \arrow[dr, "1_{\mathsf{u}X}"] & \\
\mathsf{u}\mathsf{T}C \arrow[r, "\mathsf{u}\mathsf{T}R"] \arrow[rr, bend right, "\mathsf{u}\overline{R}"'] & \mathsf{u}\mathsf{T}X \arrow[r, "\mathsf{u}x"] & \mathsf{u}X
\end{tikzcd}
\qquad
\begin{tikzcd}
C' \arrow[r, "S"] \arrow[d, "\eta_{C'}"'] & \mathsf{u}X' \arrow[d, "\eta_{X'}"'] \arrow[dr, "1_{\mathsf{u}X'}"] & \\
\mathsf{u}\mathsf{T}C' \arrow[r, "\mathsf{u}\mathsf{T}S"] & \mathsf{u}\mathsf{T}X' \arrow[r, "\mathsf{u}x'"] & \mathsf{u}X'
\end{tikzcd} \tag{7.13}$$

The diagram at left above shows that the triangle at left in (7.12) commutes. Uniqueness of $\overline{R}$ follows from the uniqueness of mates noted in Remark 2.22.

Next, the strict T-map $\overline{S}$ will be defined using the universal property of the pushout (7.5). Consider the following diagram in $\mathcal{K}$, explained below.

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "\eta_C"'] \arrow[ddddrr, "R"', rounded corners, to path={-- ([xshift=-8ex]\tikztostart.west) |- ([yshift=-4ex]\tikztotarget.south) \tikztonodes -- (\tikztotarget.south)}] & C' \arrow[dr, "\eta_{C'}"] \arrow[dddrr, "S", rounded corners, to path={-- ([xshift=30ex]\tikztostart.east) |- (\tikztotarget.east) \tikztonodes}] & & \\
& \mathsf{u}\mathsf{T}C \arrow[r, "\mathsf{u}\mathsf{T}\phi"] \arrow[d, "\mathsf{u}\zeta^\flat"'] & \mathsf{u}\mathsf{T}C' \arrow[dr, "\mathsf{u}\mathsf{T}S"] & \\
\mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\zeta_{\mathsf{T}C}"] \arrow[dd, "\mathsf{u}\mathsf{T}R"'] \arrow[ddrr, "\mathsf{u}\overline{R}"'] & \mathsf{u}\mathsf{i}\mathsf{Q}\mathsf{T}C \arrow[dr, "\mathsf{u}\mathsf{i}\mathsf{Q}\overline{R}"'] & \star & \mathsf{u}\mathsf{T}X' \arrow[d, "\mathsf{u}x'"] \\
& & \mathsf{u}\mathsf{i}\mathsf{Q}X \arrow[r, "\mathsf{u}f^\perp"'] & \mathsf{u}X' \\
\mathsf{u}\mathsf{T}X \arrow[rr, "\mathsf{u}x"'] & & \mathsf{u}X \arrow[u, squiggly, "\mathsf{u}\zeta_X"] \arrow[ur, squiggly, "\mathsf{u}f"'] &
\end{tikzcd} \tag{7.14}$$

In the above diagram, the two upper-left quadrilateral regions commute by (7.4) and naturality of η, respectively. The lower left triangle commutes by definition of $\overline{R}$ in (7.13). In the lower right triangle, $f^\perp$ is the strict mate of f in (4.4) and hence the triangle commutes by definition. The lower trapezoid region commutes by naturality of ζ, and the two outer regions commute by (7.13). The outer diagram commutes by the hypothesis $\mathsf{u}f \circ R = S \circ \phi$ in (7.12).

Referring to the region $\star$ in (7.14) above, let

$$h_1 = f^\perp \circ (\mathsf{Q}\overline{R}) \circ \zeta^\flat \quad \text{and} \quad h_2 = x' \circ (\mathsf{T}S) \circ (\mathsf{T}\phi).$$

The above argument, together with 2-functoriality of u, shows that $\mathsf{u}h_1 \circ \eta_C = \mathsf{u}h_2 \circ \eta_C$. Therefore, because h_1 and h_2 are strict T-maps, we conclude $h_1 = h_2$ by the uniqueness of mates noted in Remark 2.22.

The strict T-maps h_1 and h_2 are the two composites around the boundary of the diagram in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$ shown below. Since these are equal, there is a unique strict T-map $\overline{S}$ induced by the universal property of the pushout (7.5).

$$\begin{tikzcd}
\mathsf{T}C \arrow[r, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & \mathsf{T}C' \arrow[d, "\kappa"'] \arrow[dr, "\mathsf{T}S"] & \\
\mathsf{QT}C \arrow[r, "\widehat{\phi}"] \arrow[dr, "\mathsf{Q}\overline{R}"'] & \mathsf{T}(C',\phi) \arrow[dr, dashed, "\overline{S}", "\exists!"'] & \mathsf{T}X' \arrow[d, "x'"] \\
 & \mathsf{Q}X \arrow[r, "f^\perp"'] & X'
\end{tikzcd} \tag{7.15}$$

The construction of $\overline{S}$ then shows the following two equalities required for $\overline{S}$. First, using the definition $\widetilde{\phi} = \widehat{\phi}\zeta_{\mathsf{T}C}$ in (7.6), the lower left parallelogram in (7.15), naturality of ζ, and the equality $f^\perp \zeta_C = f$ in (4.4), we have

$$\begin{aligned}
\overline{S}\,\widetilde{\phi} &= \overline{S}\,\widehat{\phi}\,\zeta_{\mathsf{T}C} \\
&= f^\perp\,(\mathsf{Q}\overline{R})\,\zeta_{\mathsf{T}C} \\
&= f^\perp\,\zeta_X\,\overline{R} \\
&= f\,\overline{R}.
\end{aligned}$$

Second, using the definition $\kappa_\phi = \mathsf{u}\kappa \circ \eta_{C'}$ from (7.6), 2-functoriality of u, the lower right parallelogram in (7.15), and the equality $S = (\mathsf{u}x') \circ (\mathsf{uT}S) \circ \eta_{C'}$ from the diagram at right in (7.13), we have

$$\begin{aligned}
(\mathsf{u}\overline{S})\,\kappa_\phi &= (\mathsf{u}\overline{S})\,(\mathsf{u}\kappa)\,\eta_{C'} \\
&= (\mathsf{u}x')\,(\mathsf{uT}S)\,\eta_{C'} \\
&= S.
\end{aligned}$$

This completes the construction of $\overline{S}$ and the proof that it satisfies the required equalities. Uniqueness of $\overline{S}$ is proved in Lemma 7.7. This completes the proof. ☐

8 The equivalence Δ

In this section we assume that

- T has an effective pseudomorphism classifier (Definition 4.7) and
- T admits universal pseudomorphisms (Definition 6.5).

This section contains two results showing that the canonical comparison (6.25)

$$\Delta\colon \mathsf{T}(C',\phi) \longrightarrow \mathsf{T}C'$$

is a surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. Its inverse is the strict T-map in Notation 6.12

$$\kappa\colon \mathsf{T}C' \longrightarrow \mathsf{T}(C',\phi),$$

defined as the mate of $\kappa_\phi\colon C' \longrightarrow \mathsf{uT}(C',\phi)$.

Theorem 8.1. *Suppose* T *is a 2-monad on* $\mathcal{K}$ *that admits an effective pseudomorphism classifier* $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ *and universal pseudomorphisms* $\widetilde{\phi}$. *Suppose, moreover, that* $\mathcal{K}$ *admits cotensors of the form* $\{\mathbb{I}, -\}$. *Then the strict* T*-map*

$$\Delta = \overline{\eta_{C'}} \colon \mathsf{T}(C', \phi) \longrightarrow \mathsf{T}C'$$

in (6.25) is a surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ *with inverse*

$$\kappa \colon \mathsf{T}C' \longrightarrow \mathsf{T}(C', \phi)$$

in (6.13).

Proof. This argument consists of the following two steps.

i. Show that $\Delta\kappa = 1_{\mathsf{T}C'}$.

ii. Define an invertible T-algebra 2-cell

$$\overline{\beta} \colon \kappa\Delta \cong 1_{\mathsf{T}(C',\phi)}.$$

To begin, recall κ_ϕ from (6.7) is part of the unit morphism for $\widetilde{\phi}$. The strict T-map κ is uniquely determined such that the outer triangle of the following diagram commutes in $\mathcal{K}$.

$$\begin{tikzcd}
C' \arrow[rr, "\kappa_\phi"] \arrow[dr, "\eta_{C'}"'] & & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dl, shift right, "\mathsf{u}\Delta"'] \\
& \mathsf{u}\mathsf{T}C' \arrow[ur, shift right, "\mathsf{u}\kappa"'] &
\end{tikzcd} \tag{8.2}$$

The definition of Δ (6.26) implies that the inner triangle above also commutes in $\mathcal{K}$. Together these give the following equalities:

$$\begin{aligned}
\eta_{C'} &= \mathsf{u}\Delta \circ \kappa_\phi \\
&= \mathsf{u}\Delta \circ \mathsf{u}\kappa \circ \eta_{C'} \\
&= \mathsf{u}(\Delta \circ \kappa) \circ \eta_{C'}.
\end{aligned}$$

Since Δ and κ are both strict T-maps, the uniqueness of mates (Remark 2.22) implies

$$\Delta \circ \kappa = 1_{\mathsf{T}C'} \tag{8.3}$$

as desired.

Now we give the construction of β. By hypothesis, there is a T-map (6.6)

$$\widetilde{\phi} \colon \mathsf{T}C \rightsquigarrow \mathsf{T}(C', \phi) \quad \text{in} \quad \mathsf{T}\text{-}\mathsf{Alg}$$

satisfying the universal property (6.8). Applying Lemma 5.12 gives an isomorphism

$$\Gamma \colon \widetilde{\phi} \xrightarrow{\cong} \widetilde{\phi}^\flat \tag{8.4}$$

such that $\mathsf{u}\Gamma * \eta_C = 1$. Now consider the following computation, beginning with Lemma 5.12 (*i*) and continuing with the indicated justifications.

$$\begin{aligned}
\mathsf{u}\widetilde{\phi}^\flat \circ \eta_C &= \mathsf{u}\widetilde{\phi} \circ \eta_C \\
&= \kappa_\phi \circ \phi && \text{by (6.10) top} \\
&= \mathsf{u}\kappa \circ \eta_{C'} \circ \phi && \text{by (8.2)} \\
&= \mathsf{u}\kappa \circ \mathsf{u}\mathsf{T}\phi \circ \eta_C && \text{by naturality of } \eta \\
&= \mathsf{u}(\kappa \circ \mathsf{T}\phi) \circ \eta_C && \text{by functoriality of } \mathsf{u}
\end{aligned} \tag{8.5}$$

Hence, uniqueness of mates implies

$$\widetilde{\phi}^\flat = \kappa \circ \mathsf{T}\phi. \tag{8.6}$$

The equalities

$$\mathsf{u}\widetilde{\phi}^\flat \circ \eta_C = \mathsf{u}\widetilde{\phi} \circ \eta_C = \kappa_\phi \circ \phi$$

in (8.5) also show that (η_C, κ_ϕ) defines a morphism in $\mathcal{K}^{\mathbb{2}}$ from ϕ to $\mathsf{u}\widetilde{\phi}^\flat$. Applying the universal property of $\widetilde{\phi}$ (6.8) determines a morphism $(1, \overline{\kappa_\phi})$ in $\mathsf{T}\text{-}\mathsf{Alg}^{\mathbb{2},s}$, as shown in the following diagram.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[dd, "\eta_C"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[dd, "\kappa_\phi"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[dl, dashed, "1"', "\exists!"] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[dr, dashed, "\mathsf{u}\overline{\kappa_\phi}", "\exists!"'] & \\
\mathsf{u}\mathsf{T}C \arrow[rrr, "\mathsf{u}\widetilde{\phi}^\flat"] & & & \mathsf{u}\mathsf{T}(C', \phi)
\end{tikzcd} \tag{8.7}$$

Now observe that the outer diagram above can also be filled as below.

$$\begin{tikzcd}
C \arrow[rrr, "\phi"] \arrow[dr, "\eta_C"] \arrow[ddd, "\eta_C"'] & & & C' \arrow[dl, "\kappa_\phi"'] \arrow[ddd, "\kappa_\phi"] \arrow[ddl, bend left, "\eta_{C'}"] \\
& \mathsf{u}\mathsf{T}C \arrow[r, squiggly, "\mathsf{u}\widetilde{\phi}"] \arrow[ddl, dashed, "1"', "\exists!"] & \mathsf{u}\mathsf{T}(C', \phi) \arrow[d, "\mathsf{u}\Delta"'] & \\
& & \mathsf{u}\mathsf{T}C' \arrow[dr, "\mathsf{u}\kappa"] & \\
\mathsf{u}\mathsf{T}C \arrow[urr, "\mathsf{u}\mathsf{T}\phi"] \arrow[rrr, "\mathsf{u}\widetilde{\phi}^\flat"'] & & & \mathsf{u}\mathsf{T}(C', \phi)
\end{tikzcd} \tag{8.8}$$

The triangles at right and bottom commute by (8.2) and (8.6), respectively. The remaining interior is that of (6.26) defining Δ. By universality of $\widetilde{\phi}$, (6.8), we conclude

$$\kappa \circ \Delta = \overline{\kappa_\phi}.$$

Finally, we use the hypothesis that $\mathcal{K}$ admits cotensors of the form $\{\mathbb{I}, -\}$ and apply Lemma 6.21 (*ii*) to the diagram at left below, where β is the identity 2-cell of κ_ϕ. This application yields a 2-cell $\overline{\beta}$ as shown in the diagram at right below.

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "\eta"'] & C' \arrow[d, bend right=50, "\kappa_\phi"', ""{name=A}] \arrow[d, bend left=50, "\kappa_\phi", ""'{name=B}] \\
\mathsf{u}\mathsf{T}C \arrow[r, bend left, squiggly, "\mathsf{u}\widetilde{\phi}", ""'{name=C}] \arrow[r, bend right, "\mathsf{u}\widetilde{\phi}^\flat"', ""{name=D}] & \mathsf{u}\mathsf{T}(C', \phi)
\arrow[from=A, to=B, Rightarrow, "\beta"]
\arrow[from=C, to=D, Rightarrow, "\mathsf{u}\Gamma"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{T}C \arrow[r, squiggly, "\widetilde{\phi}"] \arrow[d, "1"'] & \mathsf{T}(C', \phi) \arrow[d, bend right=50, "1"', ""{name=A}] \arrow[d, bend left=50, "\overline{\kappa_\phi}", ""'{name=B}] \\
\mathsf{T}C \arrow[r, bend left, squiggly, "\widetilde{\phi}", ""'{name=C}] \arrow[r, bend right, "\widetilde{\phi}^\flat"', ""{name=D}] & \mathsf{T}(C', \phi)
\arrow[from=A, to=B, Rightarrow, "\overline{\beta}"]
\arrow[from=C, to=D, Rightarrow, "\Gamma"]
\end{tikzcd}$$

Since Γ is an isomorphism and $\beta = 1$, the resulting $\overline{\beta}$ is an invertible T-algebra 2-cell

$$\overline{\beta} \colon 1 \cong \overline{\kappa_\phi} = \kappa \circ \Delta.$$

This completes the proof that Δ and κ are inverse equivalences in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. □

Theorem 8.9. *Suppose* T *is a 2-monad on* $\mathcal{K}$ *that admits an effective pseudomorphism classifier* $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ *and universal pseudomorphisms* $\widetilde{\phi}$*. If* $\mathsf{T}(C', \phi)$ *is constructed as the pushout (7.5) in* $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$*, then*

$$\Delta = \overline{\eta_{C'}} \colon \mathsf{T}(C', \phi) \longrightarrow \mathsf{T}C'$$

in (6.25) is an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$*.*

Proof. Consider the following diagram, where the upper square is the pushout (7.5) and ω is described below.

$$\begin{tikzcd}
\mathsf{T}C \arrow[r, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & \mathsf{T}C' \arrow[d, "\kappa"] \arrow[ddr, bend left, "1"] & & \\
\mathtt{i}\mathsf{Q}\mathsf{T}C \arrow[r, "\widehat{\phi}"] \arrow[d, "\delta"'] \arrow[dd, bend right=60, "1"'] \arrow[dd, phantom, shift right=3ex, "\overset{\Theta^\flat}{\Leftarrow}"] & \mathsf{T}(C', \phi) \arrow[dr, dashed, "\exists!", "\omega"'] & & \\
\mathsf{T}C \arrow[rr, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & & \mathsf{T}C' \arrow[dr, "\kappa"] & \\
\mathtt{i}\mathsf{Q}\mathsf{T}C \arrow[rrr, "\widehat{\phi}"] & & & \mathsf{T}(C', \phi)
\end{tikzcd} \tag{8.10}$$

Here, $(\zeta^\flat, \delta, \Theta^\flat)$ is the adjoint surjective equivalence of Lemmas 5.2 and 5.12, with $\psi = \zeta$ in the latter. In particular, we have

$$\delta\zeta^\flat = 1_{\mathsf{T}C} \quad \text{and} \quad \Theta^\flat * \zeta^\flat = 1_{\zeta^\flat}. \tag{8.11}$$

The left hand side of (8.11) implies that the two solid arrow composites from $\mathsf{T}C$ in the upper left to the lower right instance of $\mathsf{T}C'$ in (8.10) are equal. Hence, we define ω as the induced strict T-map out of the pushout $\mathsf{T}(C', \phi)$, indicated by the dashed arrow in (8.10). Note that $\omega\kappa = 1$ by construction.

Next, whiskering $\widehat{\phi}$ with the isomorphism $\Theta^\flat$ gives an isomorphism

$$\kappa\omega\widehat{\phi} = \widehat{\phi}\zeta^\flat\delta \xrightarrow[\cong]{\widehat{\phi} * \Theta^\flat} \widehat{\phi} \quad \text{with} \quad \big(\widehat{\phi} * \Theta^\flat\big) * \zeta^\flat = \widehat{\phi} * 1_{\zeta^\flat} = 1_{\widetilde{\phi}}, \tag{8.12}$$

by the right hand side of (8.11) and the left hand side of (7.6). Thus, the two-dimensional aspect of the pushout implies that there is an isomorphism

$$\Psi \colon \kappa\omega \xrightarrow{\cong} 1_{\mathsf{T}(C', \phi)}$$

such that $\widehat{\phi} * \Psi = \Theta^\flat * \widehat{\phi}$ and $\Psi * \kappa = 1_\kappa$.

This shows that (κ, ω, Ψ) is an adjoint surjective equivalence in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$. From uniqueness of $\Delta = \overline{\mathtt{u}\eta'_C}$ in (6.26), it follows that $\omega = \Delta$. □

Remark 8.13. Note that Theorems 8.1 and 8.9 require slightly different hypotheses. Theorem 8.1 requires certain limits in $\mathcal{K}$, in the form of cotensors, and Theorem 8.9 requires certain colimits in $\mathsf{T}\text{-}\mathsf{Alg}_\mathsf{s}$, in the form of pushouts. ◇

Remark 8.14 (Consideration of lax coherence). The theory of pseudomorphism classifiers from Section 4 has a parallel variant for *lax morphism classifiers*, and some of the development in Section 5 can be generalized to the lax case. One can likewise generalize much of Section 6 to a notion of *universal lax morphism.*

However, the efficacy Θ for an effective lax morphism classifier is generally not invertible. The construction of $\Theta^\flat$ in (5.10) requires invertibility of Θ, and this is used in the proofs of Lemmas 5.2 and 5.12. The proofs of Theorems 8.1 and 8.9 above depend crucially on Lemmas 5.2 and 5.12, and hence do not apparently generalize to the lax case. ◇

9 Constructing Q via universal pseudomorphisms

Throughout this section, we suppose that T admits universal pseudomorphisms (Definition 6.5). The goal of this section is to show that this hypothesis determines a pseudomorphism classifier for T via certain coequalizers in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$. We first recall *reflexive pairs* of morphisms, and then introduce the more specialized notion of P*-free pairs* in Definition 9.13.

Definition 9.1. A *reflexive pair* in a category C is a pair of parallel morphisms f and g with a common section t,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right=60, "t"'] \end{tikzcd} \quad \text{so that} \quad gt = ft = 1_Y. \tag{9.2}$$

◇

Remark 9.3. Recall from Example 3.14 that each T-algebra (X, x) is the coequalizer of a canonical u-split pair, with splittings below and the forgetful u suppressed.

$$\begin{tikzcd} \mathsf{T}^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}x"'] & \mathsf{T}X \arrow[l, bend right=60, "\eta_{\mathsf{T}X}"'] \arrow[r, "x"'] & X \arrow[l, bend right=50, "\eta_X"'] \end{tikzcd}$$

Furthermore, $\mathsf{T}\eta_X$ provides a common splitting for μ and $\mathsf{T}x$, so that the following is a reflexive pair in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$.

$$\begin{tikzcd} \mathsf{T}^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}x"'] & \mathsf{T}X \arrow[l, bend right=60, "\mathsf{T}\eta_X"'] \end{tikzcd} \tag{9.4}$$

◇

Definition 9.5. For each object $C \in \mathcal{K}$, define

$$\mathsf{P}C = \mathsf{T}(C, 1_C) \tag{9.6}$$

as in (6.6), with $\phi = 1_C$. For a T-map $f\colon \mathsf{T}C \rightsquigarrow \mathsf{T}C'$, with $C, C' \in \mathcal{K}$, define

$$\mathsf{P}f = \overline{S} \quad \text{for} \quad S = \widetilde{1_{C'}} \circ \big(\mathsf{u}f\big) \circ \eta_C. \tag{9.7}$$

That is, $\overline{S}$ is the unique strict T-map determined by the universal property (6.10), as shown in the following diagram.

$$\begin{tikzcd} C \arrow[rrrr, "1_C"] \arrow[dr, "\eta_C"] \arrow[ddd, "\eta_C"'] & & & & C \arrow[dl, "\kappa_{1_C}"'] \arrow[d, "\eta_C"] \\ & \mathsf{u}\mathsf{T}C \arrow[rr, squiggly, "\mathsf{u}\widetilde{1_C}"] \arrow[ddl, dashed, "1"'] & & \mathsf{u}\mathsf{T}(C, 1_C) \arrow[ddr, dashed, "\mathsf{u}\overline{S}", "\exists !"'] & \mathsf{u}\mathsf{T}C \arrow[d, squiggly, "\mathsf{u}f"] \\ & & & & \mathsf{u}\mathsf{T}C' \arrow[d, "\mathsf{u}\widetilde{1_{C'}}"] \\ \mathsf{u}\mathsf{T}C \arrow[rr, squiggly, "\mathsf{u}f"] & & \mathsf{u}\mathsf{T}C' \arrow[rr, squiggly, "\mathsf{u}\widetilde{1_{C'}}"] & & \mathsf{u}\mathsf{T}(C', 1_{C'}) \end{tikzcd} \tag{9.8}$$

◇

Notation 9.9. Recalling Notation 3.4, we use

$$\mathsf{T}\text{-}\mathsf{Alg}_0 \quad \text{and} \quad \mathsf{T}\text{-}\mathsf{Alg}_{s0}$$

below to denote the underlying 1-categories of $\mathsf{T}\text{-}\mathsf{Alg}$ and $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$, respectively. ◇

Definition 9.10. Let $\mathcal{F}$ denote the category whose objects are 0-cells of $\mathcal{K}$ and with hom sets

$$\mathcal{F}(C, C') = \mathsf{T}\text{-}\mathsf{Alg}_0(\mathsf{T}C, \mathsf{T}C') \quad \text{for} \quad C, C' \in \mathcal{K}.$$

◇

Remark 9.11. We note that $\mathcal{F}$ is similar to the Kleisli category for the underlying monad T_0 on $\mathcal{K}_0$, but has T-maps as morphisms instead of strict T-maps. ◇

Proposition 9.12. *Let $I\colon \mathcal{F} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_0$ denote the functor given by T on objects and the identity on morphisms. Then, in the context of Definition 9.5, P defines a functor*

$$\mathsf{P}\colon \mathcal{F} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_{s0}\,.$$

Furthermore, the components $\widetilde{1}_C\colon \mathsf{T}C \longrightarrow \mathsf{T}(C, 1_C) = \mathsf{P}C$ in (9.8) define a natural transformation

$$\widetilde{1}\colon I \longrightarrow \mathtt{i}\mathsf{P}.$$

Proof. Functoriality of P follows from uniqueness of $\overline{S}$ in (9.8). Naturality of $\widetilde{1}$ follows from the commutativity of the lower trapezoid in (9.8). ☐

Definition 9.13. Suppose that (X, x) is a T-algebra. Recall from (9.4) that $(\mu, \mathsf{T}x)$ is a reflexive pair in $\mathcal{F}$. The P-*free pair* associated to (X, x) is the pair of strict T-maps $(\mathsf{P}\mu, \mathsf{P}\mathsf{T}x)$:

$$\begin{tikzcd} \mathsf{P}\mathsf{T}X \arrow[r, shift left, "\mathsf{P}\mu"] \arrow[r, shift right, "\mathsf{P}\mathsf{T}x"'] & \mathsf{P}X \end{tikzcd} \tag{9.14}$$

We say that $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ *admits coequalizers of* P-*free pairs* if there is a coequalizer of (9.14) in $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ for each T-algebra (X, x). ◇

Remark 9.15. In the context of Definition 9.13, the pair $(\mu, \mathsf{T}x)$ is a reflexive pair, and thus the same holds for $(\mathsf{P}\mu, \mathsf{P}\mathsf{T}x)$. Thus, if $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ admits coequalizers of reflexive pairs, then $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ admits coequalizers of P-free pairs in particular. ◇

Definition 9.16. Suppose that $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ admits coequalizers of P-free pairs, and suppose (X, x) is a T-algebra. Define $\mathsf{Q}X$ as the following coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$.

$$\begin{tikzcd} \mathsf{P}\mathsf{T}X \arrow[r, shift left, "\mathsf{P}\mu"] \arrow[r, shift right, "\mathsf{P}\mathsf{T}x"'] & \mathsf{P}X \arrow[r, dashed] & \mathsf{Q}X \end{tikzcd} \tag{9.17}$$

◇

Recalling Example 3.14 and Proposition 3.25, each T-algebra (X, x) is the coequalizer, in both $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ and $\mathsf{T}\text{-}\mathsf{Alg}$ and their respective underlying categories, of the pair $(\mu, \mathsf{T}x)$.

Definition 9.18. For each $X \in \mathsf{T}\text{-}\mathsf{Alg}_0$, define a morphism ζ_X to be the unique T-map induced by the universal property of (X, x) as the coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_0$, as shown in the following diagram with $\mathtt{i}$ suppressed. The squares at left commute by naturality of $\widetilde{1}$ in Proposition 9.12.

$$\begin{tikzcd} \mathsf{T}^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}x"'] \arrow[d, squiggly, "\widetilde{1}"'] & \mathsf{T}X \arrow[r, "x"] \arrow[d, squiggly, "\widetilde{1}"] & X \arrow[d, dashed, squiggly, "\exists!"', "\zeta_X"] \\ \mathsf{P}\mathsf{T}X \arrow[r, shift left, "\mathsf{P}\mu"] \arrow[r, shift right, "\mathsf{P}\mathsf{T}x"'] & \mathsf{P}X \arrow[r] & \mathsf{Q}X \end{tikzcd} \tag{9.19}$$

◇

Definition 9.20. For each $Y \in \mathsf{T}\text{-}\mathsf{Alg}_{s0}$, define a strict T-map

$$\delta_Y\colon \mathsf{Q}Y \longrightarrow Y$$

as follows. Recall from (6.25) the strict T-maps

$$\Delta = \overline{\eta_{C'}}\colon \mathsf{T}(C',\phi) \longrightarrow \mathsf{T}C' \quad \text{for} \quad \phi\colon C \longrightarrow C' \in \mathcal{K}.$$

In the case $\phi = 1_C$, this gives a strict T-map

$$\Delta_C\colon \mathsf{P}C \longrightarrow \mathsf{T}C. \tag{9.21}$$

Naturality of the components Δ_C with respect to strict T-maps $h\colon \mathsf{T}C \longrightarrow \mathsf{T}C'$ follows from the definition of $\mathsf{P}h$ (9.8) and uniqueness of $\overline{S}$ in the universal property (6.10) with $\phi = 1_C$, $f = h$, and $S = \Delta \circ \widetilde{1} \circ h \circ \eta_C$.

Define δ_Y as the unique strict T-map induced by the universal property of Q as the coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$, as shown in the following diagram with u suppressed. The squares at left commute by naturality of Δ.

$$\begin{tikzcd}
\mathsf{PT}Y \arrow[r, shift left, "\mathsf{P}\mu"] \arrow[r, shift right, "\mathsf{PT}y"'] \arrow[d, "\Delta_{\mathsf{T}Y}"'] & \mathsf{P}Y \arrow[r] \arrow[d, "\Delta_Y"'] & \mathsf{Q}Y \arrow[d, dashed, "\delta_Y"'] \\
\mathsf{T}^2Y \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}y"'] & \mathsf{T}Y \arrow[r, "y"] & Y
\end{tikzcd} \tag{9.22}$$

◇

Lemma 9.23. *Given a T-map $f\colon X \rightsquigarrow X'$, there are unique strict T-maps $\overline{f}$ and $f^{\perp}$ that make the following diagram commute in $\mathsf{T}\text{-}\mathsf{Alg}_0$, with u and i suppressed.*

$$\begin{tikzcd}
\mathsf{T}^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathsf{T}x"'] \arrow[d, squiggly, "\widetilde{1}"'] & \mathsf{T}X \arrow[r, "x"] \arrow[d, squiggly, "\widetilde{1}"] & X \arrow[d, squiggly, "\zeta_X"'] \arrow[dr, squiggly, "f"] & \\
\mathsf{PT}X \arrow[r, shift left, "\mathsf{P}\mu"] \arrow[r, shift right, "\mathsf{PT}x"'] & \mathsf{P}X \arrow[r] \arrow[rr, dashed, bend right, "\exists!", "\overline{f}"'] & \mathsf{Q}X \arrow[r, dashed, "f^{\perp}", "\exists!"'] & X'
\end{tikzcd} \tag{9.24}$$

Here,

- *$\overline{f}$ is the unique strict T-map such that $\overline{f} \circ \widetilde{1} = f \circ x$ and*
- *$f^{\perp}$ is the unique strict T-map such that $f^{\perp} \circ \zeta_X = f$.*

In particular, if $f = 1_Y$, then $f^{\perp} = \delta_Y$ by uniqueness.

Proof. The strict T-map $\overline{f}$ in (9.24) is induced by the universal property (6.10) with $(R, S) = (\mathsf{u}1_X, \mathsf{u}f)$. The asserted uniqueness of $\overline{f}$ is that of (6.10).

The universal property for $\mathsf{PT}X = \mathsf{T}(\mathsf{T}X, 1_{\mathsf{T}X})$ implies that $\overline{f}$ coequalizes $\mathsf{P}\mu$ and $\mathsf{PT}x$. The strict T-map $f^{\perp}$ is thus induced by universality of $\mathsf{Q}X$ as the coequalizer in $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$. The equality $f^{\perp} \circ \zeta_X = f$, in the triangle at right in (9.24), follows by commutativity of the right-hand square in (9.24) and universality of X as the coequalizer of $(\mu, \mathsf{T}x)$.

The asserted uniqueness of $f^{\perp}$ follows from the uniqueness of $\overline{f}$ and uniqueness in the universal property of $\mathsf{Q}X$. Indeed, suppose $f^{\dagger}\colon \mathsf{Q}X \longrightarrow X'$ is any strict T-map such that $f^{\dagger} \circ \zeta_X = f$, and let $\ell\colon \mathsf{P}X \longrightarrow \mathsf{Q}X$ denote the structure morphism in (9.24). Commutativity of the triangle and square at right in (9.24) implies $f^{\dagger} \circ \ell \circ \widetilde{1} = f \circ x$, and so $f^{\dagger} \circ \ell$ is equal to $\overline{f}$ by uniqueness. This, in turn, implies $f^{\dagger} = f^{\perp}$ by uniqueness in the universal property of $\mathsf{Q}X$. □

Definition 9.25. Given a T-map $f\colon X \rightsquigarrow X'$, define a strict T-map

$$\mathsf{Q}f = \big(\zeta_{X'} \circ f\big)^{\perp} \colon \mathsf{Q}X \longrightarrow \mathsf{Q}X' \tag{9.26}$$

as the unique strict T-map of Lemma 9.23 associated to the composite $\zeta_{X'} \circ f$. Thus, $\mathsf{Q}f$ is the unique strict T-map such that the following diagram commutes in $\mathsf{T}\text{-}\mathsf{Alg}_0$.

$$\begin{tikzcd} X \arrow[dr, squiggly, "f"] \arrow[dd, squiggly, "\zeta_X"'] & & \\ & X' \arrow[dr, squiggly, "\zeta_{X'}"] & \\ \mathsf{Q}X \arrow[rr, dashed, "\exists!", "{\mathsf{Q}f = \big(\zeta_{X'} \circ f\big)^{\perp}}"'] & & \mathsf{Q}X' \end{tikzcd} \tag{9.27}$$

◇

Proposition 9.28. *There is a functor*

$$\mathsf{Q}\colon \mathsf{T}\text{-}\mathsf{Alg}_0 \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_{s0} \tag{9.29}$$

with object and morphism assignments given respectively by (9.17) and (9.26). Furthermore, the components of (9.19) and (9.22) define respective natural transformations

$$\zeta\colon 1_{\mathsf{T}\text{-}\mathsf{Alg}_0} \longrightarrow \mathtt{i}\mathsf{Q} \quad \textit{and} \quad \delta\colon \mathsf{Q}\mathtt{i} \longrightarrow 1_{\mathsf{T}\text{-}\mathsf{Alg}_{s0}} \tag{9.30}$$

Proof. Functoriality of Q follows from uniqueness of the strict T-maps $\mathsf{Q}f = \big(\zeta_{X'} \circ f\big)^{\perp}$ in (9.27). Naturality of ζ with respect to T-maps f holds by definition of $\mathsf{Q}f$, since the triangle (9.27) is the naturality square for ζ. Naturality of δ with respect to strict T-maps $g\colon Y \longrightarrow Y'$ follows from naturality of ζ, the equality $\delta_X \circ \zeta_X = 1_X$ in Lemma 9.23, and uniqueness of the strict T-maps $f^{\perp}$ in (9.24). □

Theorem 9.31. *Suppose T is a 2-monad on $\mathcal{K}$ that admits universal pseudomorphisms $\widetilde{\phi}$. Suppose that $\mathcal{K}$ admits cotensors of the form $\{\mathbb{2}, X\}$ and suppose that $\mathsf{T}\text{-}\mathsf{Alg}_{s0}$ admits coequalizers of P-free pairs (Definition 9.13). Then the functor Q, together with unit ζ and counit δ, in Proposition 9.28 extends to a 2-functor that is left 2-adjoint to $\mathtt{i}$.*

Proof. Recalling Proposition 3.7 (*i*) and (*ii*), with $V = \mathtt{i}$, it suffices to show $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ is an adjunction of underlying 1-categories.

$$\begin{tikzcd} \mathsf{T}\text{-}\mathsf{Alg}_0 \arrow[r, bend left, "\mathsf{Q}"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-}\mathsf{Alg}_{s0} \arrow[l, bend left, "\mathtt{i}"] \end{tikzcd}$$

To do this, first recall from Lemma 9.23 that, for each T-map $f\colon X \rightsquigarrow X'$ there is a unique strict T-map $f^{\perp}\colon \mathsf{Q}X \longrightarrow X'$ such that $f^{\perp} \circ \zeta_X = f$. The existence and uniqueness $f^{\perp}$ shows that composition with components of ζ induces a bijection of morphism sets

$$\mathsf{T}\text{-}\mathsf{Alg}_{s0}(\mathsf{Q}X, X') \xrightarrow[\cong]{- \circ \zeta_X} \mathsf{T}\text{-}\mathsf{Alg}_0(X, \mathtt{i}X')$$

for each pair of T-algebras X and X'. Naturality of such a bijection follows from associativity of 1-cell composition and naturality of ζ. Therefore, $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ is an adjunction of underlying 1-categories, as desired. □

Part III: Applications to strict monoidal structures

10 Formal diagrams

This section develops the context for formal diagrams in the case $\mathcal{K} = \mathcal{C}at$, the 2-category of small categories. Recall, for a monad T that admits universal pseudomorphisms, the counit (6.15) at a T-map $f\colon X \rightsquigarrow X'$ is

$$\widetilde{\varepsilon}_f = (\varepsilon_X, \overline{1_{X'}}).$$

Here, $\varepsilon_X = x$ is the algebra structure morphism for X and $\overline{1_{X'}}$ is the unique strict T-map such that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{u}X \arrow[rrr, "\mathsf{u}f"] \arrow[dr, "\eta_X"] \arrow[dd, "1_X"'] & & & \mathsf{u}X' \arrow[dl, "\kappa_f"'] \arrow[dd, "1_{X'}"] \\
& \mathsf{u}\mathsf{T}X \arrow[r, squiggly, "\mathsf{u}\widetilde{f}"] \arrow[dl, dashed, "\exists!"', "\mathsf{u}\varepsilon_X = \mathsf{u}x"] & \mathsf{u}\mathsf{T}(X', f) \arrow[dr, dashed, "\exists!", "\mathsf{u}\overline{1_{X'}}"'] & \\
\mathsf{u}X \arrow[rrr, squiggly, "\mathsf{u}f"] & & & \mathsf{u}X'
\end{tikzcd} \tag{10.1}$$

Over $\mathcal{K} = \mathcal{C}at$, each T-algebra X has an underlying set of objects, $\mathsf{ob}X$. Thus, we have the following.

Definition 10.2. Suppose T is a 2-monad on $\mathcal{C}at$ that admits universal pseudomorphisms (Definition 6.5). For each T-map

$$f\colon (X, x) \rightsquigarrow (X', x'),$$

define a strict T-map Λ as the composite below,

$$\begin{tikzcd}
\mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}) \arrow[rr, "\Lambda"] \arrow[dr] & & X' \\
& \mathsf{T}(X', f) \arrow[ur, "\overline{1_{X'}}"'] &
\end{tikzcd} \tag{10.3}$$

where f_{ob} denotes the restriction of f to objects, the unlabeled strict T-map is induced by the inclusion of objects $\mathsf{ob}X' \hookrightarrow X'$, and $\overline{1_{X'}}$ is part of the counit $\widetilde{\varepsilon}_f$ in (10.1). Equivalently, Λ is the unique strict T-map induced by the universal property (6.10) in the following diagram, where the unlabeled arrows are induced by inclusion of objects.

$$\begin{tikzcd}
\mathsf{ob}X \arrow[rrr, "f_{\mathsf{ob}}"] \arrow[dr, "\eta_X"] \arrow[ddd] & & & \mathsf{ob}X' \arrow[dl, "\kappa_f"'] \arrow[ddd] \\
& \mathsf{u}\mathsf{T}(\mathsf{ob}X) \arrow[r, squiggly, "\mathsf{u}\widetilde{f_{\mathsf{ob}}}"] \arrow[d] \arrow[ddl, dashed, "\exists!"] & \mathsf{u}\mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}) \arrow[d] \arrow[ddr, dashed, "\Lambda", "\exists!"'] & \\
& \mathsf{u}\mathsf{T}X \arrow[r, squiggly, "\mathsf{u}\widetilde{f}"] \arrow[dl, "x"] & \mathsf{u}\mathsf{T}(X', f) \arrow[dr, "\overline{1_{X'}}"'] & \\
\mathsf{u}X \arrow[rrr, squiggly, "\mathsf{u}f"] & & & \mathsf{u}X'
\end{tikzcd} \tag{10.4}$$

◇

Remark 10.5. Note, in the context of Definition 10.2, that

$$\Lambda\colon \mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}) \longrightarrow X' \tag{10.6}$$

is generally distinct from the following composite of x' with the canonical comparison Δ of (6.25), where the unlabeled arrow is again induced by inclusion of objects:

$$\mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}) \longrightarrow \mathsf{T}(X', f) \xrightarrow{\Delta} \mathsf{T}X' \xrightarrow{x'} X'. \tag{10.7}$$

Indeed, if f is a strict T-map, so that the algebra constraint $f_{\bullet}$ in (2.6) is an identity, then uniqueness of Λ in (10.4) will imply that (10.6) and (10.7) are equal. In general however, they are distinct, and their difference is a key feature of our examples in Section 15. ◇

Definition 10.8. Suppose T is a 2-monad on $\mathcal{C}at$ and (X, x) is a T-algebra. In the following, the unlabeled arrows are induced by inclusions of objects

$$\mathsf{ob}X \hookrightarrow X \quad \text{and} \quad \mathsf{ob}X' \hookrightarrow X'.$$

Diagram: A *diagram* $(\mathbb{D}, D)$ in X consists of a small category $\mathbb{D}$ and a functor $D \colon \mathbb{D} \longrightarrow X$. We consider a morphism $s \colon a \longrightarrow b$ in X as a diagram by taking $\mathbb{D} = \mathbb{2}$, with D sending the unique morphism of $\mathbb{2}$ to s.

Formal diagram: A diagram $(\mathbb{D}, D)$ in X is called a *formal diagram for* X or an X*-formal diagram* if there is a lift $\widetilde{D}$ such that the following commutes in $\mathcal{C}at$. In this case, $\widetilde{D}$ is called an X*-formal lift of* $(\mathbb{D}, D)$.

$$\begin{tikzcd}
& \mathsf{T}(\mathsf{ob}X) \arrow[d] \\
& \mathsf{T}X \arrow[d, "x"] \\
\mathbb{D} \arrow[r, "D"] \arrow[uur, dashed, "\widetilde{D}"] & X
\end{tikzcd} \tag{10.9}$$

Formal diagram for a T-map: Suppose that T admits universal pseudomorphisms (6.8), and suppose that $f \colon (X, x) \rightsquigarrow (X', x')$ is a T-map. A diagram $(\mathbb{D}, D)$ in X' is called a *formal diagram for* f or an f*-formal diagram* if there is a lift $\widetilde{D}$ such that the triangle at left below commutes in $\mathcal{C}at$, where f_{ob} denotes the restriction of f to objects and Λ is defined in (10.3). In this case, $\widetilde{D}$ is called an f*-formal lift of* $(\mathbb{D}, D)$.

$$\begin{tikzcd}
& \mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}) \arrow[r, "\Delta"] \arrow[d] \arrow[dd, bend left=60, "\Lambda"] & \mathsf{T}(\mathsf{ob}X') \arrow[d] \\
& \mathsf{T}(X', f) \arrow[d, "\overline{1_{X'}}"'] & \mathsf{T}X' \arrow[d, "x'"] \\
\mathbb{D} \arrow[r, "D"] \arrow[uur, dashed, "\widetilde{D}"] & X' & X'
\end{tikzcd} \tag{10.10}$$

Dissolution: If $(\mathbb{D}, D)$ is a formal diagram for f with lift $\widetilde{D}$ as in (10.10), the *dissolution* of $\widetilde{D}$, denoted $|D|$, is the composite

$$|D| = \Delta \circ \widetilde{D} \colon \mathbb{D} \longrightarrow \mathsf{T}(\mathsf{ob}X').$$

Finite generation: In the above contexts, a lift $\widetilde{D}$ for a formal diagram is said to be *finitely generated* if there is a finite set of objects $G \subset \mathsf{ob}X$ such that $\widetilde{D}$ factors through, respectively, the strict T-map

$$\mathsf{T}G \longrightarrow \mathsf{T}(\mathsf{ob}X) \quad \text{or} \quad \mathsf{T}(G', f_G) \longrightarrow \mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}),$$

induced by inclusion of objects, where f_G denotes the restriction of f_{ob} to G.

In any of the above cases, we say that a diagram $(\mathbb{D}, D)$ *commutes* if we have $D(u) = D(v)$ for every parallel pair of morphisms u and v in $\mathbb{D}$. ◇

Remark 10.11 (Using dissolution diagrams). Suppose, in the context of Definition 10.8, that $(\mathbb{D}, D)$ is a formal diagram for f, with lift $\widetilde{D}$ to $\mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}})$. Suppose, furthermore, that Δ is an equivalence, as in Theorems 1.5 and 1.9.

Then, for each pair of parallel morphisms u and v in $\mathbb{D}$, the lifts $\widetilde{D}(u)$ and $\widetilde{D}(v)$ are equal in $\mathsf{T}(\mathsf{ob}X', \phi)$ if and only if their dissolutions $|D|(u)$ and $|D|(v)$ are equal in $\mathsf{T}(\mathsf{ob}X')$. Hence, the diagram $(\mathbb{D}, \widetilde{D})$ commutes in $\mathsf{T}(\mathsf{ob}X', \phi)$ if and only if the dissolution diagram $(\mathbb{D}, |D|)$ commutes in $\mathsf{T}(\mathsf{ob}X')$. Furthermore, commutativity of $(\mathbb{D}, \widetilde{D})$ implies that of the original diagram $(\mathbb{D}, D)$.

Note, however, that the distinction in Remark 10.5 implies D and $|D|$ generally give *distinct* diagrams in X'. That is, for each morphism u in $\mathbb{D}$, the morphisms in X' determined by $D(u)$ and $|D|(u)$—composing the latter along the right hand side of (10.10)—are generally not equal in X'.

Thus, if Δ is an equivalence, the dissolution diagram $(\mathbb{D}, |D|)$ is a diagram that is generally different from the given diagram $(\mathbb{D}, D)$, and yet commutativity of the former implies that of the latter. Section 15 contains a variety of examples that demonstrate this phenomenon. ◇

Remark 10.12 (Formal diagrams that factor through κ). In the context of Definition 10.8, recall from (6.13) the strict T-map

$$\kappa \colon \mathsf{T}(\mathsf{ob}X') \longrightarrow \mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}})$$

is the mate of

$$\kappa_{f_{\mathsf{ob}}} \colon \mathsf{ob}X' \longrightarrow \mathsf{T}(\mathsf{ob}X', f_{\mathsf{ob}}).$$

Note that the composite $\Delta \circ \kappa$ is equal to the identity $1_{\mathsf{T}(\mathsf{ob}X')}$, as in (8.3).

Each X'-formal diagram is trivially an f-formal diagram by composing its lift $\widetilde{D}$ with κ. In such a case, for the dissolution diagram $|D|$ obtained by composing with Δ, we have

$$|D| = \Delta \circ (\kappa \circ \widetilde{D}) = \widetilde{D}.$$

We will say that an f-formal lift *reduces to an X'-formal lift* if it factors through κ. ◇

11 Strict monoidal structures

We use the following notations for the 2-monads on $\mathcal{K} = \mathcal{C}at$ whose algebras are general or strict monoidal structures in the plain, symmetric, and braided monoidal cases. For basic definitions and properties, we refer the reader to [ML98, Chapter XI], [JS93], and [Yau24, Chapter 1].

Here, we give a brief description of the relevant 2-monads. See, e.g., [Lac02, Section 4]. More detailed descriptions will not be required, but can be found in operadic presentations such as, e.g., [Yau21, Part 4] or [JY24, Chapters 11 and 12]. We use a superscript g to denote the *general* monoidal cases, and use unadorned notation for the strict monoidal cases.

Notation 11.1 (Monads for monoidal structures).

Plain monoidal: Let M^{g} denote the 2-monad whose algebras are monoidal categories. Let M denote the 2-monad whose algebras are strict monoidal categories.

For a category C, the free strict monoidal category $\mathsf{M}C$ has objects given by tuples $\langle a \rangle = (a_1, \ldots, a_n)$, for $n \geq 0$, with $a_i \in C$ for $i \in \{1, \ldots, n\}$. The morphisms of $\mathsf{M}C$ are tuples of morphisms, so that the underlying category of $\mathsf{M}C$ is $\coprod_n C^n$. The monoidal product is given by concatenation and the monoidal unit is the empty tuple.

Symmetric monoidal: Let S^{g} denote the 2-monad whose algebras are symmetric monoidal categories. Let S denote the 2-monad whose algebras are symmetric strict monoidal categories, also known as *permutative categories*.

For a category C, the free symmetric strict monoidal category $\mathsf{S}C$ has the same objects and monoidal structure as $\mathsf{M}C$. The morphisms of $\mathsf{S}C$ are generated by those of $\mathsf{M}C$, together with permutations of the tuples $\langle a \rangle$. In particular, for a single object a, the free symmetric strict monoidal category $\mathsf{S}\{a\}$ has an object for each natural number n, corresponding to the n-tuple $(a, \ldots, a)$. The hom sets are given by

$$\big(\mathsf{S}\{a\}\big)(m,n) \cong \begin{cases} \emptyset, & \text{if } \; m \neq n, \\ \Sigma_m, & \text{if } \; m = n, \end{cases} \tag{11.2}$$

where the symmetry isomorphism $\beta_{a,a}$ is identified with the transposition $(1\ 2)$.

Braided monoidal: Let B^{g} denote the 2-monad whose algebras are braided monoidal categories. Let B denote the 2-monad whose algebras are braided strict monoidal categories.

For a category C, the free braided strict monoidal category $\mathsf{B}C$ has the same objects and monoidal structure as $\mathsf{M}C$ and $\mathsf{S}C$. The morphisms of $\mathsf{B}C$ are generated by those of $\mathsf{M}C$ together with braidings of strands labeled by the entries of the tuples $\langle a \rangle$.

In the cases $\mathsf{T} = \mathsf{M}, \mathsf{S}, \mathsf{B}$, respectively, the T^{g}-maps and T-maps are plain, symmetric, and braided monoidal functors. These are also sometimes called plain/symmetric/braided *strong* monoidal functors. We will suppress the additional adjective except where it is useful to emphasize the distinction with *strict* T- or T^{g}-maps. The latter are the plain/symmetric/braided *strict* monoidal functors, so they have identity monoidal and unit constraints.

In both the symmetric and braided cases, a T-map $f\colon A \rightsquigarrow B$ satisfies an additional *braid axiom*, expressed as commutativity of the following diagram for $a, a' \in A$. Here, $\bullet$ and β denote the monoidal products and symmetry/braid isomorphisms, respectively, in both A and B.

$$\begin{tikzcd} f(a) \bullet f(a') \arrow[r, "\beta_{f(a),f(a')}"] \arrow[d, "f_2"'] & f(a') \bullet f(a) \arrow[d, "f_2"] \\ f(a \bullet a') \arrow[r, "f(\beta_{a,a'})"] & f(a' \bullet a) \end{tikzcd} \tag{11.3}$$

In all three cases $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$, the T-algebra 2-cells are monoidal transformations. ◇

In each case of Notation 11.1, algebras for the strict monoidal monads T are also algebras for the general monoidal monads T^{g}, with $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$. There is a morphism of monads

$$\theta^{\mathsf{T}} \colon \mathsf{T}^{\mathsf{g}} \longrightarrow \mathsf{T}$$

for each T, and changing monad structure along this morphism is the forgetful functor from the strict to general variants.

The statements in the following result are equivalent to the general coherence theorems [ML98, VII.2, Corollary], [ML98, XI.1, Theorem 1], and [JS93, Corollary 2.6], respectively.

Theorem 11.4 (Monoidal Strictification). *Suppose C is a category. Each of*

$$\theta^{\mathsf{M}} \colon \mathsf{M}^{\mathsf{g}}C \longrightarrow \mathsf{M}C$$
$$\theta^{\mathsf{S}} \colon \mathsf{S}^{\mathsf{g}}C \longrightarrow \mathsf{S}C$$
$$\theta^{\mathsf{B}} \colon \mathsf{B}^{\mathsf{g}}C \longrightarrow \mathsf{B}C$$

is a plain, respectively symmetric, respectively braided, strict monoidal functor, and is an equivalence.

Corollary 11.5. *For each monad $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$, commutativity of a formal diagram $(\mathbb{D}, D)$ with lift*

$$\widetilde{D} \colon \mathbb{D} \longrightarrow \mathsf{T}^{\mathsf{g}}(\mathsf{ob}X)$$

is determined by that of the composite

$$\mathbb{D} \xrightarrow{\widetilde{D}} \mathsf{T}^{\mathsf{g}}(\mathsf{ob}X) \xrightarrow{\theta^{\mathsf{T}}} \mathsf{T}(\mathsf{ob}X).$$

Diagrammatic coherence for strict monoidal structures

Our applications to coherence for strong monoidal functors in Section 15 will make use of the corresponding coherence theorems for monoidal structures on categories. We recall these in Theorem 11.9 below, making use of the following concepts.

Definition 11.6. Suppose G is a set, regarded as a discrete category.

Underlying braids: Each morphism $s\colon \langle a\rangle \longrightarrow \langle b\rangle$ in the braided strict monoidal category $\mathsf{B}G$ has an *underlying braid* $\upsilon(s)$ determined as follows.

- For an identity morphism, $\upsilon(1) = 1$, the identity braid.
- For a composite, $\upsilon(s's) = \upsilon(s')\upsilon(s)$, the composition of braids.
- For a concatenation, $\upsilon(s' + s) = \upsilon(s') \oplus \upsilon(s)$, the block sum of braids.
- For the braid isomorphism, $\upsilon(\beta_{\langle a\rangle,\langle a'\rangle})$ is the elementary block braid that passes the block of strands labeled by $\langle a\rangle$ under the block of strands labeled by $\langle a'\rangle$, without braiding within either block.

Underlying permutations: Each morphism $s\colon \langle a\rangle \longrightarrow \langle b\rangle$ in the symmetric strict monoidal category $\mathsf{S}G$ has an *underlying permutation* $\pi(s)$ defined as the underlying permutation of the underlying braid $\upsilon(s)$.

Underlying permutations, respectively braids, in the more general $\mathsf{S}^{\mathsf{g}}G$, respectively $\mathsf{B}^{\mathsf{g}}G$, are defined via the equivalences θ^{S}, respectively θ^{B}. ◇

Notation 11.7. Let

$$\mathbb{P} = \Big\{ \begin{tikzcd} 0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 1 \end{tikzcd} \Big\} \tag{11.8}$$

denote the free parallel arrow category, consisting of two objects and two parallel morphisms, s and t, between them. ◇

Theorem 11.9 (Monoidal Coherence). *Suppose A is a monoidal, respectively symmetric monoidal, respectively braided monoidal category. Suppose $(\mathbb{P}, D)$ is a formal diagram with lift $\widetilde{D}$, classifying a pair of parallel morphisms Ds and Dt in A.*

i. In the plain monoidal case, $\mathsf{M}^{\mathsf{g}}(\mathsf{ob}A)$ has at most one morphism between any pair of objects, so $\widetilde{D}s = \widetilde{D}t$ and hence $Ds = Dt$ [ML98, VII.2].

ii. In the symmetric case, if the underlying permutations $\pi(\widetilde{D}s)$ and $\pi(\widetilde{D}t)$ are equal, then $\widetilde{D}s = \widetilde{D}t$ and hence $Ds = Dt$ [ML98, XI.1].

iii. In the braided case, if the underlying braids $\upsilon(\widetilde{D}s)$ and $\upsilon(\widetilde{D}t)$ are equal, then $\widetilde{D}s = \widetilde{D}t$ and hence $Ds = Dt$ [JS93, Corollary 2.6].

12 Diagrammatic coherence in the symmetric case

In the symmetric case $\mathsf{T} = \mathsf{S}$ in Section 11, there is a simplification for formal diagrams that are finitely generated—a condition which holds in all diagrammatic coherence applications known to the authors. The simplification makes use of the following result that finite coproducts and finite products of symmetric strict monoidal categories are equivalent.

Theorem 12.1 ([GJO24, Theorem 14.27]). *Suppose given symmetric strict monoidal categories A_i for $i \in \{1, \ldots, n\}$. There is a symmetric strict monoidal functor I*

$$\coprod_{i=1}^{n} A_i \xrightarrow{I} \prod_{i=1}^{n} A_i \tag{12.2}$$

such that the following statements hold.

i. Each composite with the canonical morphisms

$$A_i \longrightarrow \coprod_{i=1}^{n} A_i \xrightarrow{I} \prod_{i=1}^{n} A_i \longrightarrow A_j$$

is the identity on A_i if $i = j$ and constant at the monoidal unit of A_j otherwise.

ii. I is an equivalence of symmetric strict monoidal categories.

Remark 12.3. In Theorem 12.1, I is a symmetric strict monoidal functor, and it is an equivalence, but it does not have a strict monoidal inverse. See [GJO24, Remark 14.25] for further explanation of this point. The proof of Theorem 12.1 depends on an analysis of coproducts for symmetric strict monoidal categories that specializes the Gray tensor product of 2-categories. ◇

Recall that S is left adjoint to the forgetful u, and therefore commutes with colimits, particularly coproducts.

Definition 12.4. Suppose G is a finite set. Define a strict monoidal functor $\widetilde{I}$, and strict monoidal functors I_a for each $a \in G$, as the composites described below.

$$\begin{tikzcd}
\mathsf{S}\Big(\coprod_{b\in G}\{b\}\Big) \arrow[r, "\cong"] \arrow[d, equal] & \coprod_{b\in G}\mathsf{S}\{b\} \arrow[r, "I", "\simeq"'] & \prod_{b\in G}\mathsf{S}\{b\} \arrow[d] \\
\mathsf{S}G \arrow[urr, "\widetilde{I}"] \arrow[rr, "I_a"'] & & \mathsf{S}\{a\}
\end{tikzcd} \tag{12.5}$$

In the above diagram, the isomorphism is given by commuting S with coproducts, the equivalence I is that of (12.2), and the unlabeled arrow is projection from the product. ◇

Recall from Definition 10.8 that a finitely generated formal diagram is one that factors through a free algebra on a finite set.

Definition 12.6. Suppose A is a symmetric strict monoidal category and suppose that $(\mathbb{D}, D)$ is a diagram in A that is formal and finitely generated, with lift $\widetilde{D} \colon \mathbb{D} \longrightarrow \mathsf{S}G$ for a finite set $G \subset \mathsf{ob}A$. For each morphism s in $\mathbb{D}$ and each $a \in G$, define the permutation $\pi_a^{\widetilde{D}}(s)$ as the underlying permutation of the image of s in $\mathsf{S}\{a\}$. That is,

$$\pi_a^{\widetilde{D}}(s) = \pi\big((I_a\widetilde{D})(s)\big).$$

We call $\pi_a^{\widetilde{D}}(s)$ the *a-permutation of s* or the *self-permutation of a*. ◇

Theorem 12.7. *Suppose $(\mathbb{P}, D)$ is a formal diagram classifying a pair of parallel morphisms Ds and Dt in a symmetric strict monoidal category A. Suppose, moreover, that there is a finitely generated lift $\widetilde{D}$, factoring through $\mathsf{S}G$ for a finite set G, such that*

$$\pi_a^{\widetilde{D}}(s) = \pi_a^{\widetilde{D}}(t) \quad \text{for each} \quad a \in G. \tag{12.8}$$

Then $Ds = Dt$ in A.

Proof. The hypotheses of the theorem establish the following context, where the left hand triangle is that of the formal diagram $(\mathbb{P}, D)$ and the finitely generated lift $\widetilde{D}$. The right hand triangle is (12.5). Recall that $\widetilde{I}$ is an equivalence by Theorem 12.1.

$$\begin{tikzcd}
& \mathsf{S}G \arrow[r, "\widetilde{I}", "\simeq"'] \arrow[d] \arrow[dr, "I_a"] & \prod_{b\in G}\mathsf{S}\{b\} \arrow[d] \\
& \mathsf{S}(\mathsf{ob}A) \arrow[d] & \mathsf{S}\{a\} \\
& \mathsf{S}A \arrow[d] & \\
\mathbb{P} \arrow[r, "D"] \arrow[uuur, "\widetilde{D}"] & A &
\end{tikzcd}$$

By the universal property of the product, the equalities (12.8) imply that the morphisms $\widetilde{I}\widetilde{D}s$ and $\widetilde{I}\widetilde{D}t$ are equal in $\prod_{b\in G}\mathsf{S}\{b\}$. Since $\widetilde{I}$ is an equivalence, we have

$$\widetilde{D}s = \widetilde{D}t \quad \text{in} \quad \mathsf{S}G,$$

and hence $Ds = Dt$ as desired. □

Remark 12.9. It is instructive to compare the statement of Theorem 12.7 with the more familiar statement for vectors in a vector space V over a field k. If V has finite dimension n, then choosing a basis for V provides an isomorphism $V \cong k^{\oplus n}$. Thus, two vectors $v, w \in V$ are equal if an only if their components in $k^{\oplus n}$ are equal. The self-permutations $\pi_a^{\widetilde{D}}(s)$ provide the same condition: $\widetilde{I}$ is an equivalence and, therefore, two underlying permutations $\pi^{\widetilde{D}}(s)$ and $\pi^{\widetilde{D}}(t)$ are equal if and only if their a-permutations are equal for each generating object a. ◇

Several examples of Theorem 12.7 are given in Section 16. In particular, see Remark 16.8, Non-Example 16.10 and Remark 16.13.

13 Explication: Pseudomorphism classifiers

In this section we give an explicit description of the pseudomorphism classifiers

$$\mathsf{Q}\colon \mathsf{T}\text{-}\mathsf{Alg} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg_s}$$

for each 2-monad $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$ of Notation 11.1. We present a unified construction, noting minor differences in the three cases where appropriate. In these applications, we work with the strict monoidal 2-monads T, instead of the general T^{g}, in order to highlight the essential features. Equivalent results hold for the general monoidal variants by Corollary 11.5. Here and in Section 14 we make use of the following.

Notation 13.1. Suppose $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$ and suppose $(A, \bullet, e)$ is a T-algebra with monoidal unit e and multiplication denoted as $\bullet$ or with juxtaposition. Recall from Notation 11.1 that the objects of $\mathsf{T}A$ are given by tuples of objects from A. The morphisms of $\mathsf{T}A$ are generated by tuples of morphisms from A together with, in the symmetric and braided cases, permutations and braidings, respectively.

We will use the following notation for objects and morphisms in $\mathsf{T}A$ that are given by tuples of objects and morphisms in A:

$$\begin{aligned} \langle a\rangle &= \langle a_i\rangle_{i=1}^n = (a_1, \ldots, a_n) \\ \langle s\rangle &= \langle s_i\rangle_{i=1}^n = (s_1, \ldots, s_n) \end{aligned} \tag{13.2}$$

where a_i and s_i are objects and morphisms, respectively, in A and $n \geq 0$.

- The number n is called the *length* of $\langle a\rangle$.
- The empty tuple is denoted $\langle\rangle$ and has length 0.
- For a tuple $\langle a\rangle$ of length n, we write
$$a_\bullet = a_1 \bullet\bullet\bullet a_n$$
to denote the product in A of the entries a_i.
- For tuples $\langle a^1\rangle$ and $\langle a^2\rangle$ of length n_1 and n_2, respectively, we denote concatenation with a semicolon ; and write
$$\langle a^{1;2}\rangle = \langle a^1\rangle; \langle a^2\rangle = \langle a^1; a^2\rangle$$
to denote the tuple whose first n_1 entries are those of $\langle a^1\rangle$ and whose final n_2 entries are those of $\langle a^2\rangle$.

This same terminology and notation is used for tuples of morphisms $\langle s\rangle$.

We also denote the image of a general morphism t under the multiplication $\mathsf{T}A \xrightarrow{\bullet} A$ as $t_\bullet$. For example, t may be a permutation or braiding if $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$. In such a case, $t_\bullet$ is the corresponding symmetry or braid isomorphism in A.

Thus, the composite

$$\mathsf{T}A \xrightarrow{\bullet} A \xrightarrow{\eta_A} \mathsf{T}A$$

is denoted as a length-one tuple with subscript $\bullet$, so we write

$$\begin{aligned} \langle a\rangle &\longmapsto (a_\bullet), \\ \langle s\rangle &\longmapsto (s_\bullet), \quad \text{and} \\ t &\longmapsto (t_\bullet) \end{aligned} \tag{13.3}$$

where $\langle a\rangle$ and $\langle s\rangle$ are tuples of objects and morphisms, respectively, and t is a general morphism of $\mathsf{T}A$. ◇

Using Notation 13.1, we now define the pseudomorphism classifier Q for each of the three cases $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$.

Definition 13.4. Suppose A is a category. Define a T-algebra $\mathsf{Q}A$ as follows.

Objects: The objects of $\mathsf{Q}A$ are those of $\mathsf{T}A$.

Free morphisms: The morphisms of $\mathsf{T}A$ are included as morphisms of $\mathsf{Q}A$, and are called *free morphisms* there. The inclusion of objects and free morphisms is denoted

$$\iota\colon \mathsf{T}A \hookrightarrow \mathsf{Q}A. \tag{13.5}$$

When describing individual objects or morphisms, we will often suppress ι and identify objects and morphisms of $\mathsf{T}A$ with their images in $\mathsf{Q}A$.

Adjoined isomorphisms: For each object $\langle a\rangle$ in $\mathsf{ob}(\mathsf{Q}A) = \mathsf{ob}(\mathsf{T}A)$, there is an *adjoined isomorphism*

$$\mathsf{q}_{\langle a\rangle} : \langle a\rangle \xrightarrow{\cong} (a_\bullet) \quad \text{in} \quad \mathsf{Q}A.$$

The morphisms of $\mathsf{Q}A$ are generated under composition and concatenation by the free morphisms and adjoined isomorphisms, subject to the following axioms. In the symmetric or braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, the symmetry or braiding isomorphism of $\mathsf{Q}A$ is given by the corresponding free morphism from $\mathsf{T}A$.

Free composites and products: The inclusion ι is a strict T-map. Thus, composites or products of free morphisms are given by those of $\mathsf{T}A$.

Naturality of q: The adjoined isomorphisms q are natural with respect to free morphisms. That is, using the notation (13.3) and suppressing ι, the following diagram commutes for each morphism $t\colon \langle a_i\rangle_{i=1}^n \longrightarrow \langle a'_i\rangle_{i=1}^n$ in $\mathsf{T}A$.

$$\begin{tikzcd} \langle a\rangle \arrow[r, "t"] \arrow[d, "\mathsf{q}_{\langle a\rangle}"'] & \langle a'\rangle \arrow[d, "\mathsf{q}_{\langle a'\rangle}"] \\ (a_\bullet) \arrow[r, "(t_\bullet)"] & (a'_\bullet) \end{tikzcd} \tag{13.6}$$

Associativity of q: The following diagrams commute for tuples $\langle a^1\rangle$, $\langle a^2\rangle$, and $\langle a^3\rangle$ in $\mathsf{Q}A$, where the diagram at left uses the fact that e is a strict unit for A.

$$\begin{tikzcd}
\langle a^1\rangle;\langle\rangle;\langle a^2\rangle \arrow[r, equal] \arrow[d, "1;\mathsf{q}_{\langle\rangle};1"'] & \langle a^1\rangle;\langle a^2\rangle \arrow[d, "\mathsf{q}"] \\
\langle a^1\rangle;(e);\langle a^2\rangle \arrow[r, "\mathsf{q}"] & (a^{1;2}_\bullet)
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle a^1\rangle;\langle a^2\rangle;\langle a^3\rangle \arrow[r, "1;\mathsf{q}_{\langle a^{2;3}\rangle}"] \arrow[d, "\mathsf{q}_{\langle a^{1;2}\rangle};1"'] \arrow[dr, "\mathsf{q}"] & \langle a^1\rangle;(a^{2;3}_\bullet) \arrow[d, "\mathsf{q}"] \\
(a^{1;2}_\bullet);\langle a^3\rangle \arrow[r, "\mathsf{q}"] & (a^{1;2;3}_\bullet)
\end{tikzcd}
\tag{13.7}$$

Normality of q: For a tuple of length one, (a) with $a \in A$, we have

$$\mathsf{q}_{(a)} = 1_{(a)} = (1_a). \tag{13.8}$$

◇

Definition 13.9. Suppose given a T-map $f\colon A \rightsquigarrow B$ between T-algebras A and B. Define a strict T-map

$$\mathsf{Q}f\colon \mathsf{Q}A \longrightarrow \mathsf{Q}B$$

as follows. For a tuple of objects $\langle a\rangle$, define

$$(\mathsf{Q}f)\langle a_i\rangle_{i=1}^n = \langle f(a_i)\rangle_{i=1}^n.$$

For a free morphism $t\colon \langle a_i\rangle_{i=1}^n \longrightarrow \langle a'_i\rangle_{i=1}^n$, define $\mathsf{Q}f$ as $\mathsf{T}f$. That is, define

$$\big(\mathsf{Q}f\big)(\iota t) = \iota\big(\big(\mathsf{T}f\big)t\big)\colon \langle f(a_i)\rangle_{i=1}^n \longrightarrow \langle f(a'_i)\rangle_{i=1}^n.$$

For an adjoined isomorphism $\mathsf{q}_{\langle a\rangle}$, where $\langle a\rangle = \langle a_i\rangle_{i=1}^n$, define $\big(\mathsf{Q}f\big)\mathsf{q}_{\langle a\rangle}$ as the composite

$$\langle f(a_i)\rangle \xrightarrow{\mathsf{q}_{\langle f(a_i)\rangle}} ([f(a_i)]_\bullet) \xrightarrow{(f_\bullet)} (f(a_\bullet)), \tag{13.10}$$

where $[f(a_i)]_\bullet$ denotes the product of the entries $f(a_i)$ and

$$f_\bullet\colon [f(a_i)]_\bullet \longrightarrow f(a_\bullet)$$

is the notation of (2.6) to indicate any composite of, respectively,

- monoidal constraints f_2, if $n \geq 2$,
- unit constraints f_0, if $n = 0$, or
- identities $1_{f(a)}$, if $n = 1$.

This defines $\mathsf{Q}f$ on the objects and generating morphisms of $\mathsf{Q}A$. Then, $\mathsf{Q}f$ is defined to be functorial with respect to formal composition $\circ$ and strict monoidal with respect to concatenation ; in $\mathsf{Q}A$ and $\mathsf{Q}B$. In the symmetric or braided cases, $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, the definition of $\mathsf{Q}f$ on free morphisms implies that $\mathsf{Q}f$ satisfies the additional braid axiom (11.3) of a T-map. In all three cases for T, we have $\mathsf{Q}f \circ \iota = \iota \circ \mathsf{T}f$ as strict T-maps.

To verify that $\mathsf{Q}f$ is well defined with respect to the relations (13.6) through (13.8), one uses the corresponding relations in the codomain T-algebra B together with functoriality of f and naturality of $f_\bullet$. Furthermore, naturality of q and the definition of composition for monoidal functors shows that Q is functorial with respect to identities and composites of T-maps. ◇

Definition 13.11. In each of the cases $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$, the T-algebra 2-cells are monoidal transformations. If $\alpha \colon f \longrightarrow f'$ is a monoidal transformation between T-maps $f, f' \colon A \rightsquigarrow B$, then

$$\mathsf{Q}\alpha \colon \mathsf{Q}f \longrightarrow \mathsf{Q}f'$$

is defined componentwise for objects $\langle a \rangle = \langle a_i \rangle_{i=1}^n$ by

$$\big(\mathsf{Q}\alpha\big)_{\langle a \rangle} = \langle \alpha_{a_i} \rangle_{i=1}^n. \tag{13.12}$$

The monoidal transformation axioms for $\mathsf{Q}\alpha$ hold because concatenation of tuples is strictly associative and unital. Similarly, 2-functoriality of Q with respect to identities and horizontal or vertical composites of monoidal transformations is verified componentwise. ◇

Together, Definitions 13.4, 13.9, and 13.11 define a 2-functor

$$\mathsf{Q} \colon \mathsf{T}\text{-}\mathsf{Alg} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}\,.$$

Recall from Definition 4.7 that a pseudomorphism classifier $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ is effective if the unit/counit pair (ζ, δ) is componentwise an adjoint surjective equivalence. The following will be used in Proposition 13.14 below to show that Q is an effective pseudomorphism classifier for T.

Definition 13.13. In the context of Definitions 13.4, 13.9, and 13.11 above, there are 2-natural transformations ζ and δ together with an invertible monoidal transformation Θ defined as follows.

Unit: For a T-algebra A, define a T-map

$$\zeta_A \colon A \rightsquigarrow \mathtt{i}\mathsf{Q}A$$

by sending each object and morphism of A to the corresponding length-one tuple in $\mathsf{Q}A$. The monoidal and unit constraints of ζ are given by the adjoined isomorphisms q. Thus, in the symmetric and braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, ζ_A satisfies the braid axiom (11.3). Naturality of ζ with respect to T-maps f holds because $\mathsf{Q}f$ is defined by $\mathsf{T}f$ on tuples $\langle a \rangle$ and free morphisms t. Likewise, 2-naturality with respect to monoidal transformations follows from (13.12).

Counit: For a T-algebra B, define a strict T-map

$$\delta_B \colon \mathsf{Q}\mathtt{i}B \longrightarrow B$$

by sending each tuple of objects $\langle a \rangle$ to their product $a_\bullet$ in B, each free morphism t to $t_\bullet$, and each adjoined isomorphism q to an identity. Thus, in the symmetric or braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, δ_B satisfies the braid axiom (11.3). This is a strict T-map because the monoidal product in B is strictly associative and unital. Naturality of δ with respect to strict T-maps holds because such T-maps strictly preserve monoidal units and products.

Efficacy: For each T-algebra B, define an invertible monoidal transformation

$$\Theta \colon \zeta_B \delta_B \xrightarrow{\cong} 1_{\mathsf{Q}B}$$

with components

$$\Theta_{\langle b \rangle} = \mathsf{q}_{\langle b \rangle}^{-1} \colon (b_\bullet) \longrightarrow \langle b \rangle \quad \text{for} \quad \langle b \rangle \in \mathsf{Q}B.$$

Monoidal naturality of q, and hence also Θ, is equivalent to the conditions (13.6) and (13.7). ◇

Proposition 13.14. *For each* $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$*, the 2-functor*

$$\mathsf{Q} \colon \mathsf{T}\text{-}\mathsf{Alg} \longrightarrow \mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$$

is an effective pseudomorphism classifier for T.

Proof. The 2-functor Q, unit ζ, counit δ, and isomorphism Θ are given in Definitions 13.4, 13.9, 13.11, and 13.13. For T-algebras A and B, the definitions of δ and ζ yield the following computations:

$$\begin{aligned} \delta_B\big(\zeta_B(b)\big) &= b && \text{for} \quad b \in B \\ \delta_{\mathsf{Q}A}\big((\mathsf{Q}\zeta_A)\langle a\rangle\big) &= \delta_{\mathsf{Q}A}\langle(a_i)\rangle_{i=1}^n = \langle a\rangle && \text{for} \quad \langle a\rangle = \langle a_i\rangle_{i=1}^n \in \mathsf{Q}A. \end{aligned}$$

A similar computation holds for morphisms, using the fact that the monoidal constraints of ζ are the adjoined isomorphisms q. Thus, ζ and δ satisfy the triangle identities

$$\delta_B \circ \zeta_B = 1_B \quad \text{and} \quad \delta_{\mathsf{Q}A} \circ (\mathsf{Q}\zeta_A) = 1_{\mathsf{Q}A}.$$

so that $(\mathsf{Q}, \mathtt{i}, \zeta, \delta)$ is a 2-adjunction.

Furthermore, the normality condition (13.8) for q implies

$$\Theta * \zeta_B = 1_{\zeta_B}. \tag{13.15}$$

This completes the proof. □

The explicit description of Q, above, will be helpful in Section 14 below. The following alternative description of Q is more abstract, but highlights some of its characteristic properties.

Remark 13.16. The strict T-map $\iota\colon \mathsf{T}A \longrightarrow \mathsf{Q}A$ from (13.5) is the identity on objects and factors the monad structure morphism $\mathsf{T}A \longrightarrow A$ as shown at left below. Furthermore, there is a T-map $\zeta_A\colon A \rightsquigarrow \mathsf{Q}A$ such that the adjoined isomorphisms q are the components of an invertible monoidal transformation as shown at right below.

$$\begin{tikzcd} \mathsf{T}A \arrow[rr, "\bullet"] \arrow[drr, "\iota"'] & & A \\ & & \mathsf{Q}A \arrow[u, "\delta_A"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{T}A \arrow[d, "\bullet"'] \arrow[drr, "\iota"] \arrow[dr, phantom, "\mathsf{q}\,\Swarrow"] & & \\ A \arrow[rr, squiggly, "\zeta_A"'] & & \mathsf{Q}A \end{tikzcd} \tag{13.17}$$

The normality condition (13.8) for q is equivalent to the equality

$$\mathsf{q} * \eta_A = 1_{\zeta_A}. \tag{13.18}$$

That is, the whiskering of q with the unit $\eta_A\colon A \longrightarrow \mathsf{T}A$ is the identity transformation of ζ_A. In this context, the requirement in Definition 13.13, that δ sends the adjoined isomorphisms q to identities, is equivalent to the requirement that $\delta_A\zeta_A = 1_A$ as a strict T-map. ◇

Remark 13.19. The description in Remark 13.16 indicates how the elementary presentation above in Definitions 13.4, 13.9, and 13.11 relates to the method of Power [Pow89, Theorem 3.4], which constructs a pseudomorphism classifier Q in greater generality by factoring the multiplication morphism of a T-algebra (or pseudo algebra) (X, x) as a *bijective-on-objects* functor ι followed by a *full and faithful* functor δ_A. In our applications, the left side of (13.17) provides this factorization. Power's work can be extended in greater generality via Lack's codescent for pseudo-algebras [Lac02, Theorem 4.10]. ◇

Definition 13.20. In the context of Definitions 13.4, 13.9, and 13.11, the following associated constructions are of interest. These are special cases of the general constructions in Definition 4.3, Lemma 5.2, and Remark 5.11.

i. Referring to the adjunction $\mathsf{Q} \dashv \mathtt{i}$, each T-map $f\colon A \rightsquigarrow B$, has a unique strict mate $f^{\perp}$, making the triangle at left below commute in $\mathsf{T}\text{-}\mathsf{Alg}$. Recalling Definitions 13.9 and 13.13, one verifies that $f^{\perp}$ is defined by the triangle at right below.

$$\begin{tikzcd} \mathsf{Q}A \arrow[r, "f^{\perp}"] & B \\ A \arrow[u, squiggly, "\zeta_A"] \arrow[ur, squiggly, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Q}B \arrow[d, "\delta_B"] \\ \mathsf{Q}A \arrow[ur, "\mathsf{Q}f"] \arrow[r, "f^{\perp}"] & B \end{tikzcd}$$

ii. In the case $A = \mathsf{T}C$ for a category C, there is a strict T-map

$$\zeta^\flat \colon \mathsf{T}C \longrightarrow \mathsf{QT}C \tag{13.21}$$

that sends a tuple of objects $\langle a_i \rangle_{i=1}^n$ in $\mathsf{T}C$ to the corresponding tuple of length-one tuples $\langle (a_i) \rangle_{i=1}^n$ in $\mathsf{QT}C$. In the case $n = 0$, $\zeta^\flat$ sends the empty tuple $\langle\rangle \in \mathsf{T}C$ to the empty tuple $\langle\rangle \in \mathsf{QT}C$. The assignment on morphisms is given in the same way, and $\zeta^\flat$ is a strict monoidal functor.

iii. There is an invertible monoidal transformation

$$\Theta^\flat \colon \zeta^\flat_{\mathsf{T}C} \delta_{\mathsf{T}C} \xrightarrow{\cong} 1_{\mathsf{QT}C}$$

defined as in (5.10). For an object

$$\langle w \rangle = \langle w_j \rangle_{j=1}^m \in \mathsf{QT}C,$$

where $w_j = \langle a_i^j \rangle_{i=1}^{n_j}$ is an object of $\mathsf{T}C$ for each $j \in \{1, \ldots, m\}$, the component

$$\Theta^\flat_{\langle w \rangle} \colon \zeta^\flat \delta \langle w \rangle \longrightarrow \langle w \rangle$$

is given by the composite

$$\zeta^\flat \delta \langle w \rangle \xrightarrow{\mathsf{q}} (\langle w_\bullet \rangle) \xrightarrow{\mathsf{q}^{-1}} \langle w \rangle \tag{13.22}$$

Here, each q is one of the adjoined isomorphisms in $\mathsf{QT}C$, the object $(\langle w_\bullet \rangle)$ is the length-one tuple whose entry is the concatenation in $\mathsf{T}C$ of the tuples $w_j = \langle a^j \rangle$, and $\zeta^\flat \delta \langle w \rangle = \langle (a_i^j) \rangle_{j,i}$ is the tuple of length $N = \sum_j n_j$ whose entries are the length-one tuples (a_i^j).

If $\langle w \rangle = \zeta^\flat \langle a \rangle$ for $\langle a \rangle \in \mathsf{T}C$, then the two components of q appearing in (13.22) are the same. Hence, $\Theta^\flat * \zeta^\flat = 1_{\zeta^\flat}$ as required. ◇

14 Explication: Universal pseudomorphisms

The hypotheses of Theorem 1.5 hold for $\mathcal{K} = \mathcal{C}at$ and each of the 2-monads for monoidal structures T^{g} and T in Notation 11.1, with $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$. Therefore, the comparison strict T-maps

$$\mathsf{T}^{\mathsf{g}}(C', \phi) \xrightarrow{\Delta} \mathsf{T}^{\mathsf{g}} C' \quad \text{and} \quad \mathsf{T}(C', \phi) \xrightarrow{\Delta} \mathsf{T}C'$$

are equivalences for each $\phi \colon C \longrightarrow C'$ in $\mathcal{C}at$.

This section gives an explicit description of $\mathsf{T}(G', \phi)$ in Explanation 14.4, where $\phi \colon G \longrightarrow G'$ is a function between sets, treated as discrete categories. Then, the universal T-map $\widetilde{\phi}$ for $\mathsf{T}(G', \phi)$ and the comparison Δ are described in Explanations 14.11 and 14.13, respectively. In applications, ϕ is the underlying function-on-objects of a T-map f. In that case, the strict T-map Λ of (10.3) is described in Explanation 14.17.

To begin, it will be useful to record the following.

Definition 14.1. Let $\mathcal{M}on$ denote the category of monoids in $\mathcal{S}et$. The *set-of-objects* functor

$$\mathsf{ob} \colon \mathsf{M}\text{-}\mathsf{Alg} \longrightarrow \mathcal{M}on$$

has both left and right adjoints

$$\mathsf{disc} \dashv \mathsf{ob} \dashv \mathsf{indisc} \tag{14.2}$$

defined as follows.

- $\mathsf{disc}\colon \mathcal{M}on \longrightarrow \mathsf{M}\text{-}\mathsf{Alg}_\mathsf{s}$ is the *discrete* M*-algebra* functor, sending a monoid G to the M-algebra with underlying monoid G and identity morphisms.
- $\mathsf{indisc}\colon \mathcal{M}on \longrightarrow \mathsf{M}\text{-}\mathsf{Alg}_\mathsf{s}$ is the *indiscrete* M*-algebra*, sending a monoid G to the M-algebra with underlying monoid G and a unique isomorphism between every pair of objects.

Below, we will apply disc implicitly and omit the notation. ◇

Recall from Theorem 7.11 that each universal pseudomorphism for $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$ can be obtained as a pushout of strict T-maps (7.5) shown here.

$$\begin{tikzcd}
\mathsf{T}G \arrow[r, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & \mathsf{T}G' \arrow[d, "\kappa"] \\
\mathtt{i}\mathsf{QT}G \arrow[r, "\widehat{\phi}"] & \mathsf{T}(G', \phi)
\end{tikzcd} \tag{14.3}$$

Recall that Q is described in Definition 13.4 using Notation 13.1; recall $\zeta^\flat$ from (13.21). Unpacking (14.3) yields the following.

Explanation 14.4. Suppose $\phi\colon G \longrightarrow G'$ is a functor between discrete categories and $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$. The T-algebra $\mathsf{T}(G', \phi)$ in (14.3) is given as follows. We begin by describing generating objects and their relations. Then, we describe generating morphisms and their relations

The symmetric and braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$ have the same objects as the plain monoidal case. In the monoidal case, $\mathsf{T} = \mathsf{M}$, the functor ob is left adjoint to indisc in (14.2) and therefore commutes with pushouts.

Thus, in each of the cases $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$, the objects of $\mathsf{T}(G', \phi)$ are given by the pushout (14.3) on objects. Hence, the objects are generated under the monoidal product ; by those of $\mathsf{T}G'$ and $\mathsf{QT}G$, for which we use the following terms.

Free objects: The *free objects* of $\mathsf{T}(G', \phi)$ are those of $\mathsf{T}G'$. They are tuples

$$w' = \langle a' \rangle = \langle a'_i \rangle_{i=1}^{n'}$$

where $a'_i \in G'$ and $n' \geq 0$. On objects, the functor $\kappa\colon \mathsf{T}G' \longrightarrow \mathsf{T}(G', \phi)$ is the inclusion of free objects.

ϕ-Objects: The *ϕ-objects* of $\mathsf{T}(G', \phi)$ are tuples denoted

$$\big\langle [\phi] w \big\rangle = \big\langle [\phi] w_j \big\rangle_{j=1}^{m}$$

where each $\langle w \rangle$ is an object of $\mathsf{QT}G$, so $w_j = \langle a_i^j \rangle_{i=1}^{n_j}$ is an object of $\mathsf{T}G$, and $m \geq 0$. On objects, the functor $\widehat{\phi}$ sends an object $\langle w \rangle \in \mathsf{QT}G$ to the ϕ-object $\big\langle [\phi] w \big\rangle$.

These objects are subject to the following relation, identifying the two composites around (14.3).

Object pushout relation: If $\langle w \rangle = \zeta^\flat \langle a \rangle = \big\langle (a_1^j) \big\rangle_{j=1}^{m}$ is a tuple of length-one tuples, then

$$\big\langle [\phi](a_1^j) \big\rangle_{j=1}^{m} = \langle \phi(a_1^j) \rangle_{j=1}^{m}, \tag{14.5}$$

where

- the left hand side is the ϕ-object associated to the tuple $\langle w \rangle$ whose entries are length-one tuples (a_1^j), and
- the right hand side is the free object whose entries are $\phi(a_1^j)$.

In the case that $m = 0$, the empty ϕ-object $\big\langle [\phi] \big\rangle$ is identified with the empty free object $\langle\rangle$.

This finishes the description of the objects of $\mathsf{T}(G', \phi)$.

The morphisms of $\mathsf{T}(G', \phi)$ are likewise generated by those of $\mathsf{T}G'$ and $\mathsf{QT}G$ under composition $\circ$ and the product $;$. In the symmetric and braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, there are additional formal braid isomorphisms. Thus, the morphisms of $\mathsf{T}(G', \phi)$ are generated by four types, for which we use the following terms.

Free morphisms: The *free morphisms* are those of $\mathsf{T}G'$. On morphisms, the functor κ is the inclusion of free morphisms.

ϕ-Free morphisms: The *ϕ-free morphisms* are denoted

$$[\phi]u \colon \big\langle [\phi] w \big\rangle \longrightarrow \big\langle [\phi] v \big\rangle \tag{14.6}$$

where $u \colon \langle w \rangle \longrightarrow \langle v \rangle$ is a free morphism of $\mathsf{QT}G$. Thus, u is either

- a tuple of morphisms $t_j \colon w_j \longrightarrow v_j$ in $\mathsf{T}G$;
- a permutation or braiding, in the symmetric and braided cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$; or
- a composite of such morphisms.

In the former case, since G is discrete, each t_j is either a tuple of identity morphisms or, in the cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, a permutation or braiding in $\mathsf{T}G$.

ϕ-Adjoined isomorphisms: The *ϕ-adjoined morphisms* are denoted

$$[\phi]\mathsf{q}_{\langle w \rangle} \colon \big\langle [\phi] w \big\rangle \longrightarrow \big([\phi] w_\bullet \big) \tag{14.7}$$

where $\langle w \rangle = \langle w_j \rangle_{j=1}^m$ is an object of $\mathsf{QT}G$ and $w_\bullet$ denotes the concatenation in $\mathsf{T}G$ of the tuples $w_j = \langle a_i^j \rangle_{i=1}^{n_j}$. Thus, $w_\bullet = \langle a^\bullet \rangle$ is a tuple of length $N = \sum_j n_j$ whose ℓth entry, $a_\ell^\bullet$, is a_i^J, where

$$J \in \{1, \ldots, m\} \quad \text{and} \quad i \in \{1, \ldots, n_J\}$$

are the unique natural numbers such that

$$\ell = \left[\sum_{j=1}^{J-1} n_j \right] + i. \tag{14.8}$$

Formal Morphisms: In the symmetric and braided cases, $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, there are formal permutation and braid morphisms, respectively. The formal morphisms between free objects are identified with the corresponding free morphisms given by permutation or braid morphisms in $\mathsf{T}G'$. The formal morphisms between ϕ-objects are identified with the corresponding ϕ-free morphisms given by permutation or braid morphisms in $\mathsf{QT}G$.

The morphisms of $\mathsf{T}(G', \phi)$ are freely generated under composition $\circ$ and the product ; so that the T-algebra structure on $\mathsf{T}(G', \phi)$ extends that of $\mathsf{T}G'$ and $\mathsf{QT}G$, subject to the following axioms.

Composites and products: The structure morphisms

$$\kappa \colon \mathsf{T}G' \longrightarrow \mathsf{T}(G', \phi) \quad \text{and} \quad \widehat{\phi} \colon \mathsf{QT}G \longrightarrow \mathsf{T}(G', \phi)$$

are both strict T-maps. Thus, the composites or products of free, respectively ϕ-, morphisms are given by those of $\mathsf{T}G'$, respectively $\mathsf{QT}G$.

Morphism pushout relation: For each morphism $t\colon \langle a\rangle \longrightarrow \langle b\rangle$ in $\mathsf{T}G$, where $\langle a\rangle = \langle a_i\rangle_{i=1}^n$ and $\langle b\rangle = \langle b_i\rangle_{i=1}^n$, the images of t under the two composites around (14.3) are identified. Thus, the free morphism

$$\big(\mathsf{T}\phi\big)(t)\colon \big(\mathsf{T}\phi\big)(\langle a\rangle) \longrightarrow \big(\mathsf{T}\phi\big)(\langle b\rangle)$$

is identified with the ϕ-free morphism

$$[\phi]\overline{t}\colon \big\langle [\phi](a_i)\big\rangle_{i=1}^n \longrightarrow \big\langle [\phi](b_i)\big\rangle,$$

where $\overline{t} = \zeta^\flat t$ is the free morphism induced by t, between tuples of length-one tuples $\langle (a_i)\rangle_{i=1}^n$ and $\langle (b_i)\rangle_{i=1}^n$.

Since G is discrete, this relation is trivial if $\mathsf{T} = \mathsf{M}$, in which case t is a tuple of identity morphisms. If $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, then $(\mathsf{T}\phi)t$, $\overline{t}$, and $[\phi]\overline{t}$ are the respective permutation or braiding morphisms determined by t.

This finishes the description of objects, morphisms, and T-algebra structure of $\mathsf{T}(G', \phi)$. ◇

Proposition 14.9. *The* T*-algebra described in Explanation 14.4 is a model for the pushout* $\mathsf{T}(G', \phi)$ *in (14.3).*

Now we describe the universal pseudomorphism

$$\widetilde{\phi}\colon \mathsf{T}G \rightsquigarrow \mathsf{T}(G', \phi).$$

Recalling (7.6), $\widetilde{\phi}$ is equal to the composite $\widehat{\phi} \circ \zeta$ shown below.

$$\begin{tikzcd}
\mathsf{T}G \arrow[rr, squiggly, "\widetilde{\phi}"] \arrow[dr, squiggly, "\zeta_{\mathsf{T}C}"'] & & \mathsf{T}(G', \phi) \\
& \mathsf{QT}G \arrow[ur, "\widehat{\phi}"'] &
\end{tikzcd} \tag{14.10}$$

Recall ζ is the unit in Definition 13.13.

Explanation 14.11. In the context of Explanation 14.4 and (14.12), the T-map

$$\widetilde{\phi}\colon \mathsf{T}G \rightsquigarrow \mathsf{T}(G', \phi)$$

in (14.10) is given as follows.

i. For a tuple $w = \langle a_i\rangle_{i=1}^n \in \mathsf{T}G$, with each $a_i \in G$,

$$\widetilde{\phi}w = \big([\phi]w\big)$$

is the ϕ-object of length one whose only entry is $[\phi]w$.

ii. For a morphism $t\colon w \longrightarrow v$ in $\mathsf{T}G$,

$$\widetilde{\phi}t = [\phi]t\colon \big([\phi]w\big) \longrightarrow \big([\phi]v\big)$$

is the ϕ-free morphism of length one whose entry is either the identity, if $\mathsf{T} = \mathsf{M}$, or the permutation or braid morphism corresponding to t if $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$.

iii. The unit constraint of $\widetilde{\phi}$ is given by the ϕ-adjoined isomorphism for the empty tuple:

$$[\phi]\mathsf{q}_{\langle\rangle}\colon \big\langle [\phi]\big\rangle = \langle\rangle \longrightarrow \big([\phi]\langle\rangle\big) = \big(\langle\rangle\big)$$

where, on the right hand side, $\big([\phi]\langle\rangle\big) = \big(\langle\rangle\big)$ is the ϕ-object of length one whose single entry is $[\phi]\langle\rangle = \langle\rangle$.

iv. The monoidal constraint of $\widetilde{\phi}$ is given, at a pair of objects $w_1, w_2 \in \mathsf{T}G$, by the ϕ-adjoined isomorphism for the length-two tuple $\langle w \rangle = \langle w_i \rangle_{i=1}^2$:

$$[\phi]\mathsf{q}_{\langle w \rangle} \colon \big\langle [\phi] w_i \big\rangle_{i=1}^{2} \longrightarrow \big([\phi] w_\bullet \big) = \big([\phi] \langle w_{1;2} \rangle \big).$$

The description of the unit and monoidal constraints of $\widetilde{\phi}$ follows from those of ζ in Definition 13.13.◇

Now we describe the strict T-map

$$\Delta \colon \mathsf{T}(G', \phi) \xrightarrow{\simeq} \mathsf{T}G'.$$

By Theorem 8.9, Δ is an adjoint surjective equivalence that is determined by the pushout (14.3), as indicated by the dashed arrow below.

$$\begin{tikzcd}
\mathsf{T}G \arrow[r, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & \mathsf{T}G' \arrow[d, "\kappa"] \arrow[dr, bend left, "1"] & \\
\mathtt{i}\mathsf{QT}G \arrow[r, "\widehat{\phi}"] \arrow[d, "\delta"'] & \mathsf{T}(G', \phi) \arrow[dr, dashed, "\exists!", "\Delta"'] & \\
\mathsf{T}G \arrow[rr, "\mathsf{T}\phi"] & & \mathsf{T}G'
\end{tikzcd} \tag{14.12}$$

Recall δ is the counit in Definition 13.13. Using the description of $\mathsf{T}(G', \phi)$ in Explanation 14.4 and commutativity of (14.12) yields the following.

Explanation 14.13. In the context of Explanation 14.4 and (14.12), the strict T-map

$$\Delta \colon \mathsf{T}(G', \phi) \longrightarrow \mathsf{T}G'$$

is given as follows.

i. For free objects $\langle a' \rangle, \langle b' \rangle \in \mathsf{T}G'$ and free morphisms $t' \colon \langle a' \rangle \longrightarrow \langle b' \rangle$,

$$\Delta t' = t' \colon \langle a' \rangle \longrightarrow \langle b' \rangle.$$

ii. For ϕ-objects $\big\langle [\phi] w \big\rangle = \big\langle [\phi] w_j \big\rangle_{j=1}^{m}$ with $w_j = \langle a^j \rangle = \langle a_i^j \rangle_{i=1}^{n_j} \in \mathsf{T}G$,

$$\Delta \big\langle [\phi] w \big\rangle = \big(T\phi \big) \delta \langle w \rangle = \big(T\phi \big) w_\bullet = \big\langle \phi(a_\ell^\bullet) \big\rangle_{\ell=1}^{N}$$

where $\langle a^\bullet \rangle = w_\bullet$ is the concatenation in $\mathsf{T}G$ of the tuples $w_j = \langle a^j \rangle$ as in (14.8).

iii. For ϕ-free morphisms $[\phi]u \colon \big\langle [\phi] w \big\rangle \longrightarrow \big\langle [\phi] v \big\rangle$ where $u \colon \langle w \rangle \longrightarrow \langle v \rangle$ is a free morphism of $\mathsf{QT}G$,

$$\Delta \big([\phi] u \big) = \big(T\phi \big) \delta u$$

is the corresponding identity, permutation, or braiding morphism

$$\big(T\phi \big) u \colon \big(T\phi \big) w_\bullet \longrightarrow \big(T\phi \big) v_\bullet$$

in $\mathsf{T}G'$.

iv. For ϕ-adjoined isomorphisms $[\phi]\mathsf{q}_{\langle w \rangle} \colon \langle [\phi] w \rangle \longrightarrow ([\phi] w_\bullet)$,

$$\Delta \big([\phi] \mathsf{q}_{\langle w \rangle} \big) = \big(T\phi \big) \delta \mathsf{q}_{\langle w \rangle} = 1 \colon \big(T\phi \big) w_\bullet \longrightarrow \big(T\phi \big) w_\bullet.$$

v. For formal morphisms, in the cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, Δ is a strict T-map and so it sends the formal permutation or braiding morphisms of $\mathsf{T}(G', \phi)$ to corresponding permutations or braidings in $\mathsf{T}G'$.

This completes the description of Δ. ◇

Example 14.14. In the context of Explanation 14.13, suppose given $\langle w \rangle = (w_1, w_2)$ with

$$w_1 = \langle a^1 \rangle = (a_1^1, a_2^1, a_3^1) \quad \text{and} \quad w_2 = \langle a^2 \rangle = (a_1^2, a_2^2).$$

Then $w_\bullet = \langle a^\bullet \rangle = (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2)$ and

$$\Delta \big\langle [\phi] w \big\rangle = \big(\phi(a_1^1) \ , \ \phi(a_2^1) \ , \ \phi(a_3^1) \ , \ \phi(a_1^2) \ , \ \phi(a_2^2) \big).$$

Each braiding of tuples w_j or entries a_i^j is sent by Δ to the corresponding braiding of entries $\phi(a_i^j)$. ◇

Now we describe the strict T-map from (10.3)

$$\Lambda \colon \mathsf{T}(\mathsf{ob}A', \phi) \longrightarrow A',$$

where $f \colon (A, \boldsymbol{\cdot}) \rightsquigarrow (A', \boldsymbol{\cdot})$ is a T-map and $\phi = f_{\mathsf{ob}}$ denotes the restriction of f to objects. Recalling (10.4), Λ is the unique strict T-map induced by the universal property (6.10) and the inclusions of objects. The proof of Theorem 7.11 explains how the pushout description of $\mathsf{T}(\mathsf{ob}A', \phi)$, as in (14.3), satisfies the universal property (6.10). In particular, recalling (7.15) with $\overline{S} = \Lambda$ and $\overline{R}$ being the composite $\mathsf{T}(\mathsf{ob}A) \longrightarrow \mathsf{T}A \xrightarrow{\boldsymbol{\cdot}} A$, the following diagram identifies Λ via the universal property of the pushout in $\mathsf{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$.

$$\begin{tikzcd}
\mathsf{T}(\mathsf{ob}A) \arrow[rr, "\mathsf{T}\phi"] \arrow[d, "\zeta^\flat"'] & & \mathsf{T}(\mathsf{ob}A') \arrow[d, "\kappa"'] \arrow[dr] & \\
\mathsf{QT}(\mathsf{ob}A) \arrow[rr, "\widehat{\phi}"] \arrow[dr] & & \mathsf{T}(\mathsf{ob}A', \phi) \arrow[ddr, dashed, "\Lambda", "\exists !"'] & \mathsf{T}A' \arrow[dd, "\boldsymbol{\cdot}"] \\
& \mathsf{QT}A \arrow[dr, "\mathsf{Q}\boldsymbol{\cdot}"'] & & \\
& & \mathsf{Q}A \arrow[r, "f^\perp"'] & A'
\end{tikzcd} \tag{14.15}$$

In the above diagram, $\mathsf{T}(\mathsf{ob}A', \phi)$ is described in Explanation 14.4, with $G' = \mathsf{ob}A'$. The strict T-map $\mathsf{Q}\boldsymbol{\cdot}$ is an instance of Q applied to a T-map, as in Definition 13.9. The mate $f^\perp$ (4.4) is the unique strict T-map that factors f as below.

$$\begin{tikzcd}
\mathsf{Q}A \arrow[r, "f^\perp"] & A' \\
A \arrow[u, squiggly, "\zeta_A"] \arrow[ur, squiggly, "f"'] &
\end{tikzcd} \tag{14.16}$$

For $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$, the unit ζ is described in Definition 13.13. Unpacking these, the following gives an explicit description of Λ on objects and morphisms.

Explanation 14.17. Suppose $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$ and suppose

$$f \colon (A, \boldsymbol{\cdot}) \rightsquigarrow (A', \boldsymbol{\cdot})$$

is a T-map. Let $\phi = f_{\mathsf{ob}}$ denote the restriction of f to objects, and recall from Notation 13.1 that subscripts $\bullet$ denote the image of free objects or morphisms under the multiplication $\boldsymbol{\cdot}$. Then the strict T-map Λ in (10.3) and (14.15) is given as follows.

i. For free objects $\langle a'\rangle, \langle b'\rangle \in \mathsf{T}(\mathsf{ob}A')$ and free morphisms between them, $t' \colon \langle a'\rangle \longrightarrow \langle b'\rangle$, we have

$$\Lambda t' = t'_\bullet \colon a'_\bullet \longrightarrow b'_\bullet.$$

ii. For ϕ-objects $\big\langle [\phi]w \big\rangle = \big\langle [\phi]w_j \big\rangle_{j=1}^m$ with $w_j = \langle a^j\rangle = \langle a_i^j\rangle_{i=1}^{n_j} \in \mathsf{T}(\mathsf{ob}A)$, we have

$$\Lambda\big\langle [\phi]w \big\rangle = f^\perp(\langle a_\bullet^j\rangle_{j=1}^m) = f(a_\bullet^1) \bullet\bullet\bullet f(a_\bullet^m)$$

because $f^\perp$ is strict monoidal.

iii. For ϕ-free morphisms $[\phi]u \colon \big\langle [\phi]w \big\rangle \longrightarrow \big\langle [\phi]v \big\rangle$, where

$$\langle w\rangle = \langle w_j\rangle_{j=1}^m, \quad \langle v\rangle = \langle v_j\rangle_{j=1}^m,$$

and $u \colon \langle w\rangle \longrightarrow \langle v\rangle$ is a free morphism of $\mathsf{QT}(\mathsf{ob}A)$, we have

$$\Lambda\big([\phi]u\big) = f^\perp(u_\bullet) \colon f(a_\bullet^1) \bullet\bullet\bullet f(a_\bullet^m) \longrightarrow f(b_\bullet^1) \bullet\bullet\bullet f(b_\bullet^m).$$

Here, each $w_j = \langle a^j\rangle$ and each $v_j = \langle b^j\rangle$ as above.

If u is a tuple of morphisms $t_j \colon w_j \longrightarrow v_j$ in $\mathsf{T}(\mathsf{ob}A)$, then $u_\bullet$ is their product (concatenation) in $\mathsf{T}(\mathsf{ob}A)$. If u is a permutation or braiding in $\mathsf{T}^2(\mathsf{ob}A)$, in the cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, then $u_\bullet$ is the corresponding block permutation or braiding in $\mathsf{T}(\mathsf{ob}A)$. In either case, since $f^\perp$ is strict monoidal, $[\phi]u$ is sent to either the product of the morphisms $f(t_j)$ or to the permutation or braiding of $f(a_\bullet^1) \bullet\bullet\bullet f(a_\bullet^m)$ determined by u.

iv. For ϕ-adjoined isomorphisms $[\phi]\mathsf{q}_{\langle w\rangle} \colon \langle [\phi]w\rangle \longrightarrow ([\phi]w_\bullet)$,

$$\Lambda\big([\phi]\mathsf{q}_{\langle w\rangle}\big) = f^\perp(\zeta_\bullet) = f_\bullet \colon f(a_\bullet^1) \bullet\bullet\bullet f(a_\bullet^m) \longrightarrow f(a_\bullet^1 \bullet\bullet\bullet a_\bullet^m)$$

because the morphisms $\mathsf{q}_{\langle w\rangle}$ are the monoidal and unit constraints of ζ and (14.16) is a diagram of T-maps.

v. For formal morphisms, in the cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$, Λ is a strict T-map and so it sends the formal permutation or braiding morphisms of $\mathsf{T}(\mathsf{ob}A', \phi)$ to the corresponding permutations or braidings in A'.

This completes the description of Λ. ◇

Example 14.18. Suppose, as in Explanation 14.17, that $\mathsf{T} \in \{\mathsf{M}, \mathsf{S}, \mathsf{B}\}$ and

$$f \colon (A, \bullet) \rightsquigarrow (A', \bullet)$$

is a T-map. Let $\phi = f_{\mathsf{ob}}$ denote the restriction of f to objects.

Let $\langle w\rangle = (w_1, w_2)$ as in Example 14.14, with

$$w_1 = \langle a^1\rangle = (a_1^1, a_2^1, a_3^1) \quad \text{and} \quad w_2 = \langle a^2\rangle = (a_1^2, a_2^2).$$

Then $w_\bullet = \langle a^\bullet\rangle = (a_1^1, a_2^1, a_3^1, a_1^2, a_2^2)$ and

$$\Lambda\big\langle [\phi]w \big\rangle = f(a_1^1 \bullet a_2^1 \bullet a_3^1) \bullet f(a_1^2 \bullet a_2^2).$$ ◇

15 Examples for symmetric and braided monoidal functors

In this section, we suppose that T is one of the 2-monads, S or B, for symmetric or braided monoidal structures, respectively, that are strictly associative and unital (Notation 11.1). In this section we say "monoidal" to mean strict monoidal structure for categories A and A'. Note, however, that the discussion here does apply to the corresponding general monoidal structures, S^{g} or B^{g}, by the Monoidal Strictification Theorem 11.4.

The Pseudomorphism Coherence Theorems 1.5 and 1.7 apply in these cases, and this section provides several examples using the *dissolution*, as in Theorem 1.7, to determine whether a formal diagram commutes. In these examples, we suppose given

$$f = (f, f_2, f_0) \colon (A, +, 0) \rightsquigarrow (A', \bullet, 1)$$

as follows.

i. $(A, +, 0, \beta)$ and $(A', \bullet, 1, \beta)$ are T-algebras, i.e., symmetric or braided monoidal categories with the indicated notation for monoidal products, units, and braidings.

ii. f is a T-map (Remark 2.10), and we use the zigzag arrow notation of Convention 2.21. Thus, f is a symmetric or braided strong monoidal functor.

All of our applications concern functors that are either strong or strict monoidal. We will say that f is a "monoidal functor" to mean strong monoidal functor.

Example 15.1. The following diagram in A' appears as (1.2) in the introduction. It involves f, the braiding isomorphisms of A and A', and an object $a \in A$. The two composites around the diagram apply a cyclic permutation to the objects, but combine with the monoidal constraints of f in different ways.

$$\begin{tikzcd}
f(a) \bullet f(a) \bullet f(a) \arrow[r, "f_2 \bullet 1"] \arrow[d, "f_2"'] & f(a+a) \bullet f(a) \arrow[r, "\beta"] & f(a) \bullet f(a+a) \arrow[d, "f_2"] \\
f(a+a+a) \arrow[r, "f(1+\beta)"] & f(a+a+a) \arrow[r, "f(\beta+1)"] & f(a+a+a)
\end{tikzcd} \tag{15.2}$$

One can use the naturality of f_2 and β, together with various axioms for f and β, to show that this diagram commutes.

Alternatively, (15.2) is an f-formal diagram, in the sense of Definition 10.8. The following diagram is a lift to $\mathsf{T}(\mathsf{ob}A', \phi)$, where $\phi = f_{\mathsf{ob}}$ denotes the restriction of f to objects.

$$\begin{tikzcd}
\Big([\phi](a) \,,\, [\phi](a) \,,\, [\phi](a) \Big) \arrow[r, "{[\phi]\mathsf{q} \,;\, 1}"] \arrow[d, "{[\phi]\mathsf{q}}"'] & \Big([\phi](a,a) \,,\, [\phi](a) \Big) \arrow[r, "\beta"] & \Big([\phi](a) \,,\, [\phi](a,a) \Big) \arrow[d, "{[\phi]\mathsf{q}}"] \\
\Big([\phi](a,a,a) \Big) \arrow[r, "{[\phi](1 \,,\, \beta)}"] & \Big([\phi](a,a,a) \Big) \arrow[r, "{[\phi](\beta \,,\, 1)}"] & \Big([\phi](a,a,a) \Big)
\end{tikzcd} \tag{15.3}$$

To verify that the above diagram is a lift of (15.2), one uses the descriptions of $\mathsf{T}(\mathsf{ob}A', \phi)$ and Λ in Explanation 14.4 and Explanation 14.17, respectively. In particular, the terminology of Explanation 14.4 applies as follows.

- The objects are *ϕ-objects*; each entry $[\phi](a, \ldots, a)$ is a lift of a term $f(a + \cdots + a)$.
- The morphisms $[\phi]\mathsf{q}$ are *ϕ-adjoined isomorphisms* (14.7) and are lifts of the monoidal constraints for f.
- The morphisms $[\phi](1, \beta)$ and $[\phi](\beta, 1)$ are *ϕ-free morphisms* (14.6) and are lifts of the corresponding morphisms $f(1 + \beta)$ and $f(\beta + 1)$.

- The morphism β is a *formal morphism* lifting the corresponding braiding in (15.2).

Using the description of Δ in Explanation 14.13, the *dissolution* of (15.3) is the following diagram in $\mathsf{T}(\mathsf{ob}A')$. Here, $\phi = f_{\mathsf{ob}}$ is applied separately to each object, and the ϕ-adjoined morphisms $[\phi]\mathsf{q}$ are sent to identities.

$$\begin{tikzcd}
\big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big) \arrow[r, "1"] \arrow[d, "1"'] & \big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big) \arrow[r, "\beta_{(\,f(a),f(a)\,)\,,\,f(a)}"] & \big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big) \arrow[d, "1"] \\
\big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big) \arrow[r, "(\,1\,,\,\beta\,)"] & \big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big) \arrow[r, "(\,\beta\,,\,1\,)"] & \big(\, f(a)\,,\, f(a)\,,\, f(a)\,\big)
\end{tikzcd} \tag{15.4}$$

The two composites around the above diagram have the same underlying braid of the object $f(a)$: in the left-bottom composite, the first two instances of $f(a)$ are braided past the third, one at a time, and in the top-right composite they are braided past in one step.

Therefore, the diagram (15.4) commutes in either case $\mathsf{T} = \mathsf{S}$ or $\mathsf{T} = \mathsf{B}$ by the Symmetric or Braided Coherence Theorem 11.9 (*ii*) or (*iii*), respectively. Since Δ is an equivalence by Theorem 1.5, this implies that the lift (15.3) commutes in $\mathsf{T}(\mathsf{ob}A', \phi)$ and hence diagram (15.2) commutes in A'. ◇

The key feature of Example 15.1, and of our other examples below, is that the dissolution diagram (15.4) replaces each monoidal constraint of f in (15.2) with an identity. Thus, it also replaces objects such as $f(a + a)$ with tuples $\big(f(a)\,,\, f(a)\big)$ in the free algebra $\mathsf{T}(\mathsf{ob}A')$. The lift (15.3) is what ensures that this can be done coherently.

As noted in Remark 10.11, the composites around the dissolution diagram (15.4) determine morphisms in A' that are generally *not* equal to the respective morphisms from the original diagram (15.2). Indeed, the morphisms in A' determined by the composites around (15.4) do not have the same codomain as the composites around (15.2). The purpose of the diagrammatic coherence theorems in this work is to determine:

i. how to construct a lift and corresponding dissolution of an f-formal diagram, and

ii. conditions under which Δ is an equivalence, so that commutativity of the dissolution diagram implies that of the original diagram.

Example 15.5 (Monoidal naturality of f_2). The monoidal constraint f_2 is a natural transformation with components

$$\big(f_2\big)_{a,b} \colon f(a) \bullet f(b) \longrightarrow f(a+b) \quad \text{for} \quad a, b \in A.$$

As a natural transformation, its domain and codomain are the respective composites in the following diagram, in which the products $A \times A$, $A' \times A'$ are given the componentwise monoidal structures.

$$\begin{tikzcd}
 & A' \times A' \arrow[dr, "\bullet"] & \\
A \times A \arrow[ur, "f \times f"] \arrow[dr, "+"'] & f_2 \Downarrow & A' \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

The composites above are monoidal functors, with the monoidal constraints of $+$ and $\bullet$ given by the following for $a, b, c, d \in A$ and $a', b', c', d' \in A'$:

$$a + b + c + d \xrightarrow{1 + \beta_{b,c} + 1} a + c + b + d \quad \text{and} \quad a' \bullet b' \bullet c' \bullet d' \xrightarrow{1 \bullet \beta_{b',c'} \bullet 1} a' \bullet c' \bullet b' \bullet d'.$$

The following diagram in A' is the monoidal naturality axiom at objects $(a,b),(c,d) \in A \times A$, to check whether the natural transformation f_2 is a monoidal transformation.

$$\begin{tikzcd}
f(a) \cdot f(b) \cdot f(c) \cdot f(d) \arrow[r, "f_2 \cdot f_2"] \arrow[d, "1 \cdot \beta \cdot 1"'] & f(a+b) \cdot f(c+d) \arrow[d, "f_2"] \\
f(a) \cdot f(c) \cdot f(b) \cdot f(d) \arrow[d, "f_2 \cdot f_2"'] & f(a+b+c+d) \arrow[d, "f(1+\beta+1)"] \\
f(a+c) \cdot f(b+d) \arrow[r, "f_2"] & f(a+c+b+d)
\end{tikzcd} \tag{15.6}$$

Again using Explanations 14.4 and 14.17 followed by Explanation 14.13, one can identify (15.6) as an f-formal diagram and determine the requisite lift followed by its dissolution diagram, shown below.

$$\begin{tikzcd}
\big(f(a) , f(b) , f(c) , f(d) \big) \arrow[r, "1"] \arrow[d, "(1,\beta,1)"'] & \big(f(a) , f(b) , f(c) , f(d) \big) \arrow[d, "1"] \\
\big(f(a) , f(c) , f(b) , f(d) \big) \arrow[d, "1"'] & \big(f(a) , f(b) , f(c) , f(d) \big) \arrow[d, "(1,\beta,1)"] \\
\big(f(a) , f(c) , f(b) , f(d) \big) \arrow[r, "1"] & \big(f(a) , f(c) , f(b) , f(d) \big)
\end{tikzcd} \tag{15.7}$$

The two composites around (15.7) have the same underlying braid, given by passing $f(b)$ past $f(c)$.

Therefore, (15.7) commutes in either case $\mathsf{T} = \mathsf{S}$ or $\mathsf{T} = \mathsf{B}$ by the Symmetric or Braided Coherence Theorem 11.9 (*ii*) or (*iii*), respectively. Since Δ is an equivalence by Theorem 1.5, the commutativity of the dissolution diagram (15.7) in $\mathsf{T}(\mathsf{ob}A')$ implies that the original diagram (15.6) commutes in A'.◇

If f is a symmetric or braided monoidal functor such that the monoidal constraint f_2 has components with nontrivial underlying braids, then the use of dissolution diagrams to determine commutativity of formal diagrams for f is a nontrivial simplification. For such functors (f, f_2, f_0), the underlying braids of (15.2) and (15.6) may be different from—generally more complex than—those of (15.4) and (15.7), respectively. The significance of our diagrammatic coherence, when Δ is an equivalence, is precisely this simplification, summarized in the following variant of Slogan 1.8.

Slogan 15.8. *When Δ is an equivalence, commutativity of formal diagrams for f reduces to checking commutativity of the simpler dissolution diagrams, in which the T-algebra constraints of f are replaced by identities.* ◇

Example 15.9 (Monoidal naturality of $\beta_{f,f}$). Let $f \cdot f$ denote the composite monoidal functor shown below, where *diag* is the diagonal functor:

$$A \xrightarrow{\text{diag}} A \times A \xrightarrow{f \times f} A' \times A' \xrightarrow{\cdot} A'.$$

So, $\big(f \cdot f\big)(a) = f(a) \cdot f(a)$ for objects $a \in A$, and likewise for morphisms. The monoidal constraint of $\cdot$ is $1 \cdot \beta \cdot 1$, described in Example 15.5. The diagonal functor is strict monoidal because the monoidal sum in $A \times A$ is given componentwise.

The braiding isomorphism β of A' induces a natural transformation

$$\beta_{f,f} \colon f \cdot f \longrightarrow f \cdot f \tag{15.10}$$

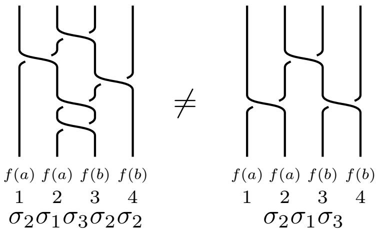
with components $\beta_{f(a),f(a)}$ for $a \in A$. The diagram below is the monoidal naturality diagram at a pair of objects $a, b \in A$ to check whether $\beta_{f,f}$ is a monoidal transformation. The left and right vertical composites are the monoidal constraints of $f \bullet f$.

$$\begin{tikzcd}
\big(f \bullet f\big)(a) \bullet \big(f \bullet f\big)(b) & \big(f \bullet f\big)(a) \bullet \big(f \bullet f\big)(b) \\
f(a) \bullet f(a) \bullet f(b) \bullet f(b) \arrow[u, equal] \arrow[r, "\beta \bullet \beta"] \arrow[d, "1 \bullet \beta \bullet 1"'] & f(a) \bullet f(a) \bullet f(b) \bullet f(b) \arrow[u, equal] \arrow[d, "1 \bullet \beta \bullet 1"] \\
f(a) \bullet f(b) \bullet f(a) \bullet f(b) \arrow[d, "f_2 \,\bullet\, f_2"'] & f(a) \bullet f(b) \bullet f(a) \bullet f(b) \arrow[d, "f_2 \,\bullet\, f_2"] \\
f(a+b) \bullet f(a+b) \arrow[r, "\beta"] \arrow[d, equal] & f(a+b) \bullet f(a+b) \arrow[d, equal] \\
\big(f \bullet f\big)(a+b) & \big(f \bullet f\big)(a+b)
\end{tikzcd} \tag{15.11}$$

The above is an f-formal diagram, and the following is a dissolution diagram for it. There, the morphism along the lower edge is the block braiding of the first two terms past the second two.

$$\begin{tikzcd}
\big(f(a)\,,\, f(a)\,,\, f(b)\,,\, f(b)\big) \arrow[r, "{\big(\beta\,,\,\beta\big)}", "\sigma_1\sigma_3"'] \arrow[d, "{\big(1\,,\,\beta\,,\,1\big)}"', "\sigma_2"] & \big(f(a)\,,\, f(a)\,,\, f(b)\,,\, f(b)\big) \arrow[d, "\sigma_2"', "{\big(1\,,\,\beta\,,\,1\big)}"] \\
\big(f(a)\,,\, f(b)\,,\, f(a)\,,\, f(b)\big) \arrow[d, "1"'] & \big(f(a)\,,\, f(b)\,,\, f(a)\,,\, f(b)\big) \arrow[d, "1"] \\
\big(f(a)\,,\, f(b)\,,\, f(a)\,,\, f(b)\big) \arrow[r, "\sigma_2\sigma_1\sigma_3\sigma_2", "\beta"'] & \big(f(a)\,,\, f(b)\,,\, f(a)\,,\, f(b)\big)
\end{tikzcd} \tag{15.12}$$

In the above diagram, the inner labels on each morphism are the underlying braids, where σ_i is the elementary braiding of strand i under strand $i+1$. The left-bottom and top-right composites around the boundary of (15.11) are shown in the following braid diagram. In these diagrams, the right-to-left composition of elementary braids is ordered bottom-to-top.



Since these braids are not equal, $\beta_{f,f}$ in (15.10) is generally not a monoidal transformation when $\mathsf{T} = \mathsf{B}$ and f is a braided monoidal functor. For example, when $A = A' = \mathsf{B}\{a, b\}$ is the free braided monoidal category on two objects and f is the identity, then $\beta_{f,f}$ is not a monoidal transformation.

However, since the underlying *permutations* around the diagram (15.11) are equal, the Symmetric Coherence Theorem 11.9 (*ii*) implies that (15.11) does commute when $\mathsf{T} = \mathsf{S}$. Thus, $\beta_{f,f}$ is a monoidal transformation when f is a symmetric monoidal functor between symmetric monoidal categories. Note, again, that this conclusion holds independently of whether the monoidal constraints f_2 have nontrivial underlying permutations. ◇

Remark 15.13. In each of the above examples, one can also check commutativity directly, using naturality of the monoidal constraints f_2. Diagram (15.11) is particularly straightforward, involving a single use of naturality to commute β with $f_2 \bullet f_2$. Formal diagrams for f are always amenable to such an approach. However, it can be a nontrivial task to determine *which* combination of naturality and other axioms will reduce the commutativity of the original diagram to a computation in $\mathsf{T}(\mathsf{ob}A')$. The advantage of the dissolution approach is that it formalizes such a reduction by systematically replacing the monoidal constraints with identities. ◇

16 Non-example via quadrupling

In this section we continue the context and conventions of Section 15, so that $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$ is the monad for (strict) symmetric or braided monoidal categories. We consider two specific functors f and h whose monoidal constraints have nontrivial underlying braids.

This section gives several examples of using Theorem 12.7, which is a refinement of the Symmetric Coherence Theorem 11.9 (*ii*). Then, Non-Example 16.10, Remark 16.14, and Lemma 16.17 discuss a formal diagram (16.12) where the only lifts of interest reduce, in the sense of Remark 10.12, to A-formal lifts. For such lifts, the dissolution diagram strategy of Section 15 does not provide any simplification.

Definition 16.1 (Doubling functor). The *doubling functor* $f\colon A \rightsquigarrow A$ with unit and monoidal constraints f_0 and f_2, respectively, is defined as follows for objects $a, b \in A$ and morphisms s in A.

$$f(a) = a + a \quad \text{and} \quad f(s) = s + s,$$

$$\begin{tikzcd}
 & f(0) \arrow[d, equal] & f(a)+f(b) \arrow[r, "f_2"] \arrow[d, equal] & f(a+b) \arrow[d, equal] \\
0 \arrow[ur, "f_0"] \arrow[r, "1_0"'] & 0+0 & a+a+b+b \arrow[r, "1_a+\beta_{a,b}+1_b"] & a+b+a+b
\end{tikzcd} \tag{16.2}$$

◇

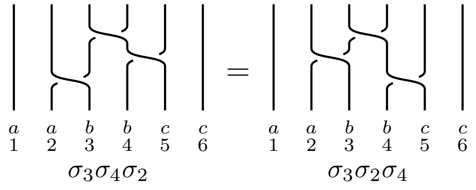
We will show below that f is a *symmetric* monoidal functor in the symmetric case, where $\mathsf{T} = \mathsf{S}$ and A is a symmetric monoidal category. In the braided case, where $\mathsf{T} = \mathsf{B}$ and A is merely braided monoidal, then f is a monoidal functor, but generally not *braided* monoidal. Although these conclusions will be familiar to experts, we include them as preparation for the calculations in Non-Example 16.10 and Remark 16.13.

The unity diagrams for the doubling functor are trivial since $f_0 = 1_0$. The following example discusses the associativity and braid axioms for f.

Example 16.3 (Axioms for doubling). Let f be the doubling functor from Definition 16.1. The associativity diagram, for objects $a, b, c \in A$, is the following.

$$\begin{tikzcd}
a+a+b+b+c+c \arrow[r, "\sigma_4"] \arrow[ddd, bend right, "\sigma_2"'] & a+a+b+c+b+c \arrow[ddd, bend left, "\sigma_3\sigma_2"] \\
f(a)+f(b)+f(c) \arrow[u, equal] \arrow[r, "1+f_2"] \arrow[d, "f_2+1"'] & f(a)+f(b+c) \arrow[u, equal] \arrow[d, "f_2"] \\
f(a+b)+f(c) \arrow[r, "f_2"] \arrow[d, equal] & f(a+b+c) \arrow[d, equal] \\
a+b+a+b+c+c \arrow[r, "\sigma_3\sigma_4"] & a+b+c+a+b+c
\end{tikzcd} \tag{16.4}$$

In the above diagram, the inner arrows are labeled as structure morphisms of f and the outer arrows are labeled by their underlying braids, where σ_i are the elementary braids as in Example 15.9. The underlying braids for the left-bottom and top-right composites around the boundary of (16.4) are shown in the following braid diagrams.



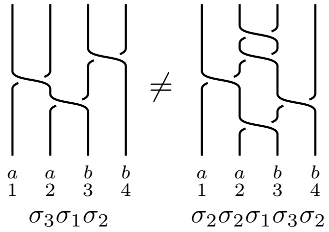
(16.5)

Since these braids are equal, the Braided Coherence Theorem 11.9 (*iii*) implies that (16.4) commutes when $\mathsf{T} = \mathsf{B}$, and thus also when $\mathsf{T} = \mathsf{S}$.

The symmetry axiom for the doubling functor, for objects $a, b \in A$, is the following.

$$\begin{tikzcd}
a+a+b+b \arrow[r, "\sigma_2\sigma_1\sigma_3\sigma_2"] \arrow[dddd, bend right, "\sigma_2"'] & b+b+a+a \arrow[dddd, bend left, "\sigma_2"] \\
f(a)+f(b) \arrow[u, equal] \arrow[r, "\beta_{f(a),f(b)}"] \arrow[d, "f_2"'] & f(b)+f(a) \arrow[u, equal] \arrow[d, "f_2"] \\
f(a+b) \arrow[r, "f(\beta_{a,b})"] \arrow[d, equal] & f(b+a) \arrow[d, equal] \\
a+b+a+b \arrow[r, "\sigma_3\sigma_1"] & b+a+b+a
\end{tikzcd} \tag{16.6}$$

The above diagram is labeled similarly to (16.4), with inner arrows labeled via structure morphisms and outer arrows labeled by their underlying braids. The following diagrams use the same conventions as above and show the underlying braids for the left-bottom and top-right composites around the boundary of (16.6).



(16.7)

Since these braids are not equal, (16.6) does not generally commute when $\mathsf{T} = \mathsf{B}$. That is, for a general braided monoidal category A, the doubling functor is not necessarily a *braided* monoidal functor, although it is a plain monoidal functor.

Note, however, that the underlying permutations of the braids above *are* equal. Thus, the Symmetric Coherence Theorem 11.9 (*ii*) implies that (16.6) does commute when $\mathsf{T} = \mathsf{S}$. That is, the doubling functor is a symmetric monoidal functor when A is a symmetric monoidal category. ◇

Remark 16.8. In the case $\mathsf{T} = \mathsf{S}$, there is a refinement of the Symmetric Coherence Theorem 11.9 (*ii*), discussed in Section 12. For finitely-generated formal diagrams in a symmetric monoidal category A, such as those of Example 16.3, Theorem 12.7 shows that it suffices to check the self-permutation of x, in the sense of Definition 12.6, for each generating object x.

In (16.4), it suffices to check the three self-permutations $\pi_a^{\widetilde{D}}$, $\pi_b^{\widetilde{D}}$, and $\pi_c^{\widetilde{D}}$, where $\widetilde{D}$ denotes the formal lift of (16.4) to the free monoidal category on three objects, $\mathsf{S}\{a, b, c\}$. Each self-permutation $\pi_x^{\widetilde{D}}$ is determined by projecting to the free symmetric monoidal category on the single object x, for $x \in \{a, b, c\}$. In the braid diagram (16.5), this corresponds to removing the strands for each object $y \neq x$, and then checking the underlying permutation of the resulting braid.

In both the left-bottom and top-right composites around (16.4), neither instance of a is permuted past the other. That is, the strands labeled a in (16.5) do not cross. Thus, $\pi_a^{\widetilde{D}} = 1$ for each composite around (16.4). Likewise, the self-permutations of b and c are identities for both composites. This is sufficient for Theorem 12.7 to imply that (16.4) commutes.

The same approach can be used for the composites around (16.6): the self-permutations of both a and b are trivial, for both composites around (16.6). This is sufficient for Theorem 12.7 to imply that (16.6) commutes. ◇

Recall from Remark 10.12, for a symmetric or braided monoidal functor $f\colon A \longrightarrow A'$, a lift $\widetilde{D}$ of an f-formal diagram is said to *reduce to an A'-formal diagram* if it factors through the inclusion of free objects and morphisms

$$\kappa\colon \mathsf{T}(\mathsf{ob}A') \longrightarrow \mathsf{T}(\mathsf{ob}A', \phi) \tag{16.9}$$

where $\phi = f_{\mathsf{ob}}$ denotes the restriction to objects. None of Examples 15.1, 15.5, and 15.9 factor through κ, because the respective lifts involve the ϕ-adjoined isomorphisms $[\phi]\mathsf{q}$, which are then mapped via Δ to identities.

The following provides an example of a monoidal naturality diagram that involves only braid isomorphisms and monoidal constraints and yet, except for certain trivialities, any lift to generating morphisms of $\mathsf{T}(\mathsf{ob}A, \phi)$ must factor through κ and hence reduce to an A-formal lift. Remark 16.14 and Lemma 16.17 below explain some details and additional subtleties related to this case.

Non-Example 16.10 (Cyclic braiding). Let h denote the *quadrupling* functor $h = f \circ f$, where f is the doubling functor from Definition 16.1 and Example 16.3. Thus, we have

$$h(a) = a + a + a + a \quad \text{for} \quad a \in A,$$

and h is a monoidal functor in either the symmetric or braided monoidal cases $\mathsf{T} \in \{\mathsf{S}, \mathsf{B}\}$. In the symmetric case, $\mathsf{T} = \mathsf{S}$, quadrupling is a symmetric monoidal functor. In other words (Remark 2.10), h is an S-map, but generally not a B-map.

There is a natural transformation γ with components given by the cyclic braiding of the first summand past the other three:

$$\gamma_a = \beta_{a,(a+a+a)}\colon h(a) = a + a + a + a \longrightarrow a + a + a + a = h(a). \tag{16.11}$$

The following is the monoidal naturality diagram for $a, b \in A$, to determine whether γ is a monoidal transformation. Here, we use the notation

$$\sigma_{i:k} = \sigma_k \sigma_{k-1} \cdots \sigma_i$$

to denote the braiding of strand i under strands $i+1$ through $k+1$.

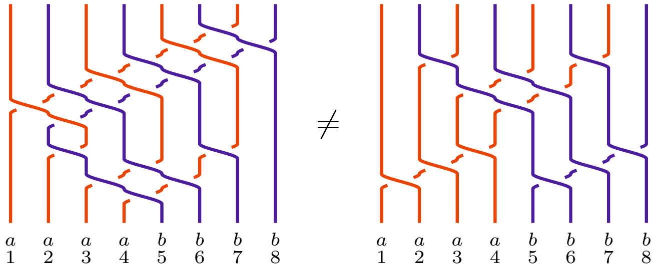
$$\begin{tikzcd}
h(a)+h(b) \arrow[d, equal] & h(a)+h(b) \arrow[d, equal] \\
a+a+a+a+b+b+b+b \arrow[r, "\gamma_a + \gamma_b", "\sigma_{1:3}\,\sigma_{5:7}"'] \arrow[d, "1+\beta_{a+a,b+b}+1"', "\sigma_{3:4}\,\sigma_{4:5}"] & a+a+a+a+b+b+b+b \arrow[d, "\sigma_{3:4}\,\sigma_{4:5}"', "1+\beta_{a+a,b+b}+1"] \\
a+a+b+b+a+a+b+b \arrow[d, "\begin{array}{c}1+\beta_{a,b}+1\\+1+\beta_{a,b}+1\end{array}"', "\sigma_2\sigma_6"] & a+a+b+b+a+a+b+b \arrow[d, "\sigma_2\sigma_6"', "\begin{array}{c}1+\beta_{a,b}+1\\+1+\beta_{a,b}+1\end{array}"] \\
a+b+a+b+a+b+a+b \arrow[r, "\sigma_{1:6}\,\sigma_{2:7}", "\gamma_{a+b}"'] \arrow[d, equal] & a+b+a+b+a+b+a+b \arrow[d, equal] \\
h(a+b) & h(a+b)
\end{tikzcd} \tag{16.12}$$

The above is an A-formal diagram, in the sense of Definition 10.8: it admits a lift to $\mathsf{T}(\mathsf{ob}A)$, with underlying braids shown on the inner labels.

Composing with the inclusion of free objects and morphisms κ (16.9), with $\phi = h_{\mathsf{ob}}$, trivially yields a diagram in $\mathsf{T}(\mathsf{ob}A, \phi)$. However, as discussed in Remark 10.12, the resulting dissolution diagram is equal to the original lift and does not yield any simplification.

The vertical morphisms in (16.12) are the monoidal constraints for $h = f \circ f$, and one would obtain a simpler dissolution diagram if these were lifted to ϕ-adjoined isomorphisms $[\phi]\mathsf{q}$. However, Lemma 16.17 below shows that such morphisms are not generally composable with lifts of γ.

Here, we use the Braided and Symmetric Coherence Theorems 11.9 and 12.7 directly on the A-formal lift of (16.12). The underlying braids of the left-bottom and top-right composites are shown below. These braids are distinct; strands 2 and 5 are linked on the left, but not on the right.



$\sigma_{1:6}\,\sigma_{2:7}\,\sigma_2\,\sigma_6\,\sigma_{3:4}\,\sigma_{4:5}$ $\sigma_2\,\sigma_6\,\sigma_{3:4}\,\sigma_{4:5}\,\sigma_{1:3}\,\sigma_{5:7}$

Therefore, the cyclic braiding γ is generally *not* a monoidal transformation in the braided case $\mathsf{T} = \mathsf{B}$. For example, if $A = \mathsf{B}\{a, b\}$ is the free braided monoidal category on two objects, then γ will not be a monoidal transformation.

In the symmetric case $\mathsf{T} = \mathsf{S}$, checking the underlying permutations in (16.12) simplifies via Theorem 12.7. The vertical composites have trivial a-permutation and hence the self-permutation of a under either the left-bottom or top-right composite is the cyclic permutation $(1\ 4\ 3\ 2)$. The same statements apply to b. This is sufficient for Theorem 12.7 to imply that (16.12) commutes. Thus, the natural transformation γ in (16.11) is monoidal natural in the symmetric case $\mathsf{T} = \mathsf{S}$, but generally not in the braided case, $\mathsf{T} = \mathsf{B}$. ◇

Remark 16.13. In the symmetric case $\mathsf{T} = \mathsf{S}$, Non-Example 16.10 can be generalized to show that any permutation $\gamma \in \Sigma_n$ determines a monoidal natural automorphism of the *n-fold sum* functor

$$h^n(a) = \underbrace{a + \cdots + a}_{n \text{ summands}} \quad \text{for} \quad a \in A.$$

Here, h^n is a symmetric monoidal functor defined inductively as $h^n = (h^2 + 1) \circ h^{n-1}$, with $h^2 = f$ being the symmetric monoidal doubling functor. For objects $a, b \in A$, both the a-permutation and the b-permutation of the monoidal constraint

$$h^n_2 \colon h^n(a) + h^n(b) \longrightarrow h^n(a + b)$$

are identities. Letting $\gamma \colon h^n \longrightarrow h^n$ also denote the natural transformation induced by $\gamma \in \Sigma_n$, both morphisms $\gamma_a + \gamma_b$ and γ_{a+b} have a-permutation equal to $\gamma \in \Sigma_n$, and likewise for b-permutations. Thus, Theorem 12.7 shows that the monoidal naturality diagram for γ commutes for each $a, b \in A$. ◇

In the following discussion, we restrict to the symmetric case, $\mathsf{T} = \mathsf{S}$, because the quadrupling functor is an S-map, but not a B-map. Thus, Definition 10.2 applies to h in the case $\mathsf{T} = \mathsf{S}$, but not in the case $\mathsf{T} = \mathsf{B}$. The details of this discussion will require the following observation and subsequent terminology to exclude certain lifts of the monoidal unit $0 \in A$ and its identity morphism.

Remark 16.14. In the context of Non-Example 16.10, there are several objects and morphisms of $\mathsf{S}(\mathsf{ob}A, \phi)$ that are nontrivial lifts of the monoidal unit 0 and its identity morphism. In particular, there are ϕ-adjoined isomorphisms that lift the unit constraint $h_0 = 1_0$; the monoidal constraints at 0,

$$(h_2)_{0,0} = 1_0 \colon h(0) + h(0) \longrightarrow h(0+0) = 0;$$

or other such combinations of unit and monoidal constraints of h at 0.

More generally, ϕ-objects of the form $\big([\phi](0, \ldots, 0)\big)$, or ; products of such objects, will be lifts of 0. Morphisms between such objects will be lifts of 1_0, and therefore do not make substantial contributions to lifts of interest for, e.g., the composites around (16.12). ◇

Definition 16.15. Suppose A is a symmetric strict monoidal category with unit 0, and $\phi\colon \mathsf{ob}A \longrightarrow \mathsf{ob}A$ is a map of sets. An object $x \in \mathsf{S}(\mathsf{ob}A, \phi)$ is called *tidy* if it has no ; factors of the form $\big([\phi](0, \ldots, 0)\big)$. A composite of morphisms in $\mathsf{S}(\mathsf{ob}A, \phi)$

$$x_0 \xrightarrow{\xi_1} x_1 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_r} x_r \quad \text{for} \quad r \geq 1 \tag{16.16}$$

is called *tidy* if each x_i is a tidy object and each morphism ξ_i is a ; product of generating morphisms.◇

Lemma 16.17. *Suppose $A = \mathsf{S}\{a, b\}$ is the free symmetric monoidal category on two objects a and b. In the context of Non-Example 16.10, any tidy lift of (16.12) to $\mathsf{S}(\mathsf{ob}A, \phi)$, with $\phi = h_{\mathsf{ob}}$, reduces to an A-formal lift.*

Proof. We will use the description of

$$\Lambda \colon \mathsf{S}(\mathsf{ob}A, \phi) \longrightarrow A \tag{16.18}$$

as shown in the following diagram, which is (14.15) applied to this case and explained further below.

$$\begin{tikzcd}
\mathsf{S}(\mathsf{ob}A) \arrow[rr, "\mathsf{S}\phi"] \arrow[d, "\zeta^\flat"'] & & \mathsf{S}(\mathsf{ob}A) \arrow[d, "\kappa"'] \arrow[drr] & & \\
\mathsf{QS}(\mathsf{ob}A) \arrow[rr, "\widehat{\phi}"] \arrow[dr] & & \mathsf{S}(\mathsf{ob}A, \phi) \arrow[ddrr, dashed, "\Lambda", "\exists!"'] & & \mathsf{S}A \arrow[dd, "+"] \\
& \mathsf{QS}A \arrow[dr, "\mathsf{Q}+"'] & & & \\
& & \mathsf{Q}A \arrow[rr, "h^\perp"'] & & A
\end{tikzcd} \tag{16.19}$$

Here, the upper left square is a pushout of symmetric monoidal categories and symmetric strict monoidal functors. The dashed arrow Λ is the unique symmetric strict monoidal functor induced by the outer composites.

Recalling Explanation 14.4, the generating morphisms of $\mathsf{S}(\mathsf{ob}A, \phi)$ consist of free morphisms, ϕ-morphisms, and formal morphisms, described as follows and explained further below.

- The *free objects* and *free morphisms* of $\mathsf{S}(\mathsf{ob}A, \phi)$ are those in the image of κ.
- The *ϕ-objects* and *ϕ-morphisms* of $\mathsf{S}(\mathsf{ob}A, \phi)$ are those in the image of $\widehat{\phi}$.
- The *formal morphisms* of $\mathsf{S}(\mathsf{ob}A, \phi)$ are symmetry isomorphisms for the product ; induced by concatenation of tuples (Notation 13.1).

Since $\mathsf{S}(\mathsf{ob}A, \phi)$ is obtained as a pushout, free objects and morphisms that are in the image of $\mathsf{S}\phi$ are identified with the corresponding ϕ-objects and ϕ-morphisms in the image of $\zeta^\flat$. Furthermore, the symmetry isomorphisms in $\mathsf{S}(\mathsf{ob}A)$ and $\mathsf{QS}(\mathsf{ob}A)$ are identified with the corresponding formal morphisms of $\mathsf{S}(\mathsf{ob}A, \phi)$. In particular, formal morphisms between free objects are identified with the corresponding permutation isomorphisms of $\mathsf{S}(\mathsf{ob}A)$.

Below, we will show that every lift of (16.12) to a tidy composite in $\mathsf{S}(\mathsf{ob}A, \phi)$ factors through κ. The argument uses the following two invariants that are associated to any map of sets $\phi\colon \mathsf{ob}A \longrightarrow \mathsf{ob}A$. The hypothesis that ϕ is given by quadrupling will be used below, when we apply these invariants to the case of interest.

- Each morphism in $A = \mathsf{S}\{a, b\}$ has an *underlying a-permutation* and an *underlying b-permutation*, described in Definition 11.6. Therefore, each morphism ξ of $\mathsf{S}(\mathsf{ob}A, \phi)$ has underlying a- and b-permutations given by those of $\Lambda\xi$.
- Each object of $\mathsf{S}(\mathsf{ob}A, \phi)$ has an *a-signature* and a *b-signature* that are elements of $\mathsf{S}(\mathbb{N})$, explained below.

For objects of A, let ν^a denote the composite

$$\mathsf{ob}A = \mathsf{ob}\big(\mathsf{S}(\{a, b\})\big) \longrightarrow \mathsf{S}(\{a\}) \longrightarrow \mathbb{N}$$

given first by sending b to 0 and then taking isomorphism classes of objects. Let ν^b denote the similar composite that first sends a to 0 and then takes isomorphism classes of objects. Each $\nu \in \{\nu^a, \nu^b\}$ induces a free functor

$$\mathsf{S}(\mathsf{ob}A) \xrightarrow{\mathsf{S}\nu} \mathsf{S}\mathbb{N}$$

that is given by applying ν entry-wise to tuples of objects of A. The free functor $\mathsf{S}\nu$ is symmetric strict monoidal with respect to the concatenation of tuples, denoted ; as in Notation 13.1. Define the *a-signature* and *b-signature* of an object $\langle c\rangle = \langle c_i\rangle_{i=1}^{n}$ in $\mathsf{S}(\mathsf{ob}A)$ as the tuples of natural numbers

$$\begin{aligned} \mathsf{sgn}^a\langle c\rangle &= (\mathsf{S}\nu^a)\langle c\rangle = \big\langle \nu^a(c_i)\big\rangle_{i=1}^{n} \quad \text{and} \\ \mathsf{sgn}^b\langle c\rangle &= (\mathsf{S}\nu^b)\langle c\rangle = \big\langle \nu^b(c_i)\big\rangle_{i=1}^{n} \end{aligned} \tag{16.20}$$

for c_i in $\mathsf{ob}A$.

To define the a- and b-signatures of general objects $x \in \mathsf{S}(\mathsf{ob}A, \phi)$, recall from Explanation 14.4 that the upper square of (16.19) remains a pushout after taking the underlying monoid of objects. That is, applying the functor

$$\mathsf{ob}\colon \mathsf{S}\text{-}\mathsf{Alg}_{\mathsf{s}} \longrightarrow \mathscr{Mon}$$

as in (14.2) preserves pushouts because it is left adjoint to indisc.

Now recall from Definition 13.4 that the objects of $\mathsf{Q}A$ are given by those of the free algebra $\mathsf{S}A$. Therefore, the monoid homomorphism $\mathsf{S}\phi$ in the following diagram of monoids induces the dashed arrow Λ', factoring

$$\mathsf{ob}\Lambda\colon \mathsf{ob}\big(\mathsf{S}(\mathsf{ob}A, \phi)\big) \longrightarrow \mathsf{ob}\big(A\big)$$

through $\mathsf{ob}\big(\mathsf{S}A\big)$. Here, the two unlabeled arrows are induced by inclusion of objects $\mathsf{ob}A \hookrightarrow A$.

$$\begin{tikzcd}
\mathsf{ob}\big(\mathsf{S}(\mathsf{ob}A)\big) \arrow[rr, "\mathsf{S}\phi"] \arrow[d, "\zeta^\flat"'] & & \mathsf{ob}\big(\mathsf{S}(\mathsf{ob}A)\big) \arrow[d, "\kappa"'] \arrow[drr] & & & \\
\mathsf{ob}\big(\mathsf{QS}(\mathsf{ob}A)\big) \arrow[rr, "\widehat{\phi}"] \arrow[dr] & & \mathsf{ob}\big(\mathsf{S}(\mathsf{ob}A, \phi)\big) \arrow[rr, dashed, "\Lambda'", "\exists!"'] & & \mathsf{ob}\big(\mathsf{S}A\big) \arrow[r, shift left, "\mathsf{S}\nu^a"] \arrow[r, shift right, "\mathsf{S}\nu^b"'] \arrow[dd, "+"] & \mathsf{ob}\big(\mathsf{S}\mathbb{N}\big) \\
& \mathsf{ob}\big(\mathsf{QS}A\big) \arrow[dr, "\mathsf{Q}+"'] & & \mathsf{ob}\big(\mathsf{S}A\big) \arrow[ur, "\mathsf{S}\phi"'] & & \\
& & \mathsf{ob}\big(\mathsf{Q}A\big) \arrow[ur, equal] \arrow[rr, "h^\perp"'] & & \mathsf{ob}\big(A\big) &
\end{tikzcd} \tag{16.21}$$

Define the *a-signature* and *b-signature* of a general object $x \in \mathsf{S}(\mathsf{ob}A, \phi)$ via the corresponding signature of $\Lambda'(x)$, as follows:

$$\mathsf{sgn}^a(x) = (\mathsf{S}\nu^a)\Lambda'(x) \quad \text{and} \quad \mathsf{sgn}^b(x) = (\mathsf{S}\nu^b)\Lambda'(x). \tag{16.22}$$

This agrees with the previous definitions (16.20) for free objects $x \in \mathsf{S}(\mathsf{ob}A)$ since commutativity of (16.21) requires that $\Lambda'\kappa$ is the identity on objects. Note that these signatures are invariants of objects only; they do not extend to all morphisms of $\mathsf{S}(\mathsf{ob}A, \phi)$. This completes the definitions of a- and b-signature.

Now we apply the underlying permutation and signature invariants to complete the proof. The following observations, for objects x and y in $\mathsf{S}(\mathsf{ob}A, \phi)$, make use of the hypothesis $\phi = h_{\mathsf{ob}}$ and details of the diagram (16.12).

(1) If x is a lift of an object in (16.12), or isomorphic to such a lift, then the sum of the entries in $\mathsf{sgn}^a(x)$, respectively the sum of the entries of $\mathsf{sgn}^b(x)$, is equal to four.

(2) If x is a ϕ-object, then each entry of $\mathsf{sgn}^a(x)$, respectively $\mathsf{sgn}^b(x)$, is divisible by four. This follows from Explanation 14.17 because h is the quadrupling functor: Λ' sends each ϕ-object to an object of $\mathsf{S}A$ for which each entry is $h(a^j_\bullet)$ for a certain object $a^j_\bullet \in A$.

(3) If x is a ϕ-object such that each entry of $\mathsf{sgn}^a(x)$ and each entry of $\mathsf{sgn}^b(x)$ is zero, then x is a ; product of objects of the form $\big([\phi](0,\ldots,0)\big)$. This follows from the same explanation of Λ' as above, because every nonzero object of $\mathsf{S}A$ has nonzero a- or b-signature.

(4) The underlying a-permutation of each composite around (16.12) is $(1\ 4\ 3\ 2)$, which is an odd permutation. The same holds for the underlying b-permutations around (16.12).

(5) If $\xi\colon x \longrightarrow y$ is a free or formal morphism of $\mathsf{S}(\mathsf{ob}A, \phi)$, then $\mathsf{sgn}^a(x)$ and $\mathsf{sgn}^a(y)$ have the same set of entries, possibly in a permuted order. A similar observation holds for $\mathsf{sgn}^b(x)$ and $\mathsf{sgn}^b(y)$.

(6) If $\xi\colon x \longrightarrow y$ is a ϕ-morphism in $\mathsf{S}(\mathsf{ob}A, \phi)$, then Explanation 14.17 shows that $\Lambda\xi$ is given either by applying h to certain permutations, or by the monoidal constraints of h. Since h is given by quadrupling, and the underlying a-permutation of the monoidal constraint h_2 is trivial, the underlying a-permutation of $\Lambda\xi$ is even in either case. Likewise, the underlying b-permutation of $\Lambda\xi$ is also even.

(7) If all the entries of $\mathsf{sgn}^a(x)$ are even, and $\xi\colon x \longrightarrow y$ is a ; product of generating morphisms of $\mathsf{S}(\mathsf{ob}A, \phi)$, then the underlying a-permutation of ξ is even. The ; factors of ξ that are free or formal morphisms have underlying a-permutations that are even because they are given by permuting entries of tuples or factors of the ; product. The ; factors of ξ that are ϕ-morphisms have underlying a-permutations that are even by observation (6). A similar observation holds if all the entries of $\mathsf{sgn}^b(x)$ are even: then the underlying b-permutation of ξ is even.

Now suppose that

$$x_0 \xrightarrow{\xi_1} x_1 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_r} x_r \tag{16.23}$$

is a tidy composite in $\mathsf{S}(\mathsf{ob}A, \phi)$ lifting either of the composites around (16.12). Recalling Definition 16.15, the assumption that (16.23) is tidy means that each ξ_i is a product of generating morphisms and none of the x_i have ; factors of the form $\big([\phi](0,\ldots,0)\big)$. The observations above lead to the following conclusions.

i. The a-signature $\mathsf{sgn}^a(x_0)$ must have at least one odd entry. If not—if all the entries of $\mathsf{sgn}^a(x_0)$ are even—then observations (2), (5), (6), and (7) imply that the underlying a-permutation of ξ_1 is even and that all the entries of $\mathsf{sgn}^a(x_1)$ are even. Repeating this reasoning, the underlying a-permutation of each ξ_i is even, but this contradicts observation (4). Likewise, $\mathsf{sgn}^b(x_0)$ must have at least one odd entry.

ii. Observations (1), (2), and (3), combined with the previous conclusion, imply that any ϕ-object factors of x_0 would have a- and b-signatures whose entries are all zeros. By (3), this would contradict the assumption that x_0 is a tidy object.

iii. Therefore, x_0 must be a free object such that each of $\mathsf{sgn}^a(x_0)$ and $\mathsf{sgn}^b(x_0)$ consists of entries that are all less than four and not all even.

iv. The previous conclusion implies that ξ_1 must be a free morphism, since a product of free objects or morphisms is free, and a formal morphism between free objects is identified with the corresponding free morphism.

v. Hence, x_1 must be a free object such that each of $\mathsf{sgn}^a(x_1)$ and $\mathsf{sgn}^b(x_1)$ consists of entries that are all less than four and not all even.

vi. Repeating the conclusions above, each morphism ξ_i and object x_i in (16.23) is free.

This completes the proof. ☐

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